Bayesian *k*-Space Estimation for FMRI

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Outline

1. Introduction

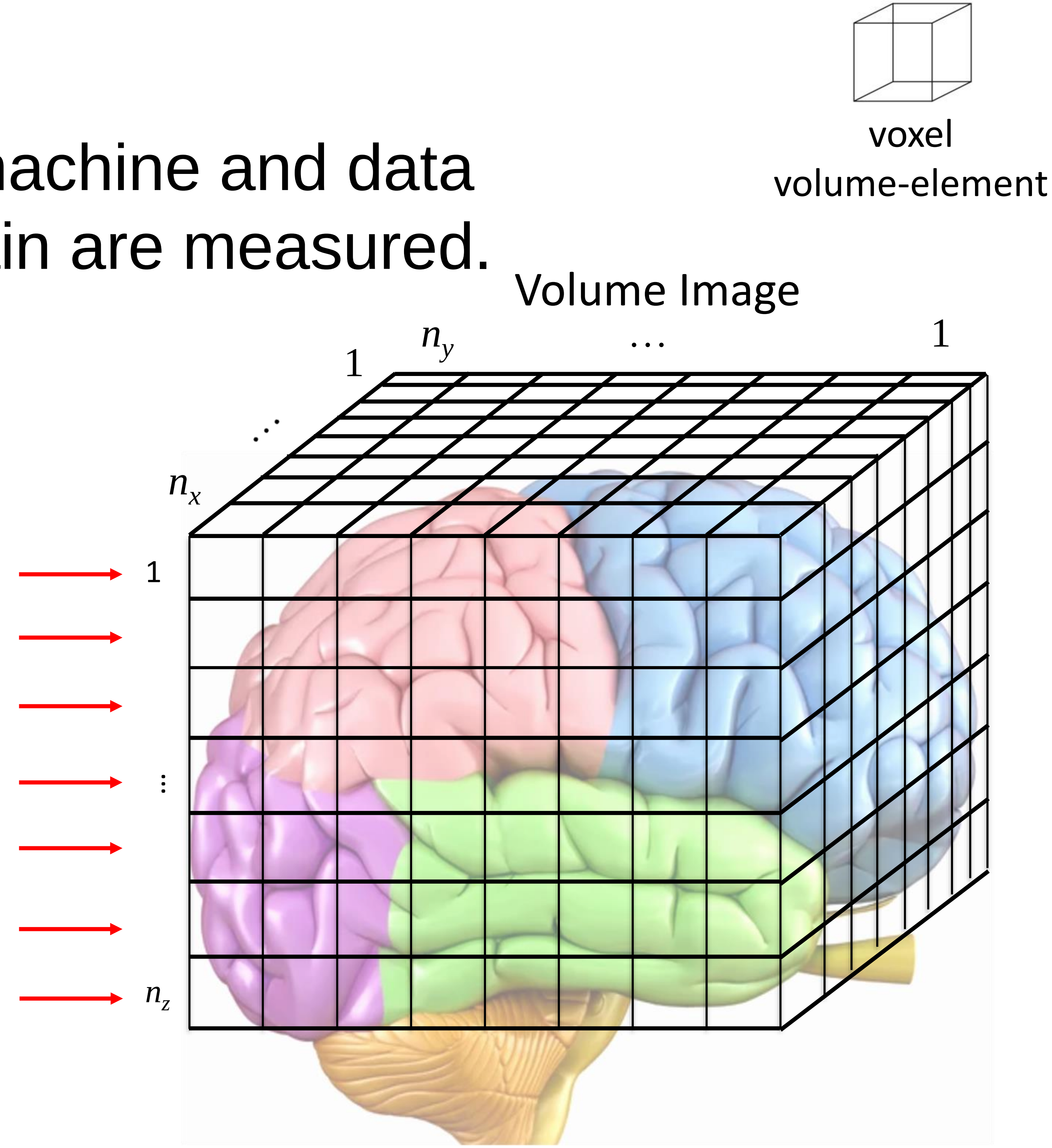
2. Methods

3. Experimental Results

4. Discussion

1. Introduction

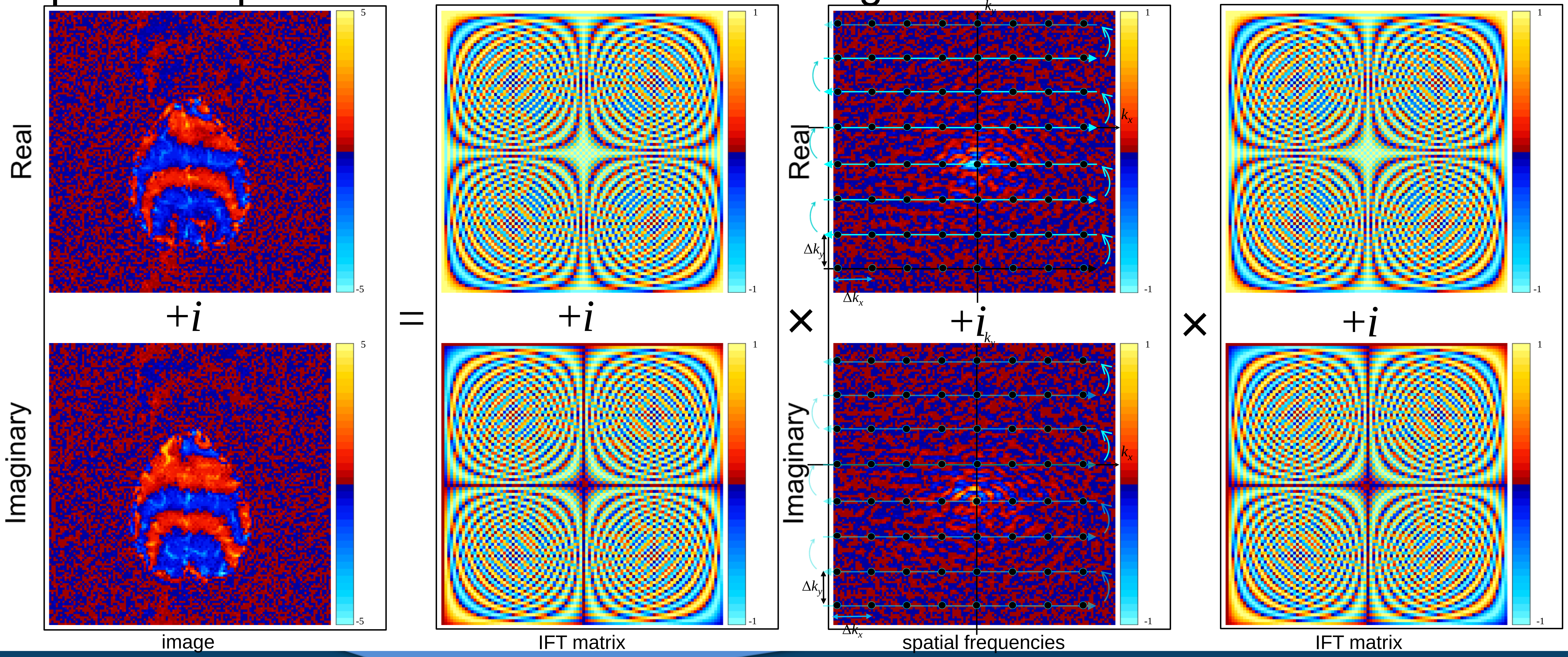
In fMRI, a subject is placed in the MRI machine and data for slice-wise volume images of their brain are measured.



1. Introduction

T_2* weighted images
n_x=n_y=96
FOV=240 mm
TR=1 image

Spatial frequencies are measured and images are IDFT reconstructed.



