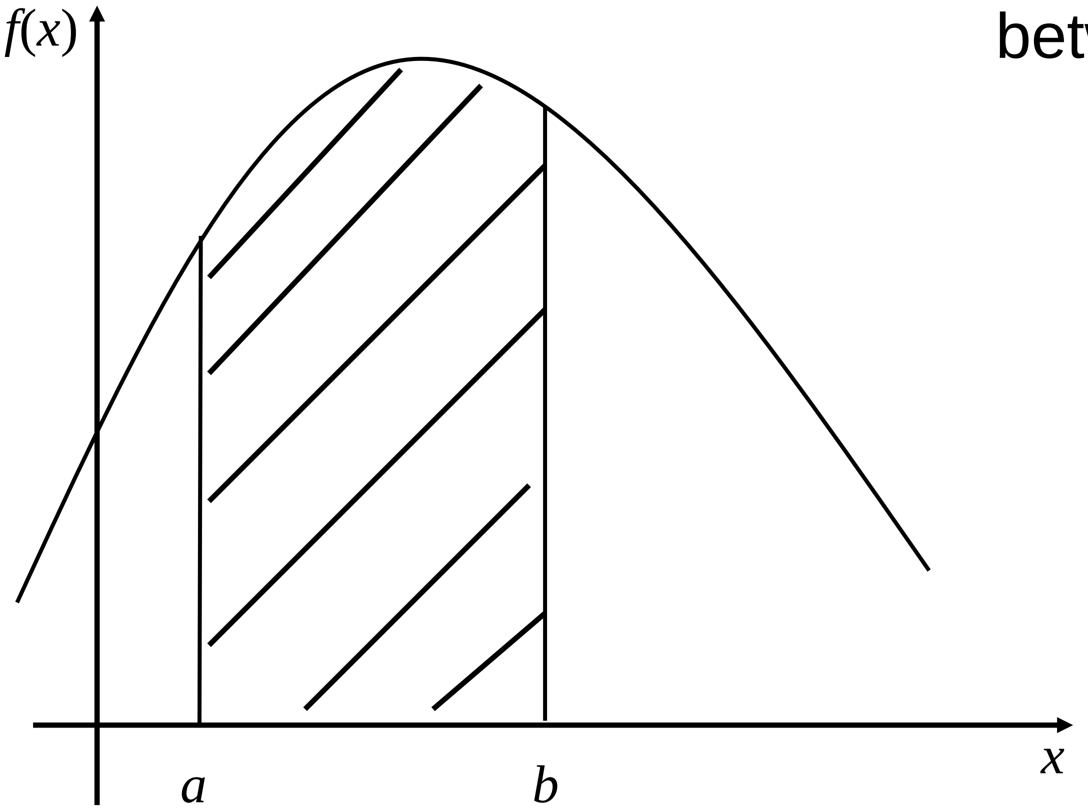
Numerical Integration

Daniel B. Rowe, Ph.D.

