Mid-semester Exam 1 Solutions

1. Every day I see the same 9 runners pass my house - three men (Alfred, Bob, and Christopher) and six women (Delilah, Edith, Frannie, Georgina, Hattie, and Idell). One of the men (Bob) and two of the women (Edith and Hattie) always run in shorts, while the rest run in pants. At any particular moment, the next runner to pass my house is equally likely to be any of the nine. I am interested in the following events:

   - A: the next runner is a man
   - B: the next runner is a woman
   - C: the next runner is wearing shorts

   (a) Compute the probabilities of A, B, and C.

   Solution: Since each simple outcome is equally likely, $P(A) = 3/9 = 1/3$, $P(B) = 6/9 = 2/3$, and $P(C) = 3/9 = 1/3$.

   (b) Are the events A and B independent?

   Solution: No, since $P(A)P(B) = 1/3*2/3 = 2/9$, and $A \cap B = \varnothing$ so $P(A \cap B) = 0$, therefore $P(A \cap B) \neq P(A)P(B)$.

   (c) Are the events A and C independent?

   Solution: Yes, since $P(A)P(C) = 1/3 * 1/3 = 1/9$ and $A \cap C = \{\text{Bob}\}$, so $P(A \cap C) = 1/9$, therefore $P(A \cap C) = P(A)P(C)$.

   (d) Write down a pair of events (from A, B, and C) that are mutually exclusive (explain your choice).

   Solution: A and B are mutually exclusive, since they don't have any outcomes in common.

2. Suppose that there are 80,000 at a ND football game, and 64,000 of them (unknown to me) have read the book "A Prayer for Owen Meany" by John Irving. I choose, at random, 15 people and ask them if they've read the book. Let x be the number who say they have.

   (a) What are the possible values for x (having '...' in your answer is fine, as long as you tell me the first few and the last few values)?

   Solution: x can be $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$

   (b) What is the expected value and the variance of x?

   Solution: This can be considered a binomial experiment with $n = 15, p = 64,000/80,000 = .8$ (note that $n = 15$, $N = 80,000$, so $n/N \leq .05$, so officially it fits into the 'rule of thumb' for choosing from a population). Therefore $\mu = 15 * .8 = 12$ and $\sigma^2 = 15 * .8 * .2 = 2.4$

(c) Compute the probability that exactly 10 have read the book.

Solution: You can either use the table and do $P(x \leq 10) - P(x \leq 9)$, or you can use the formula for a binomial probability: $P(x = 10) = C^{15}_{10}(.8)^{10}(.2)^5 = .103....$

(d) Compute the probability that x is strictly greater than its expected value. [You may use the attached table here, if you'd like.]

Solution: Since $\mu = 12$, we're looking for $P(x > 12) = 1 - P(x \leq 12) = .398...$

3. Experience tells me that when I go to the Country Bake Shop:

- 15% of the time, they'll have 0 cherry pies left;
- 35% of the time, they'll have 2 cherry pies left; and
- 50% of the time, they'll have 4 cherry pies left.

If there is at least one pie that I can buy, I'll buy one pie 80% of the time (independent of the actual number of pies remaining), and head home; the remaining 20% of the time I don't buy a pie and I head home.

Let x be the number of pies left AFTER I leave the store.

(a) Find a probability distribution for x by making a table with possible values of x and their associated probabilities.

Solution: By the independence of the choice of pie (for example, the probability of one pie remaining is the probability of two pies being there AND me buying one, hence I multiply the probabilities),

x	$p(x)$
0	.15
1	.35*.8 = .28
2	.35*.2 = .07
3	.50*.8 = .4
4	.50*.2 = .1

(b) Compute the expectation of x.

Solution: $\mu = 0(.15) + 1(.28) + 2(.07) + 3(.4) + 4(.1) = 2.02$

(c) Compute the variance of x.

Solution: $\sigma^2 = (0-2.02)^2(.15)+(1-2.02)^2(.28)+(2-2.02)^2(.4)+(3-2.02)^2(.4)+(4-2.02)^2(.1) = 1.6797...$

4. Five turtles - Alex, Billy-Bob, Cassandra, Daisy, and Edwardo - are pitted in a race. The turtles are equally matched, so all finishing orders are equally likely. There are no ties. The top two finishers get a prize.

(a) How many possible finishing orders are there?

Solution: Since the order matters, there are 5! possible finishing orders

(b) How many finishing orders are there in which neither Alex nor Billy-Bob get a prize?