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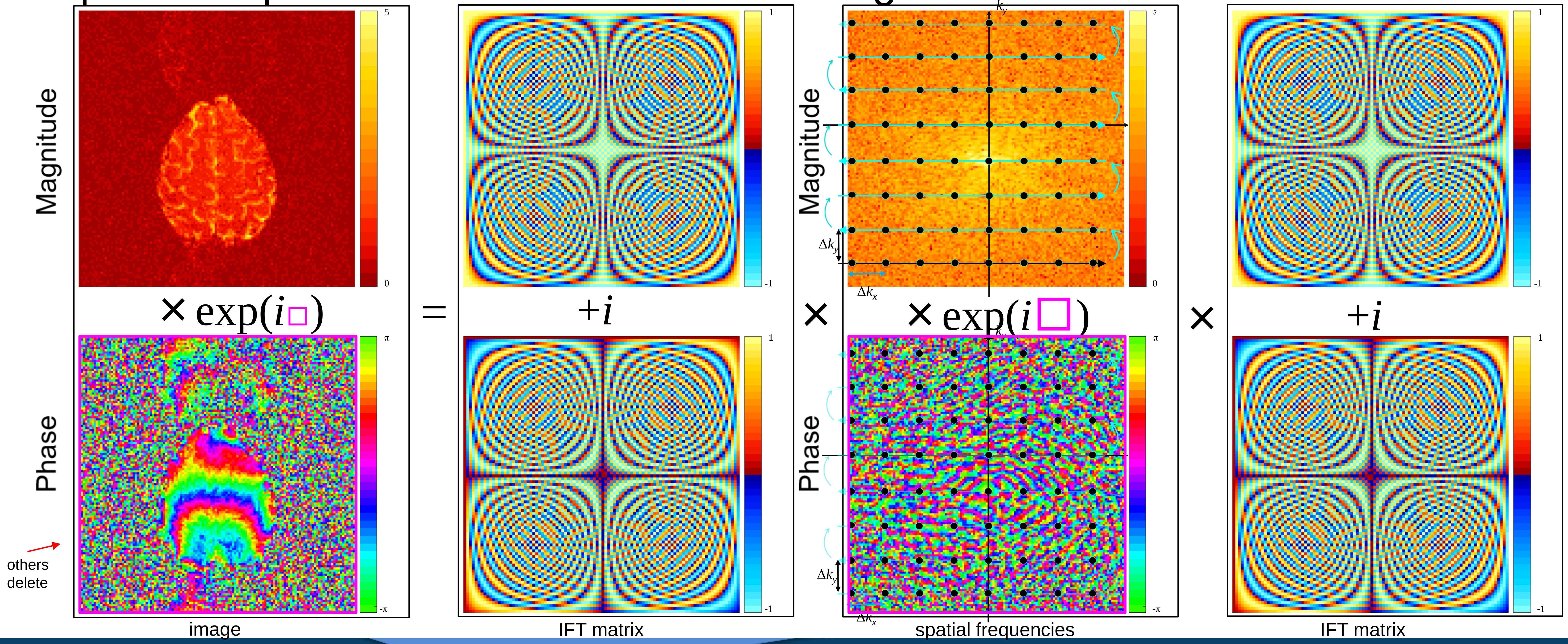
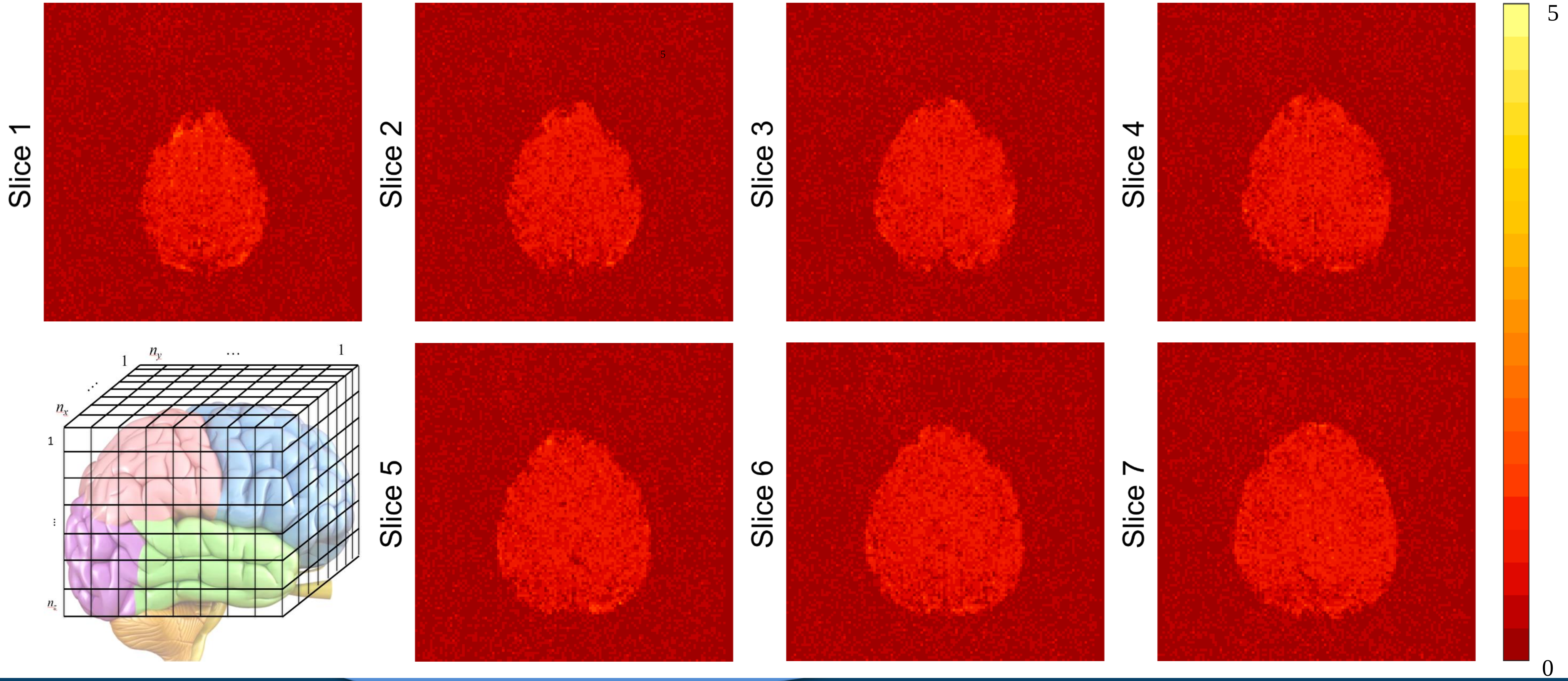