Department of Mathematical and Statistical Sciences

Integration - Area Under Curve

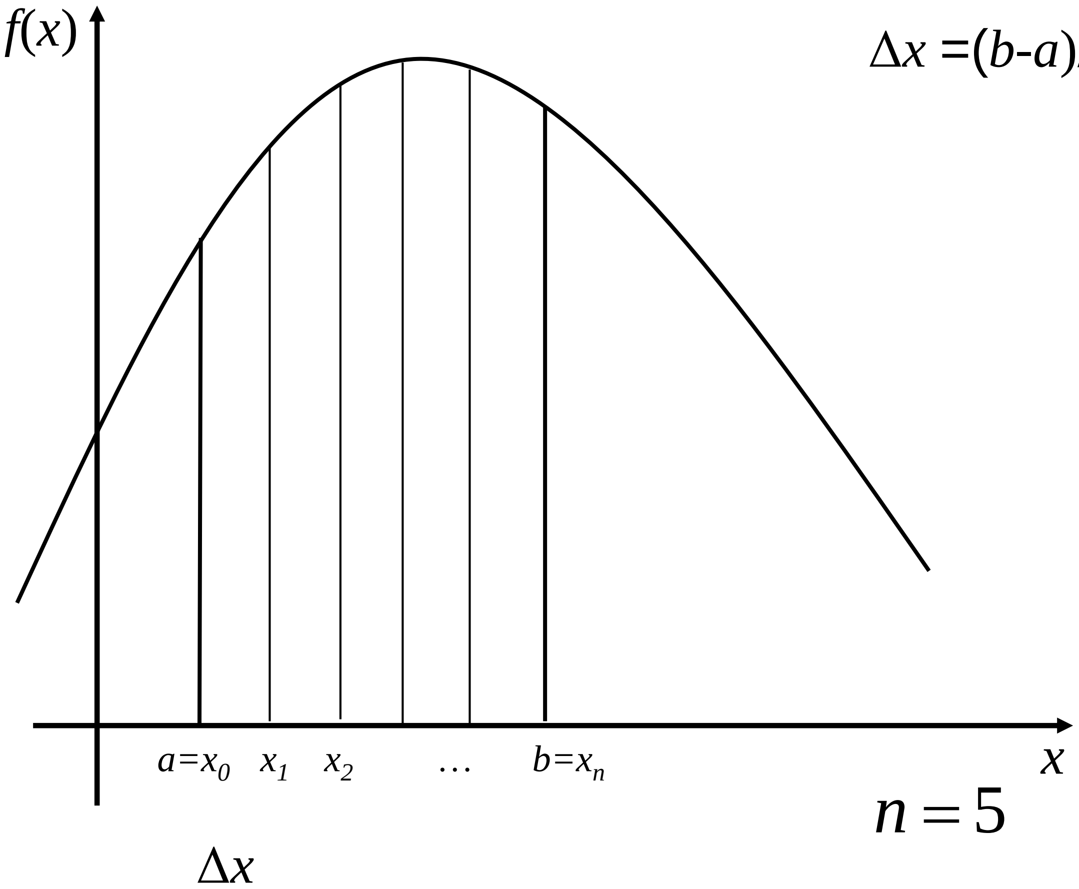


Area under curve between a and b.