Solution: We'll use the extended mn rule here. For first place, there are 3 possible choices (C,D,E). Then, for second place, there are two remaining possibilities. After that, there are three people left, and there are no more restrictions, so there are 3 possibilities for third, 2 for fourth, and 1 for fifth. In pictures,

$$\underline{3} * \underline{2} * \underline{3} * \underline{2} * \underline{1}.$$

Therefore, there are 36 finishing orders in which neither Alex nor Billy-Bob get a prize.

(c) What is the probability that neither Alex nor Billy-Bob get a prize?

Solution: If A is the event that neither Alex nor Billy-Bob get a prize,

$$P(A) = \frac{n_A}{N} = \frac{36}{120} = \frac{3}{10}.$$

5. Suppose that 2% of all the students seeking treatment at the ND Health Center are eventually diagnosed as having mononucleosis (i.e., mono). Of those who do have mono, 90% complain of a sore throat. But 30% of those not having mono also have sore throats.

(a) What is the probability that a student at the Health Center has both mono and a sore throat? [you may find a tree diagram helpful here and in the following parts]

Solution: The tree diagram might help. The algebra way to do this is (if M = having mono, S = having a sore throat),

$$P(S \cap M) = P(M)P(S|M) = (.02)(.90) = .018.$$

(b) What is the probability that a student at the Health Center has a sore throat?

Solution:

$$P(S) = P(S \cap M) + P(S \cap M^c) = .018 + P(M^c)P(S|M^c) = .018 + (.98)(.30) = .312.$$

(c) What is the probability that a student at the Health Center has mono or a sore throat (or both)?

Solution:

$$P(S \cup M) = P(S) + P(M) - P(S \cap M) = .312 + .02 - .018 = .314.$$

(d) A student in the Health Center complains of a sore throat. What is the probability that they have mono?

Solution:

$$P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{.018}{.312} = .05769...$$

6. A bleary-eyed student awakens one morning at 8:27AM. In order to make it to her 8:30AM class, she randomly pulls two socks out of a drawer that contains two red, six purple, and two yellow socks.

(a) How many ways can she pull two socks out of the drawer (regardless of their respective colors)?

Solution: Assuming that order doesn't matter (note: it can be done if you put an order on the socks that you pull out of the drawer as well - if you did it this way, you'd need to assume that order matters through the entire problem), there are 10 socks and you choose 2 of them, so there are $C_2^{10} = 45$ ways to do this.

(b) How many ways can she pull two socks out of the drawer that are the same color?

Solution: There is $C_2^2 = 1$ way to pull out two red socks, $C_2^2 = 1$ way to pull out two yellow socks, and $C_2^6 = 15$ ways to pull out two purple socks. Therefore, there are 17 ways to pull out two socks of the same color.

(c) What is the probability that she pulls two socks out of the drawer that are the same color?

Solution: If A is the event that she pulls out two socks of the same color, we have

$$P(A) = \frac{n_A}{N} = \frac{17}{45}.$$

7. **Bonus:** For reasons not entirely clear, Doomsday Airlines books a daily shuttle service from Altoona to Hoboken. They offer two flights, one on a two-engine prop plane, the other on a four-engine prop plane. Suppose that each engine on each plane will fail independently with the same probability p and that each plane will arrive safely at its destination only if at least half its engines remain in working order. Assuming that you want to continue living, for what values of p would you prefer to fly in the two-engine plane?

[Hint: find the p's for which the probability of crashing in the two-engine plane is smaller than the probability of crashing in the four-engine plane]

Solution: Following the hint, the two-engine plane crashes if both engines fail, so this happens (due to independence) with probability p^2. The four-engine plane crashes if either all 4 engines fail (which happens with probability p^4) or if one engine fails (which happens with probability $C_1^4 p^3(1-p) = 4p^3(1-p)$, as this can be considered a binomial trial). Therefore, since we want to prefer to fly in the two-engine plane, we're looking for p's with

$$p^2 < 4p^3(1-p) + p^4.$$

Noting that if $p = 0$, they both won't ever crash (and so you have no preference), we can divide by p^2 (as we know $p = 0$ doesn't make the inequality hold). So

$$1 < 4p(1-p) + p^2 = 4p - 3p^2.$$

Rearranging,

$$3p^2 - 4p + 1 < 0$$

or

$$(3p-1)(p-1) < 0.$$

Now, when $p = 1$, they both will always crash, so again we don't prefer either plane. Therefore, for all other p's we have $p - 1 < 0$, so for the inequality to hold we need $3p - 1 > 0$, or $p > 1/3$. Therefore, officially, you would prefer the two-engine plane if $1/3 < p < 1$.