1. Introduction

T_2* weighted images
n_x=n_y=96
FOV=240 mm
TR=4 image

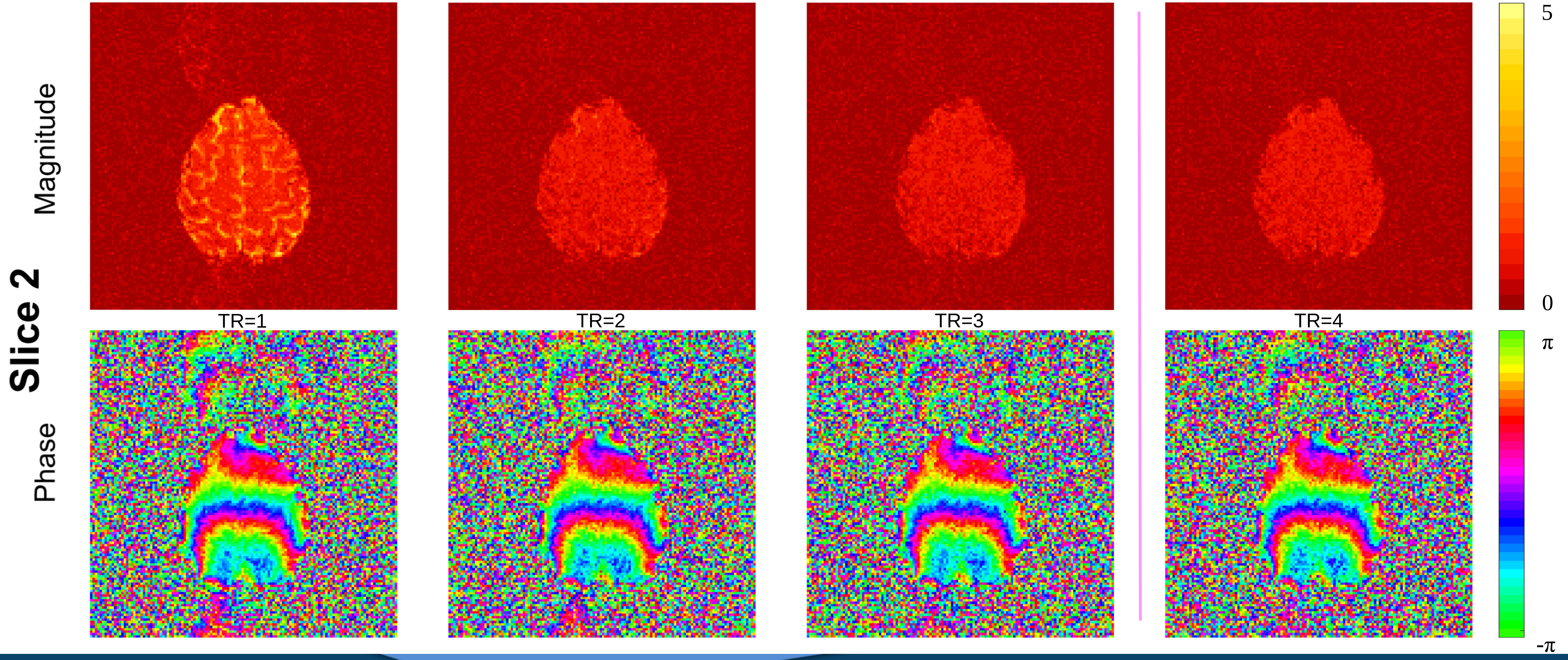
The problem is that often images have low contrast and are very noisy.



1. Introduction

T_2* weighted images
n_x=n_y=96
FOV=240 mm
TR=1 to 4 images

We can use first three discarded images to enhance images four on.

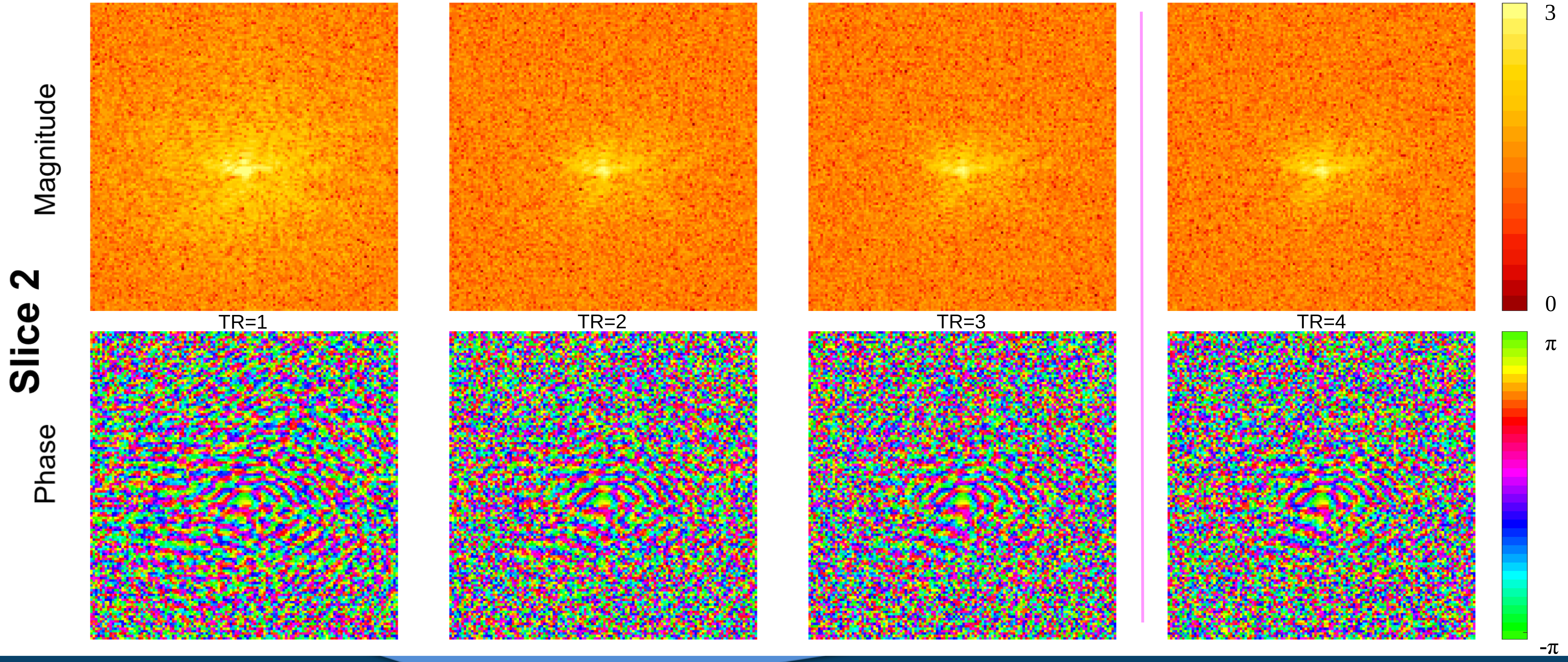


1. Introduction

$$r = (k_R^2 + k_I^2)^{1/2}$$

$$\phi = \text{atan}(k_I / k_R)$$

FMRI images are not measured directly, but their *k*-space is.



2. Methods - Likelihood

The ADCs measure independent normally distributed real and imaginary parts of a given spatial frequency coefficient as

$$f(k_R,k_I\,|\,\rho,\theta,\sigma^2)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left[-\frac{(k_R-\rho\cos\theta)^2}{2\sigma^2}\right]\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left[-\frac{(k_I-\rho\sin\theta)^2}{2\sigma^2}\right]$$

we can convert to polar coordinates as

$$f(r,\phi\,|\,\rho,\theta,\sigma^2)=\frac{r}{2\pi\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left[r^2+\rho^2-2r\rho\cos(\phi-\theta)\right]\right\}\quad .$$

Rice, 1944. Gudbjartsson and Patz, 1995l

2. Methods - Likelihood

It can be shown that the marginal distribution of the magnitude is Ricean

$$f(r \mid \rho, \sigma^2) = \frac{r}{\sigma^2} \exp\left\{ -\frac{r^2 + \rho^2}{2\sigma^2} \right\} I_0\left(\frac{r\rho}{\sigma^2} \right)$$

and the conditional distribution of the phase given the magnitude is von Mises

$$f(\phi \mid r, \rho, \theta, \sigma^2) = \frac{\exp\left\{ \dfrac{r\rho\cos(\phi - \theta)}{\sigma^2} \right\}}{2\pi I_0\left(\dfrac{r\rho}{\sigma^2} \right)} .$$

Rice, 1944. von Mises, 1918.