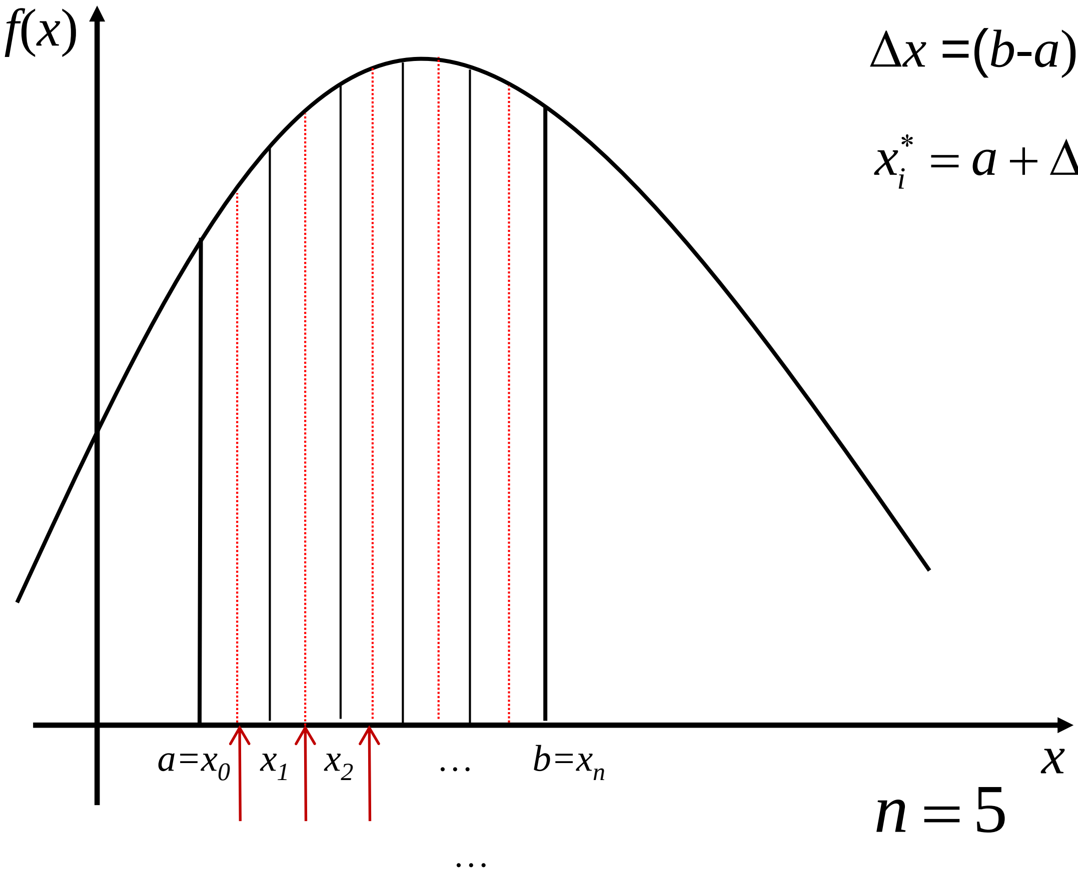
Integration - Area Under Curve

Divide into intervals: Δx small

$\Delta x = (b-a)/n$ $\quad \Delta x = x_i - x_{i-1}$



Integration - Area Under Curve



Find midpoints: Δx small

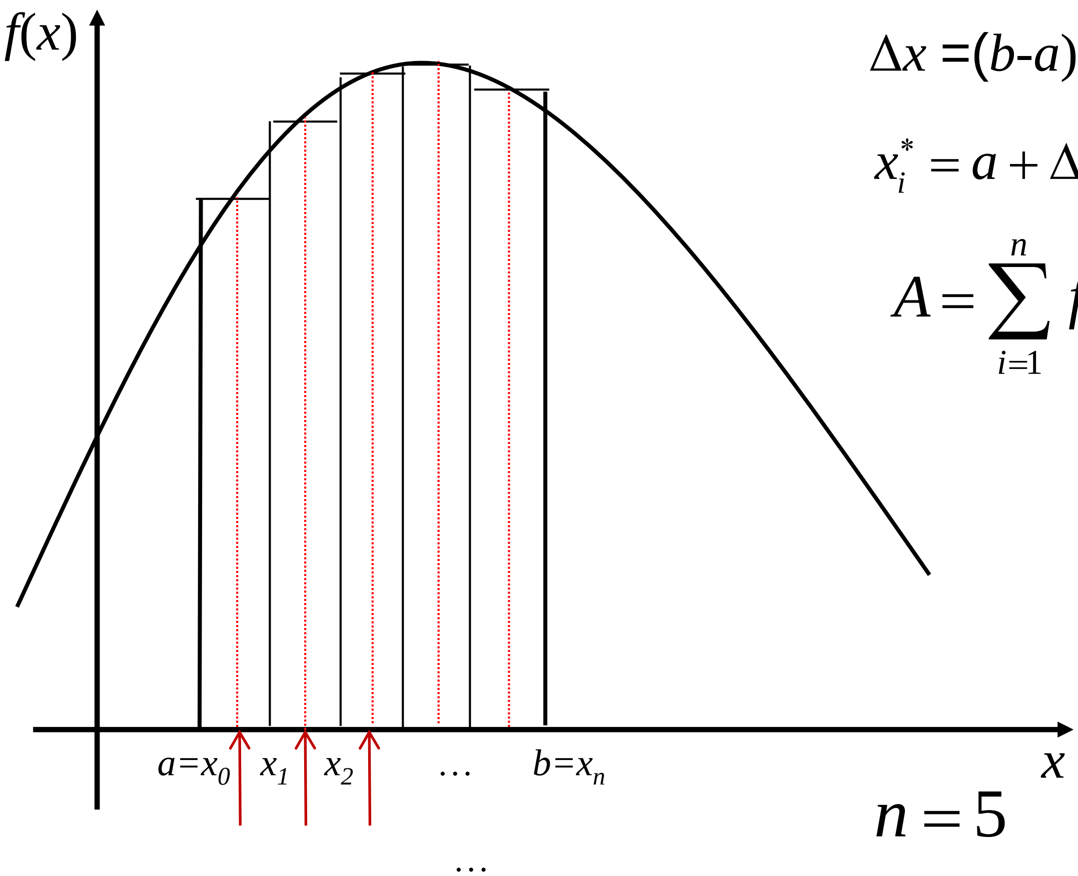
$\Delta x = (b-a)/n$ $\qquad \Delta x = x_i - x_{i-1}$

