HOMEWORK FOR LINEAR ALGEBRA - SUMMER 2010

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These homework questions are meant to help solidify your understanding. I've put some computational questions in here but have skipped others. You should consider doing even more of the problems than I have listed here; it will only help you probe the concepts better. I'm very willing to discuss any problem you try, listed or not.

- 1.2 (p 5): 2,4,5*,7

  Remark. You don't have to write out every detail, as this would involve checking all cases (e.g. for 1., you'd have to show that $0 + 0 = 0 = 0 + 0, 0 + 1 = 1 = 1 + 0, 1 + 1 = 0 = 1 + 1$ - all possible elements commute with the other ones). Just convince yourself by mentally checking through the cases, so you get a feel for the different parts of the definition.

- 1.3 (p 10): 2,3,4,5,6
- 1.4 (p 15): 1,2,3,4*, 5,6,8,10

  Remark. (*): Before doing the computation here, describe what each of these equations are geometrically, and what the solution to this system means geometrically. Mathematica might be helpful in visualizing this.

- 1.5 (p 21): 3,4,5,6,7*, 8**

  Remark. (*): We'll show this in general ($n \times n$ matrices) using vector space theory later.
  (**): $C_{11} + C_{22}$ is called the *trace* of C.

- 1.6 (p 26): 4,5,6,7,8,9
- 2.1 (p 33): 2,3,4,5,6
- 2.2 (p 39): 2,3,4,5,6,8,9
- 2.3 (p 48): 2,3,4,5,7,9,10,11,12, also:
  (1) Suppose U and W are subspaces of the vector space V with dim V=12. Further, suppose that dim $U = 4$, dim $W = 8$, and $U + W = V$. Prove that $U \cap V = \{0\}$.
  (2) Suppose U and W are both 4-dimensional subspaces of $\mathbb{R}^7$. Prove that $U \cap W \neq \{0\}$.
  (3) Let $V = \mathbb{R}[x]_3$ be the space of polynomials with coefficients in $\mathbb{R}$ of degree less than or equal to 3. Prove or disprove the following: There exists a basis of V, $\{p_0, p_1, p_2, p_3\}$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.
  (4) Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}, \sum_{i=1}^{n} a_i = 0\}$. Show that V is a subspace of $\mathbb{R}^n$, calculate $\dim V$, and find a basis for V.

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- 2.4 (p 54): 1,3,4,7
- 2.6 (p 66): 2,4,5
- 3.1 (p 73): 1,3,4,5,7,8,9,10,11,12,13
- 3.2 (p 84): 1,2,3,4,6,8,9,10,11, also:
  (1) Let f and g be functions such that $f \circ g = id$, where id is the identity function, defined on some correct domains (if you want, think of these as linear transformations, but the result is true for general functions). Show that:
    (a) f is onto (a.k.a *surjective*)
    (b) g is $1-1$ (a.k.a *injective*)
    I remember this result by thinking of f as 'outside' and g as 'inside' for $f(g(x)) = x$, so outside is onto and inside is injective (note the alliteration here).
- 3.3 (p 85): 1,4,6,7, also
  (1) Let $U, T \in L(V, V)$.
    (a) Suppose $T^2 = 0$. Show T is not invertible.
    (b) Suppose $TU = 0$. Can T be invertible? Justify.
- 3.4 (p 95): 1,3,4,5,6a)b)c),8,9,11(first paragraph only),12
- 3.5 (p 105): 1,2,3,4-I think the computation is long here, so only set up what you need to do...don't actually solve, 5,8,11,12,17(do 3×3 matrices instead of $n \times n$, but it's easy to see how this should generalize)
- 3.6 (this is actually more of a 3.5 problem...):
  (1) Suppose that V is an *infinite* dimensional vector space over a field F, and suppose that it has basis $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. Similar to what we tried before, define f_i to be the function $f_i(\alpha_j) = \delta_{ij}$. Show that $\mathcal{B}^* = \{f_1, f_2, f_3, \ldots\}$ does *not* span V^* (so is not a basis). [*Hint*: Define a linear functional $f : V \to F$ by specifying what f does each element of the basis $\mathcal{B}$ (and then extend to all of V by linearity) in such a way that f can't be a (finite) linear combination of f_i's]
- 5.2 (p 149): 1,3,4,5,6,9,12
- 5.3 (p 155): 3,4,5,6,7,8, also
  (1) If A is an $n \times n$ matrix, how are $\det(A)$ and $\det(7A)$ related?
  (2) Let $[n] = \{1, 2, \ldots n\}$. Show that a map $\sigma : [n] \to [n]$ is injective if and only if it is surjective. Thus, for σ to be in S_n (the set of all permutations on $[n]$) it is necessary and sufficient to be 1-1 or onto (this is pretty obvious - can you write something down in proof form that is convincing?).
  (3) Consider S_9, the permutations on the set $\{1, \ldots, 9\}$.
    (a) Show that S_9 contains an even number of elements.
    (b) Show that half of the permutations in S_9 are odd permutations (*Hint:* think in terms of determinants, not in terms of permutations...)
  (4) An $n \times n$ matrix is called *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$. Show that if A is nilpotent, then $\det A = 0$.
  (5) Prove that if A and B are similar $n \times n$ matrices, then $\det A = \det B$. Is the converse true? Prove or give a counterexample.
- 5.4 (p 163): 3,4,5,7,10, also:
  (1) Show that a permutation matrix is *orthogonal*, i.e. that $P^t = P^{-1}$.

- 6.2 (p 189): 2,3,5,6,7,8,9,13 (the other questions in the section are quite good, too)
- 6.3 (p 197): 1,3 (maybe use mathematica/calculator for matrix powers),4,5 (*Hint:* you might want to use long division of polynomials at some point here..., i.e. if degree $p_1(x) >$ degree $p_2(x)$, then there are polynomials $q(x), r(x)$ with $p_1(x) = q(x)p_2(x) + r(x)$, with either degree $r(x) <$ degree $p_2(x)$ or $r(x) = 0$),6,7,8
- 6.4 (p 205): 1,5,8,9,11(prove it or give a counterexample)