2. Methods - Priors

From the form of the likelihood, we can utilize priors to be Ricean for the magnitude as

$$f(\rho \mid \rho_0, \sigma^2, \gamma) = \frac{\rho}{\sigma^2 / \gamma} \exp\left\{-\frac{\rho^2 + \rho_0^2}{2\sigma^2 / \gamma}\right\} I_0\left(\frac{\rho\rho_0}{\sigma^2 / \gamma}\right),$$

von Mises for the phase conditional on the magnitude as

$$f(\theta \mid \rho, \sigma^2, \theta_0, \gamma) = \exp\left\{\frac{\rho\rho_0 \cos(\theta - \theta_0)}{\sigma^2 / \gamma}\right\} \Big/ 2\pi I_0\left(\frac{\rho\rho_0}{\sigma^2 / \gamma}\right)$$

and of course inverse gamma for the variance as

$$f(\sigma^2 \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{\beta}{\sigma^2}\right\}.$$

Similar to BIFS model which assumes real images and needs MCMC.

2. Methods - Posterior

The joint posterior distribution is not easily marginalizable. Fortunately, we can obtain posterior conditionals

$$f(\theta \mid \rho, \sigma^2, r, \phi) = \frac{\exp\{\kappa \cos(\theta - \lambda)\}}{2\pi I_0(\kappa)}$$

Von Mises

where

$$\kappa = c\rho / \sigma^2$$

$$\lambda = \arctan\left(b / a\right)$$

$$a = \rho_0 \gamma \cos\left(\theta_0\right) + r\cos\left(\phi\right)$$

$$b = \rho_0 \gamma \sin\left(\theta_0\right) + r\sin\left(\phi\right)$$

$$c = \operatorname{sign}\left(a\right)\sqrt{\left(a^2 + b^2\right)}$$

for the phase, and

$$f(\rho \mid \theta, \sigma^2, r, \phi) = \frac{2B^{A/2}\rho^{A-1}}{\Psi\left(\frac{A}{2}, \frac{C}{\sqrt{B}}\right)} \exp\left\{-B\rho^2 + C\rho\right\}$$

Modified Half Normal

$$A = 2$$

$$B = (\gamma + 1) / (2\sigma^2)$$

$$C = [\rho_0 \gamma \cos\left(\theta - \theta_0\right) + r\cos\left(\phi - \theta\right)] / \sigma^2$$

for the magnitude, and

$$f(\sigma^2 \mid \rho, \theta, r, \phi) = \frac{(\beta^*)^{\alpha^*}}{\Gamma(\alpha^*)(\sigma^2)^{\alpha^*+1}} \exp\left(-\frac{\beta^*}{\sigma^2}\right)$$

Inverse Gamma

$$\alpha^* = \alpha + 2$$

$$\beta^* = \left\{ \begin{array}{l} (\gamma + 1)\rho^2 \\ -2\rho[\rho_0 \gamma \cos\left(\theta - \theta_0\right) + r\cos\left(\phi - \theta\right)] \\ +(\gamma \rho_0^{\,2} + r^2 + 2\beta) \end{array} \right\} / 2$$

for the variance.

2. Methods

Since we have the posterior conditionals,

$\theta|\rho,\sigma^2 \sim VM(\lambda,\kappa)$, $\rho|\theta,\sigma^2 \sim MHN(A,B,C)$, and $\sigma^2|\rho,\theta \sim IG(\alpha^*,\beta^*)$

we could implement a Gibbs sampler to generate a large sample

$(\theta_{(1)},\rho_{(1)},\sigma^2_{(1)}),\ldots,(\theta_{(L)},\rho_{(L)},\sigma^2_{(L)})$

from the posterior distribution and calculate marginal posterior means.

But this is very computationally expensive, and may involve acceptance-rejection sampling for the modified half normal.

Geman and Geman, 1984; Gelfand and Smith, 1990

2. Methods

Alternatively, we can obtain maximum *a posteriori* estimates of the parameters using the *ICM* algorithm that cycles through the modes of the posterior conditionals

$$\hat{\theta} = \arctan\left(\frac{b}{a}\right)$$

$$\hat{\rho} = \frac{C + \sqrt{C^2 + 8B(A-1)}}{4B}$$

$$\hat{\sigma}^2 = \frac{\beta *}{\alpha * + 1}$$

until convergence at MAP.

$$a = \rho_0 \gamma \cos\left(\theta_0\right) + r\cos\left(\phi\right)$$
$$b = \rho_0 \gamma \sin\left(\theta_0\right) + r\sin\left(\phi\right)$$

$$A = 2$$
$$B = (\gamma + 1) / (2\sigma^2)$$
$$C = [\rho_0 \gamma \cos\left(\theta - \theta_0\right) + r\cos\left(\phi - \theta\right)] / \sigma^2$$

$$\alpha * = \alpha + 2$$
$$\beta * = \left\{ \begin{array}{l} (\gamma + 1)\rho^2 \\ -2\rho[\rho_0 \gamma \cos\left(\theta - \theta_0\right) + r\cos\left(\phi - \theta\right)] \\ +(\gamma \rho_0{}^2 + r^2 + 2\beta) \end{array} \right\} / 2$$

Lindley and Smith, 1972; O'Hagan, 1994

3. Experimental Results

FMRI data from a block design right-hand finger tapping experiment.
$n_x = n_y = 128$, $n_z = 7$, $TH = 2.5$mm, $FOV = 240$mm, $TE = 60.4$ms, $EESP = 0.832$ms, $TR = 1$s

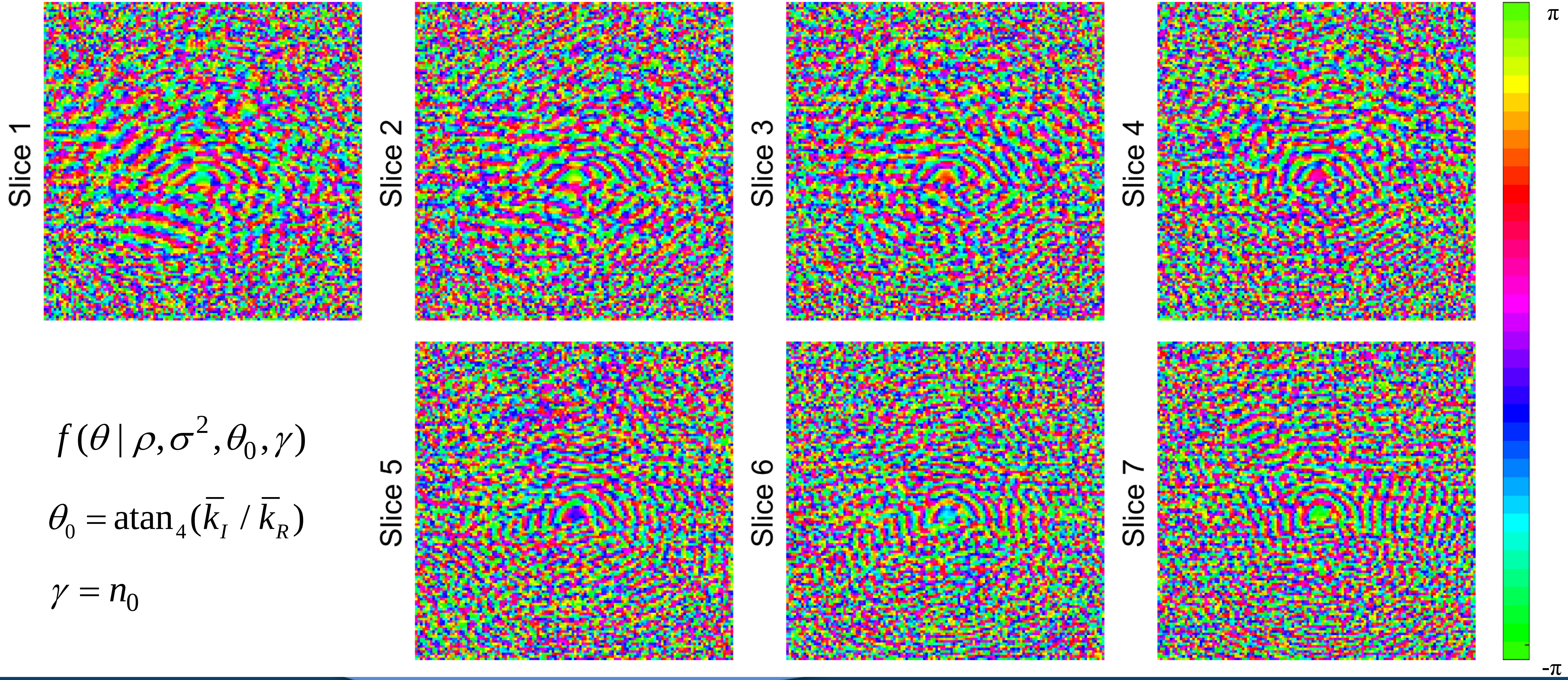
The experiment timing was an initial 16s of rest followed by 19 epochs of 16s of task alternating with 16s of rest resulting in a total of $n_t = 624$ total image volumes.