$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$

$$n = 5$$

Integration - Area Under Curve



Area by rectangles: Δx small

$\Delta x = (b-a)/n$ $\quad \Delta x = x_i - x_{i-1}$

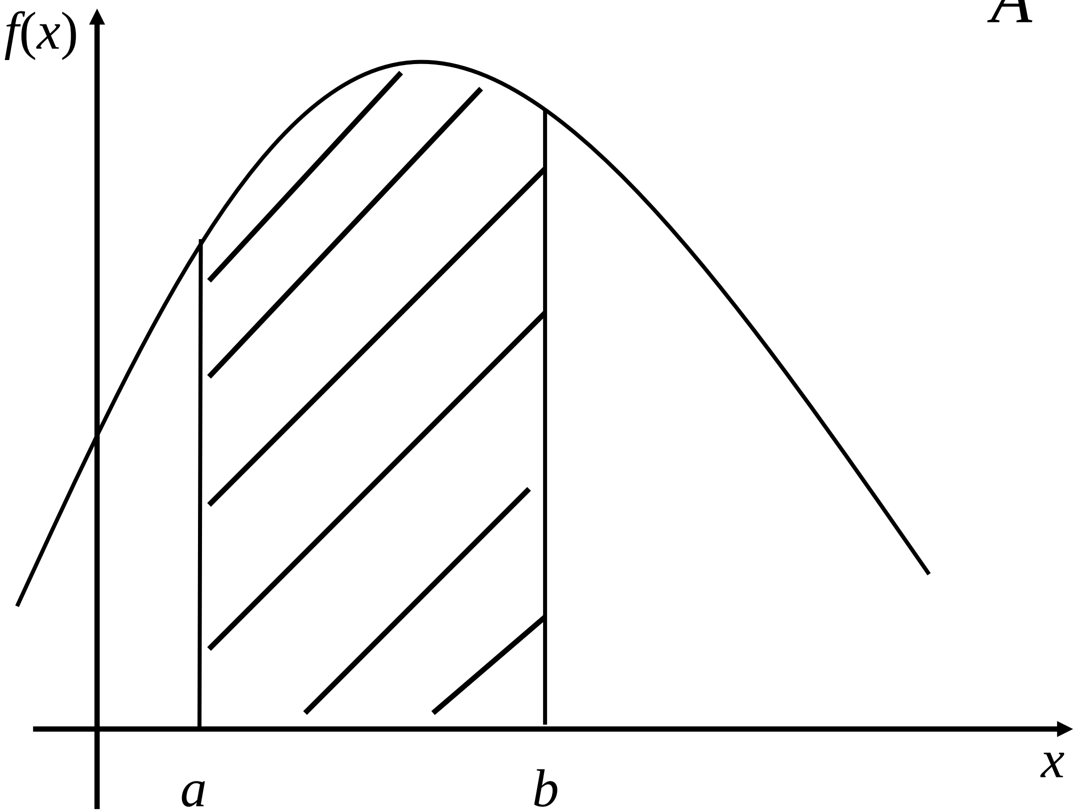
$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$n = 5$$

