Exam 1 Soultions — Fall 2010 — ACMS 20340

1. The first part is essentially asking for $P(A \cap B)$. There are two ways of doing this. The first way starts with the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. We know $P(A)$ and $P(B)$. Event $C$ is the complement of $A \cup B$, so $P(A \cup B) = P(C^c) = 1 - P(C) = 0.67$. Then some arithmetic gives $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.15 - 0.67 = 0.08$.

   The second way starts with a table:

   | | $A$ | $A^c$ | sum |
   |---|---|---|---|
   | $B$ | | | 0.15 |
   | $B^c$ | | 0.33 | |
   | sum | 0.6 | | 1 |

   Filling in the missing squares then shows $P(A \cap B) = 0.08$.

   The main point is that you cannot use $P(A \cap B) = P(A)P(B)$ without already knowing (or assuming) $A$ and $B$ are independent.

   For the second part, they are not mutually exclusive since they can occur at the same time. Why? Because $P(A \cap B) > 0$.

   For the third part, there is no way to do this without making some calculations. $A$ and $B$ are independent only if one of the following is true

   - $P(A|B) = P(A)$
   - $P(B|A) = P(B)$
   - $P(A \cap B) = P(A)P(B)$

   Using the third test, I get $P(A)P(B) = (0.6)(0.15) = 0.9 \neq 0.8 = P(A \cap B)$. Thus they are not independent.

2. A tree diagram could help organize the calculations. The possible outcomes (in dollars) are 0, 10, 20, and 30.

| $k$ | $P(X=k)$ |
|---|---|
| 0 | $(5/6)^3$ |
| 10 | $1/6$ |
| 20 | $(5/6)(1/6) = 5/36$ |
| 30 | $(5/6)(5/6)(1/6) = 25/216$ |

Use the general formulas $E(X) = \sum k \cdot p(k)$ and $Var(X) = \sum(k - \mu)^2 p(k)$. I get $E(X) = 7.92$ and $Var(X) = 113.66$. In particular, don't forget the $k = 0$ term in the computation of the variance. Some people rounded the probabilities in the first part to two decimal places, this is okay, but notice that the sum is then no longer 1. This threw off the expected value.

For the third part, we see that the player expects to win \$7.92 on average. That means the casino expects to lose \$7.92 on average. By charging \$8 the casino then expects to earn an average of \$0.08 per game, for a profit. Moreover, \$8 is the smallest whole dollar amount for which the casino expects a profit.

3. This is like the birthday problem. The easiest way is to figure out the probability of having none match, and then take the complement. In this case $P(\text{none match}) = \frac{6}{6}\frac{5}{6}\frac{4}{6} = \frac{5}{9}$. And so $P(\text{at least two match}) = 1 - \frac{5}{9} = \frac{4}{9}$.

   Another way is to directly calculate it out. There are two cases: all three dice match and exactly two dice match. The exactly two case is tricky.

$$P(\text{all three}) = \frac{6}{6}\frac{1}{6}\frac{1}{6}$$
$$P(\text{exactly 2}) = \frac{C(3,2) \cdot 6 \cdot 5}{6^3}$$

   The idea in the exactly two case is: choose two dice to match, choose a number to make be the pair, choose another number to not match. Adding gives the answer: $6/216 + 90/216 = 4/9$

4. The first part is just like the homework and examples we did in class. There is one green ball and 29 non-green balls. To draw 3 and get a green, we can draw any one of the (one) green ball, and any 2 of the non-green balls. Dividing then gives the probability

$$\frac{C(1,1)C(29,2)}{C(30,3)} = 0.1$$

To deal with my brother's urn, we need to use conditional probabilities. I thought of this by making a tree diagram, but here I do it with conditional probabilities. Let $B$ be the event that I choose my brothers urn. Let $R$ be the event that I draw two red balls. Then $P(R|B) = C(50,2)/C(75,2)$ and $P(R|B^c) = C(12,2)/C(30,2)$ come from the usual urn calculations. Now $P(B) = 0.5$, making

$$\begin{aligned} P(R) &= P(R \cap B) + P(R \cap B^c) \\ &= P(B)P(R|B) + P(B^c)P(R|B^c) \\ &= (0.5)\frac{C(50,2)}{C(75,2)} + (0.5)\frac{C(12,2)}{C(30,2)} \end{aligned}$$

I get $P(R) = 0.296$.

The third part follows from the second. Use the formula $P(B|R) = P(B \cap R)/P(R)$. I get the probability is 0.744.

5. This is a binomial distribution with $n = 20$ and $p = 1000/8300$. The first two parts come directly from formulas, so $E(X) = np = 2.41$, and $Var(X) = npq = 2.12$.

   The third part is asking for $P(X < 2.41)$. Since $X$ can only take whole numbers, this is the same as $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$. Use $P(X = k) = C(n,k)p^k q^{n-k}$ three times, ending up with the probability 0.56.

   Some people insisted on approximating this with the Poisson distribution. Sigh. This fits the criteria I mentioned in class, so in this case, okay. But the goal was to use the Binomial distribution.

6. There are 2 dolphins per hour, and the tour is 1.5 hours, so take $\mu = (2)(1.5) = 3$.

   The second part asks for $P(X \geq 2)$. Expanding this as $\sum_{k=2}^{\infty} P(X = k)$ is an infinite sum, and from a calculation perspective, it is not clear when we can stop adding terms. It is much easier to take the complement: $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$. Use the formula for the Poisson:

$$1 - \left(\frac{3^0 e^{-3}}{0!} + \frac{3e^{-3}}{1!}\right)$$

   I get $P(X \geq 2) = 0.8$.