Following standard practice the first $n_0 = 3$ volume images are omitted resulting in $n = 621$ image volumes for analysis.

However, these first omitted images are perfect for hyperparameter assessment. $\theta_0 = \text{atan}_4(\bar{k}_I / \bar{k}_R)$, $\gamma = n_0$, $\rho_0 = (\bar{k}_x + \bar{k}_y)^{1/2}$, $\alpha = n_0 - 1$, $\beta = (n_0 - 1)\sigma_0^2$, $\sigma_0^2 = \dfrac{s_R^2 + s_I^2}{2}$

3. Experimental Results

Prior k-space phase mean.



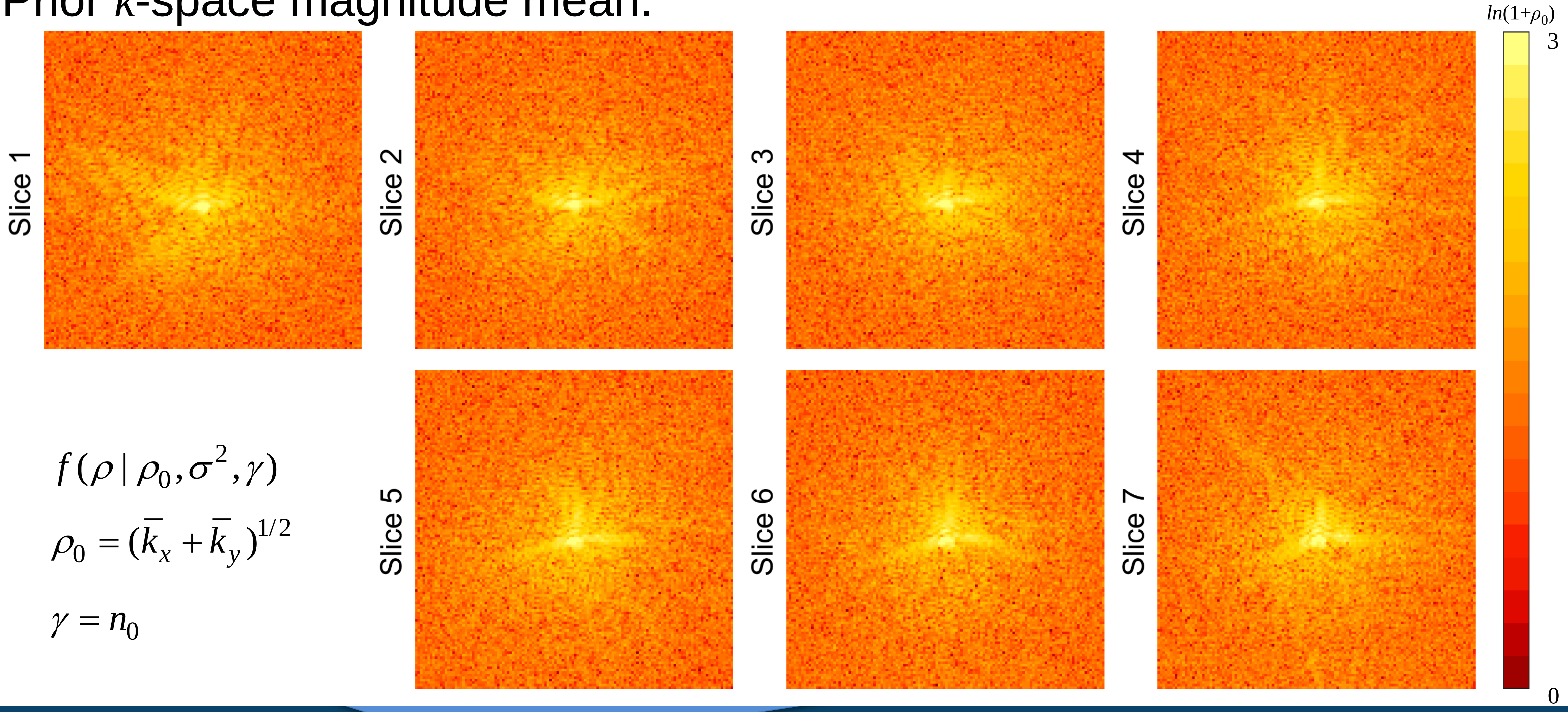
$$f(\theta \mid \rho, \sigma^2, \theta_0, \gamma)$$

$$\theta_0 = \text{atan}_4(\bar{k}_I / \bar{k}_R)$$

$$\gamma = n_0$$

3. Experimental Results

Prior k-space magnitude mean.