Integration - Area Under Curve



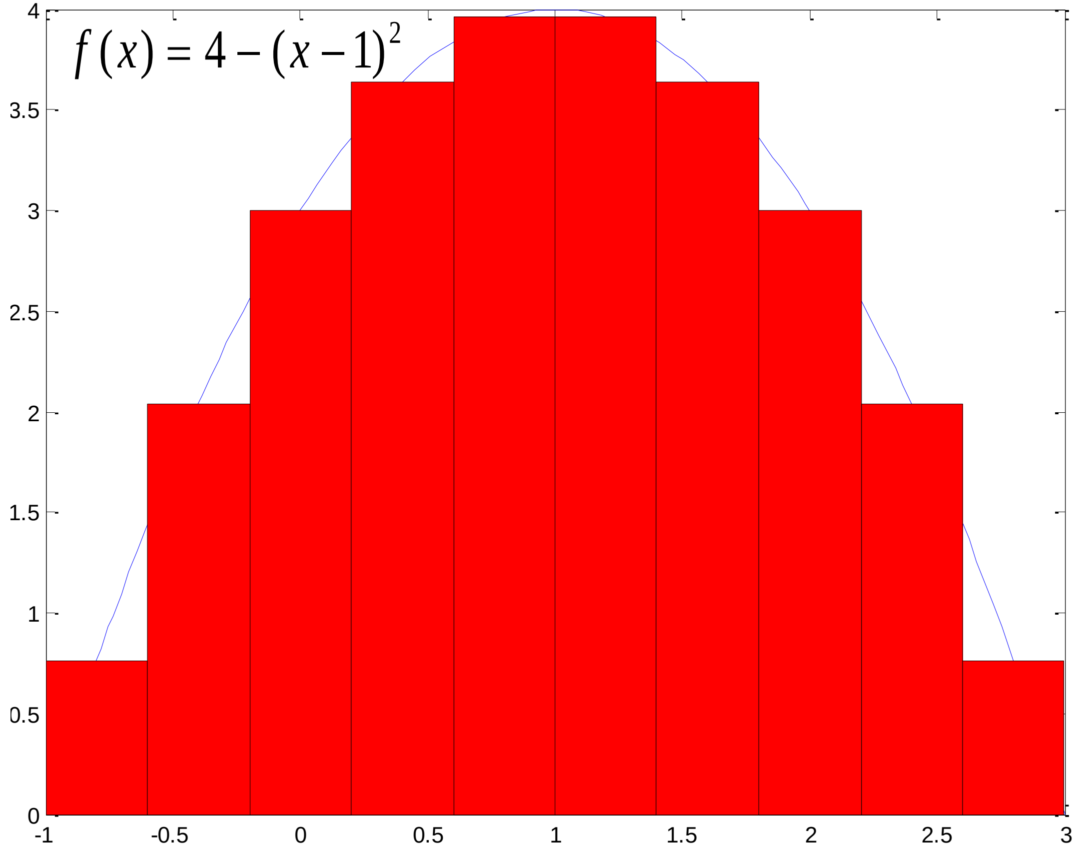
$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \int_{x=a}^{b} f(x)dx$$