$$f(\rho \mid \rho_0, \sigma^2, \gamma)$$

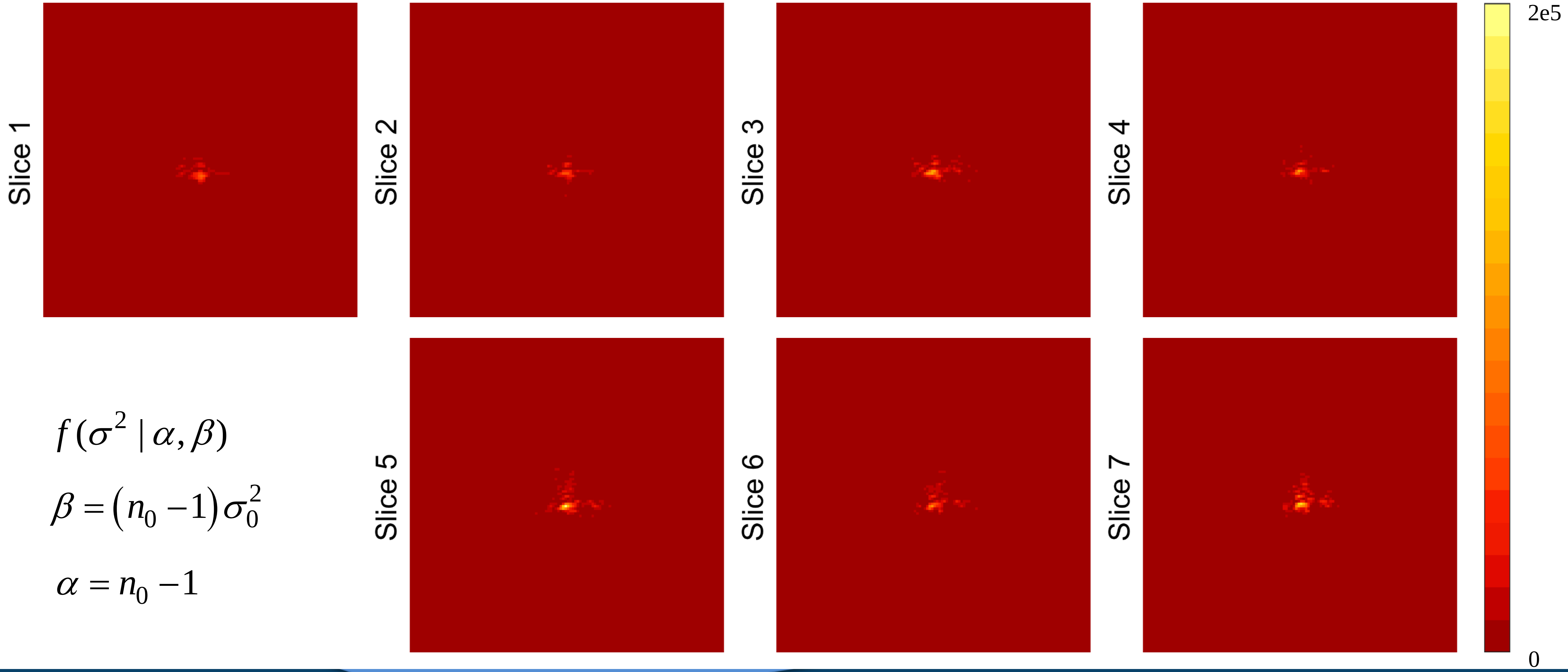
$$\rho_0 = (\bar{k}_x + \bar{k}_y)^{1/2}$$

$$\gamma = n_0$$

3. Experimental Results

$$\sigma_0^2 = \frac{s_R^2 + s_I^2}{2}$$

Prior k-space variance parameter.



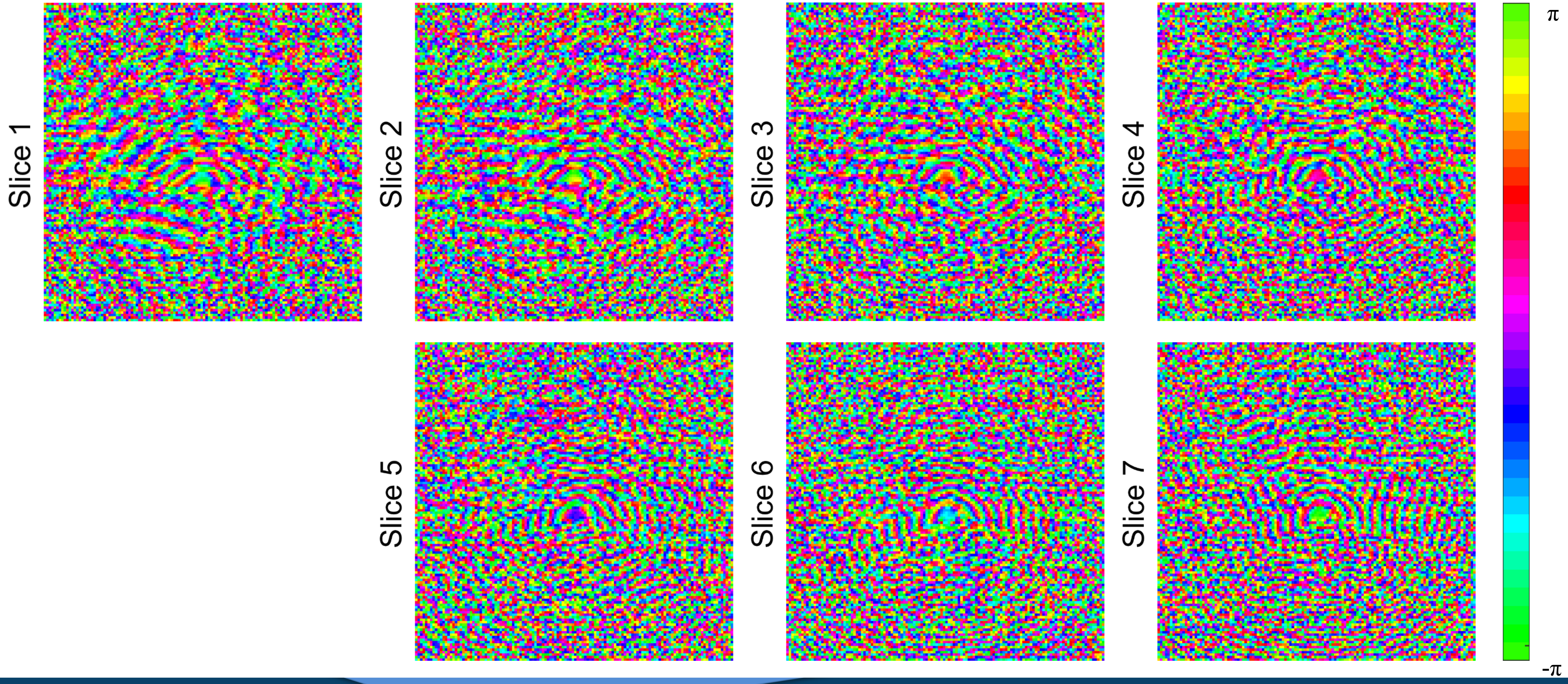
$$f(\sigma^2 \mid \alpha, \beta)$$

$$\beta = (n_0 - 1)\sigma_0^2$$

$$\alpha = n_0 - 1$$

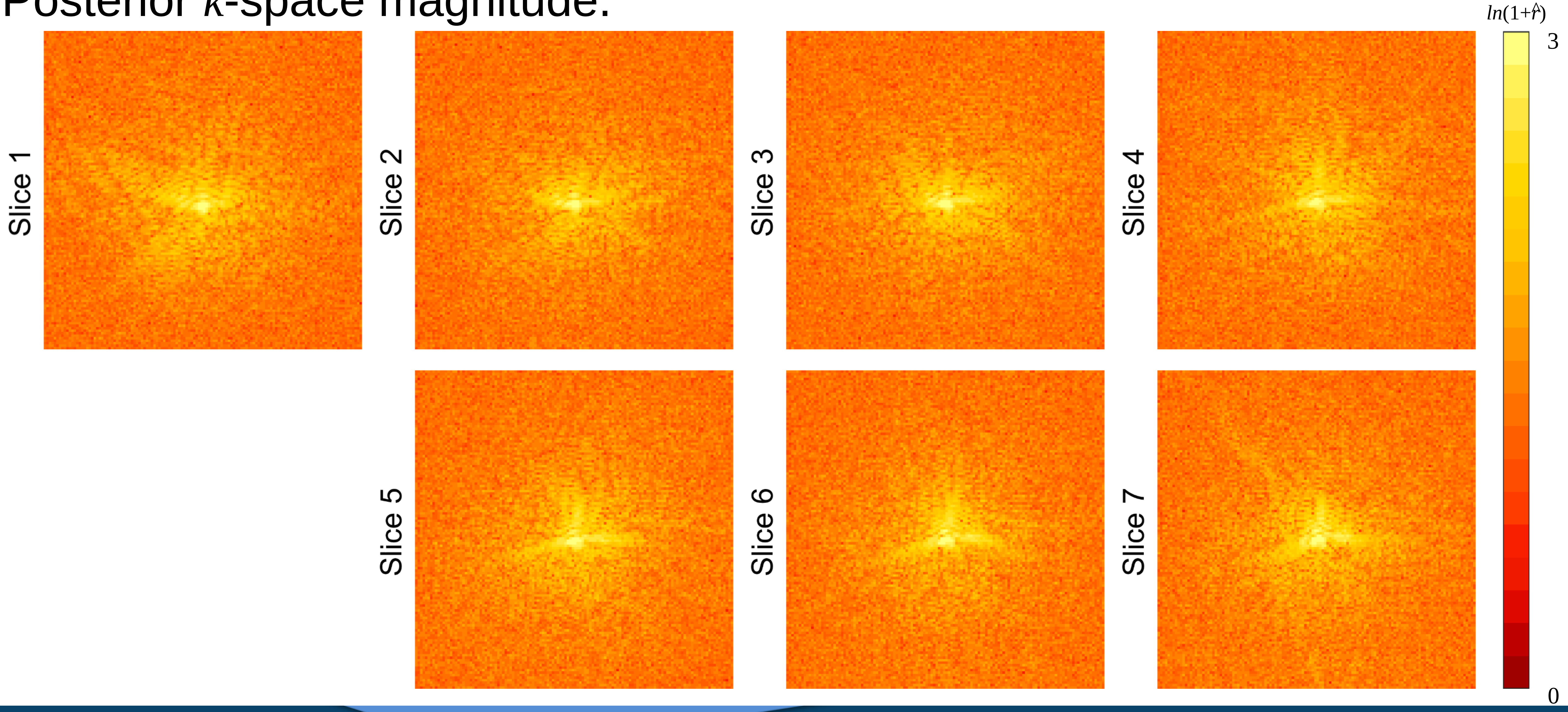
3. Experimental Results

Posterior k-space phase.



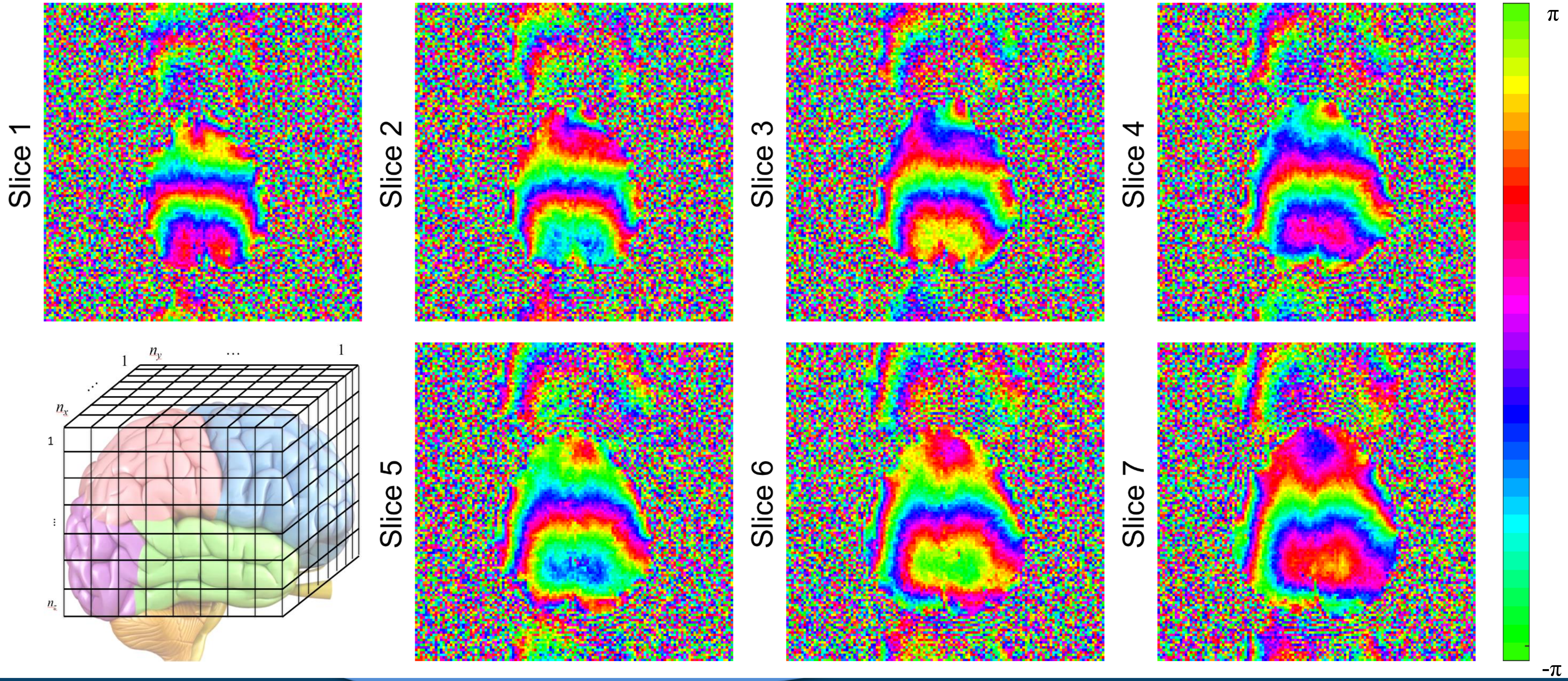
3. Experimental Results

Posterior k-space magnitude.