$$\Delta x = (b-a)/n$$

$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

Integration - Numerical Approach

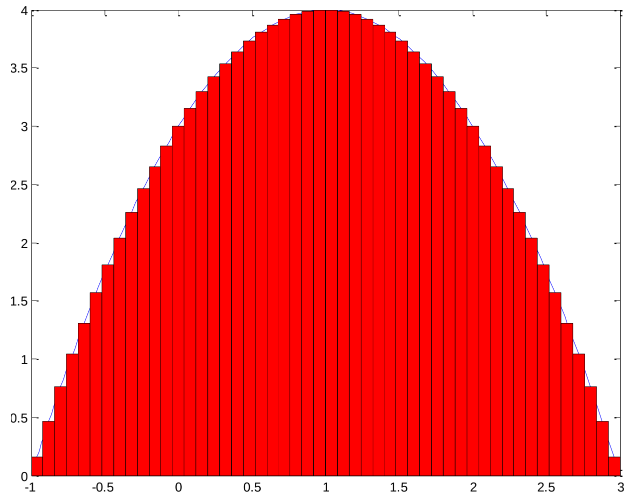


numerical

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

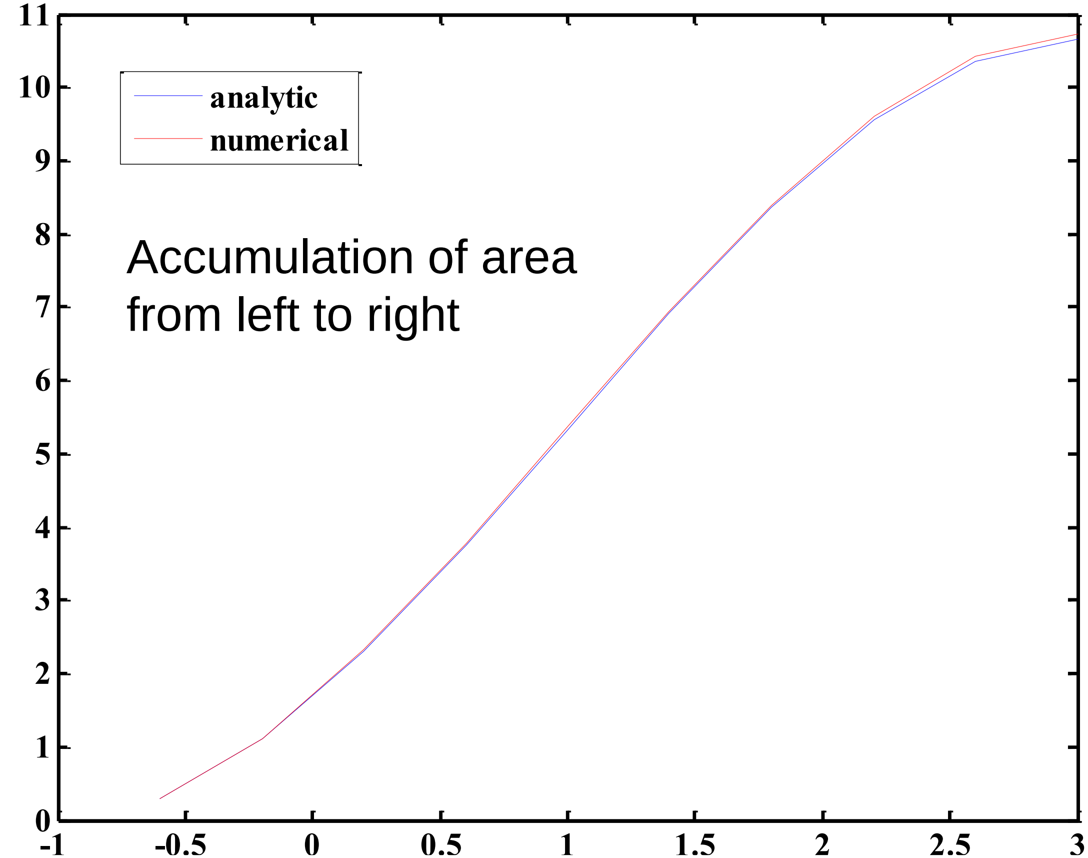
← $n=10,\ \Delta x=0.4$

↓ $n=50,\ \Delta x=0.08$



$$\int_{x=-1}^{3} [4-(x-1)^2]dx = 10.6667$$ ← analytic

Integration - Numerical Approach



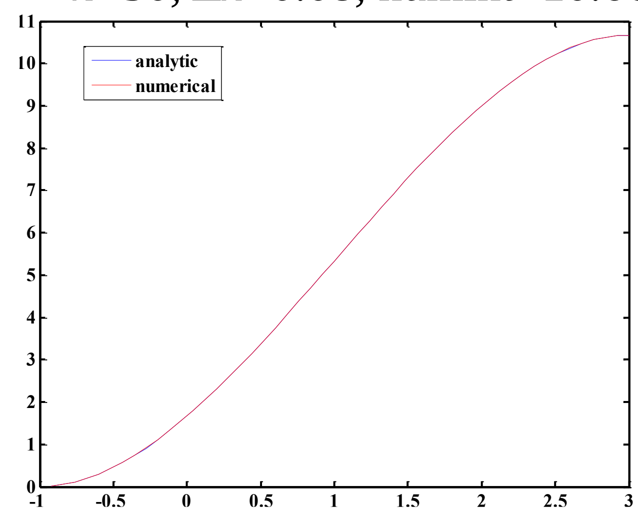
numerical

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

← n=10, Δx=0.4

numint=10.7200

↓ n=50, Δx=0.08, numint=10.6688



$$\int_{x=-1}^{3} [4-(x-1)^2]dx = 10.6667$$ ← analytic

Integration - Area Under Curve

Numerical Integration can be applied to find Expectations $E[g(x)]$

$$A = \int g(x) f(x) \; dx$$

Approximated as

$$A = \sum_{i=1}^{n} g(x_i^*) f(x_i^*) \Delta x$$

Chapter 3: Random Numbers

Daniel B. Rowe, Ph.D.

Department of Mathematical and Statistical Sciences

3.1 Pseudorandom Number Generation
"Random" (0,1) numbers

Multiplicative Congruential Method

(1) Start with seed x_0

(2) Compute $x_n = a x_{n-1}$ modulo m , n=0,1,2,…

where a and m are given positive integers

$$x_n = \text{remainder}(a x_{n-1})$$

$$x_n / m \sim U(0,1)$$

choose $m = 2^{31}-1$ and $a=7^5$ for 32 bit machines

3.1 Pseudorandom Number Generation
"Random" (0,1) numbers

Mixed Congruential Method

(1) Start with seed x_0