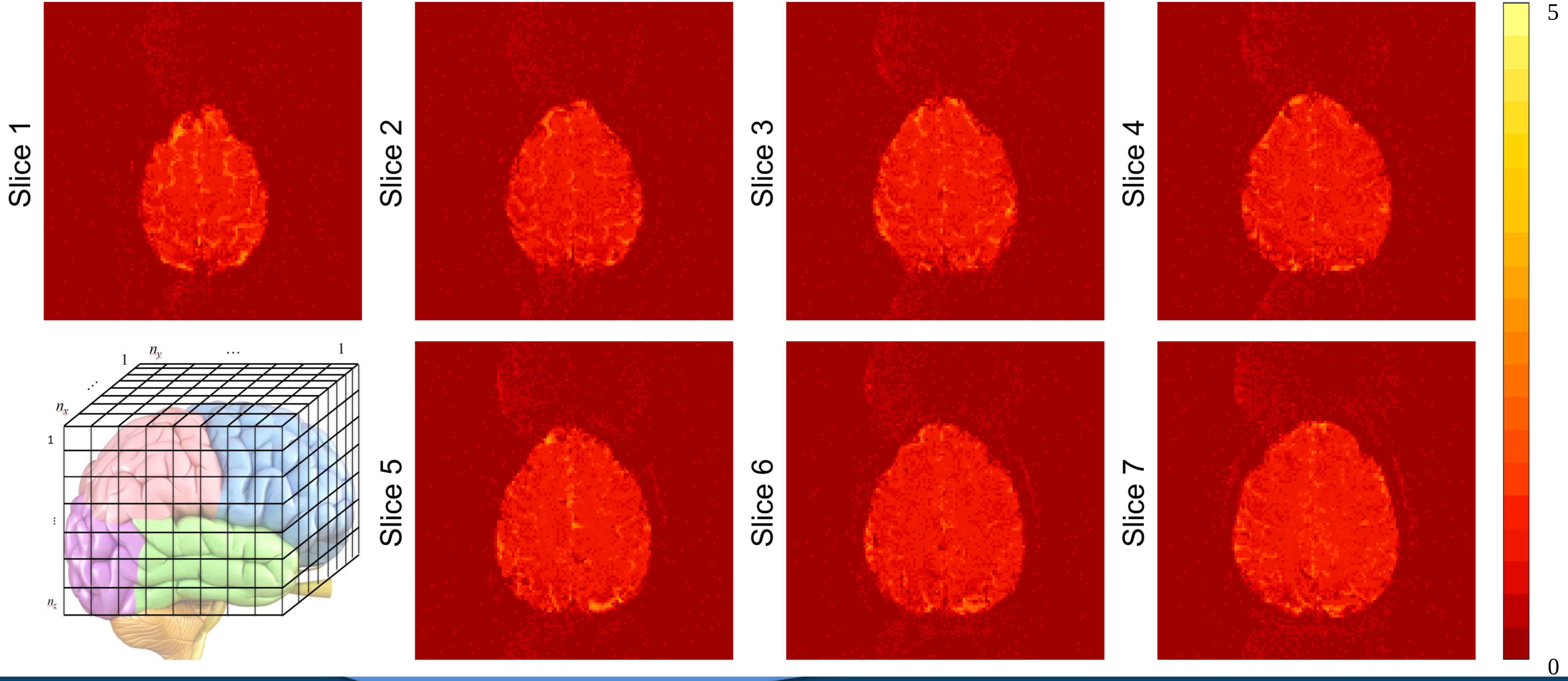
3. Experimental Results

Posterior IDFT image phase.



3. Experimental Results

Posterior IDFT image magnitude.



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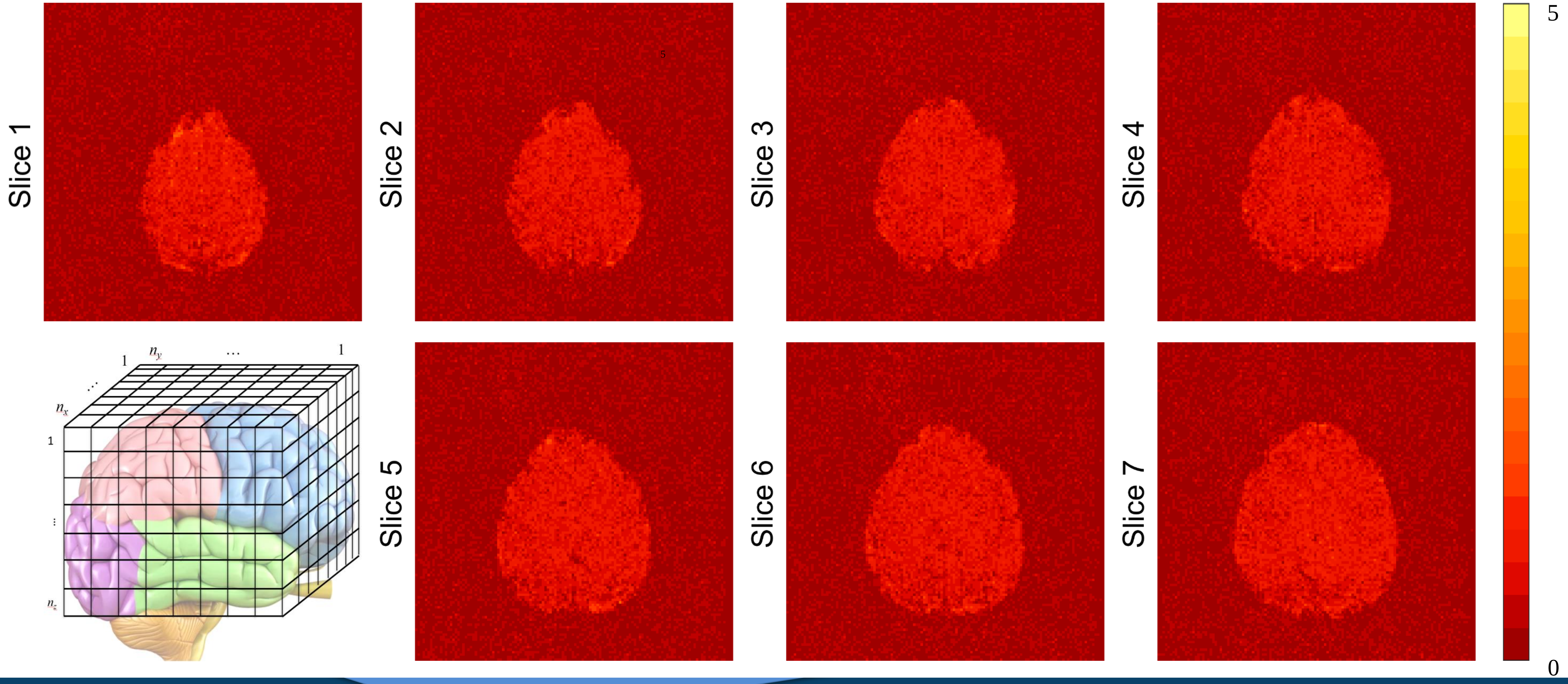
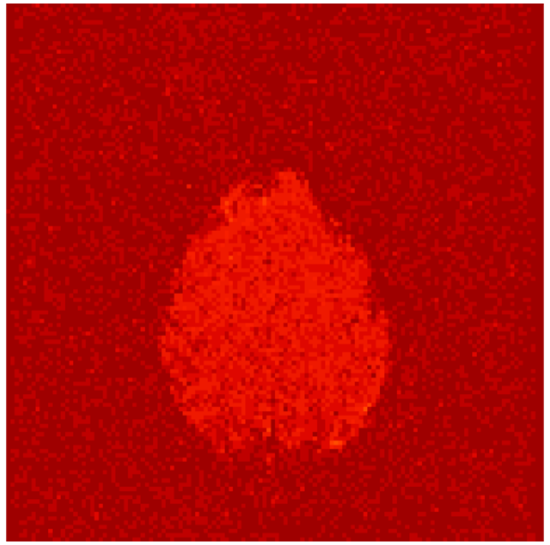


4. Discussion

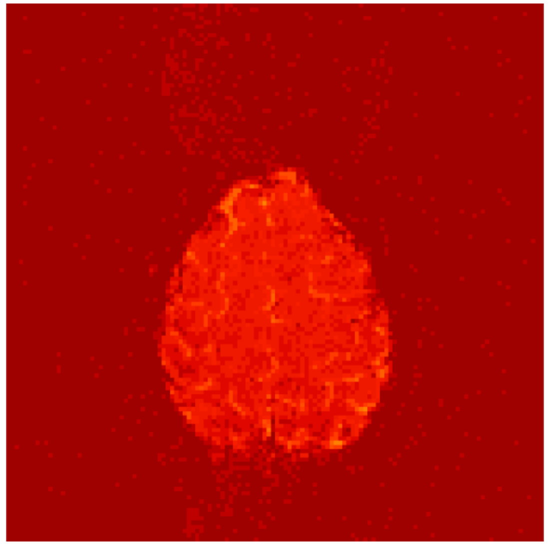
Images are noisy.



Use Bayesian Statistics to improve

$$f(\theta \mid \rho, \sigma^2, \theta_0, \gamma)$$
$$f(\rho \mid \rho_0, \sigma^2, \gamma)$$
$$f(\sigma^2 \mid \alpha, \beta)$$

Posterior images have lower noise and higher signal.



Thank You

Questions?

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