(2) Compute $x_n=(ax_{n-1}+c)$ modulo m , n=0,1,2,…

where a, c, and m are given positive integers

$$x_n = \text{remainder}((ax_{n-1}+c))$$

$$x_n / m \sim U(0,1)$$

choose m = to be the computer's word length

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

Let $g(x)$ be a function and suppose we want

$$\theta = \int_0^1 g(x)dx$$

If U is uniformly distributed over (0,1), then

$$\theta = E(g(U))$$

Think of $f(u) = \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $\theta = \int_0^1 g(u) f(u) du$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

If we generate $U_1,\ldots,U_k$ independent uniforms then $g(U_1),\ldots, g(U_k)$ will be IID with mean θ.

$$\theta = \int_0^1 g(u) f(u) du$$

$$\sum_{i=1}^{k} \frac{g(U_i)}{k} \to E[g(U)] = \theta \qquad \text{as } k \to \infty$$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

If we want $\theta = \int_a^b g(x)dx$, then we can transform x to y as y =(x -a)/(b-a), with dy =dx/(b-a)

$$\theta = \int_a^b g(x)dx \qquad dx = (b-a)dy$$

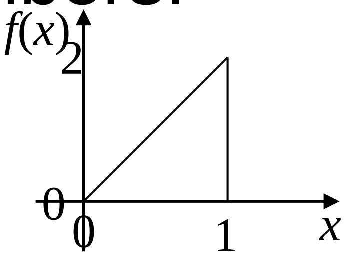
$$x = a + y[b-a]$$

$$\theta = \int_0^1 g(a+[b-a]y)(b-a)dy$$

$$\theta = \int_0^1 h(y)dy \qquad h(y) = g(a+[b-a]y)(b-a)$$

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

Integrate $\theta = \int_0^1 2x\, dx$ using $U(0,1)$ random numbers.



Analytically, we'd get $\theta = \int_0^1 2x\, dx = 2\frac{x^2}{2}\Big|_0^1 = 1$.

Or we can generate uniform numbers $u_1,\ldots,u_n$ and

Calculate $\theta \approx \frac{1}{n}\sum_{i=1}^{n}(2u_i)$.

Matlab Code:

```
rng('default')
n=10^6;
u=rand(n,1);
theta=sum(2*u)/n
```

Matlab Output:

theta = 1.0006

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

If we want $\theta = \int_0^{\infty} g(x)dx$, then we can transform x to y as y =1/(x +1), with dy =-dx/(x+1)^2=-y^2dx

$$\theta = \int_0^{\infty} g(x)dx \qquad dx = -dy / y^2 \qquad \text{limits} \quad x = 0 \rightarrow y = 1$$

$$x = 1/y - 1 \qquad x = \infty \rightarrow y = 0$$

$$\theta = \int_0^1 g(1/y - 1)dy / y^2$$

$$\theta = \int_0^1 h(y)dy \qquad h(y) = g(1/y - 1) / y^2$$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

If we want $\theta=\int_{-\infty}^{+\infty}g(x)dx$, then we can transform x to y as $y=e^x/(1+e^x)$, with $dy=dxe^x/(1+e^x)^2=ydx/(1-y)$

$$\theta=\int_{-\infty}^{+\infty}g(x)dx \qquad dx=ydy/(1-y)$$

limits $x=-\infty\rightarrow y=0$

$x=+\infty\rightarrow y=1$

$$x=\ln\left(\frac{y}{1-y}\right)$$

$$\theta=\int_0^1 g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{dy}{y(1-y)}$$

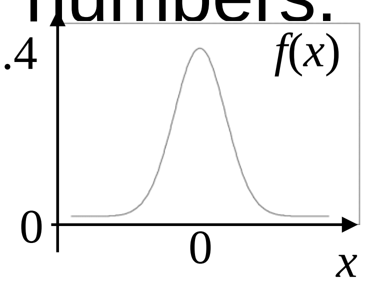
$$y=e^x/(1+e^x)$$

$$h(y)=g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{1}{y(1-y)}$$

$$\theta=\int_0^1 h(y)dy$$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

Integrate $\theta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$ using $U(0,1)$ random numbers.



We know that $\theta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$.

Or we can generate uniform numbers $u_1,\ldots,u_n$ and

Calculate $\theta \approx \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\ln\left(\frac{u_i}{1-u_i}\right)\right)^2} \frac{1}{u_i(1-u_i)} \approx 1$.

$$\theta = \int_0^1 g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{dy}{y(1-y)}$$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

Most useful in evaluating multiple integrals.

$$\theta = \int_0^1 \int_0^1 \dots \int_0^1 g(x_1, \dots, x_n) dx_1 \dots dx_n$$

The key is to use $\theta = E[g(U_1, \dots, U_n)]$, where $U_1, \dots, U_n$ are independent $U(0,1)$'s.

If we generate $U_1^{(1)}, \dots, U_n^{(1)}$ then $\theta = E[g(U_1, \dots, U_n)] \approx \sum_{i=1}^{k} \frac{g(U_1^i, \dots, U_n^i)}{k}$

$$U_1^{(2)}, \dots, U_n^{(2)}$$

$$\vdots$$

$$U_1^{(k)}, \dots, U_n^{(k)}$$

Homework 1:

$$f(x) = \exp(\exp(x)), \quad x \in [0,1]$$

0) Integrate numerically with Matlab from a=0 to b=1.

Do by pencil and paper with n=4 intervals.

$$\Delta x = 0.25 \quad \left(x_1^*, x_2^*, x_3^*, x_4^*\right) = \left(0.125, 0.375, 0.625, 0.875\right)$$

Write a Matlab program to repeat with n=4 intervals.

Change to n=100. Compare results.

Homework 1:

Chapter 3: # 1*, 3, 7#, 9, 11.

*Repeat generating 10^4 of these.

Compute mean, variance, and make a histogram.

Repeat using Matlab's rand() command.

Compare results from to Matlab's rand().

#Compare results to numerical integration in problem 0 with n=100.