Continuous Probability Density Functions

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Outline

Properties and Summaries

Assume that the continuous random variable (RV) x can take on values

$x \in (a,b)$

then, the probability density function (PDF) is given by

$f(x|\theta)$ defined for $x \in (a,b)$

where x can be defined within an infinite interval

and θ are any parameters that the PDF depends on.

Properties and Summaries

The accumulation of probability from left to right

is called the cumulative distribution function (PDF)

$$F(x|\theta) = \int_{t=-\infty}^{x} f(t|\theta)dt$$

and

1) $0 \leq f(x|\theta)$

2) $\int_x f(x|\theta) = 1$.

Properties and Summaries

Given an arbitrary continuous probability density

$f(x|\theta)$, we want to compute quantitative population

summaries of it such as:

population mean, $\mu = \int_{x=-\infty}^{\infty} x f(x|\theta) dx$

population variance, $\sigma^2 = \int_{x=-\infty}^{\infty} (x-\mu)^2 f(x|\theta) dx$

population standard deviation, $\sigma = \sqrt{\sigma^2}$.

Properties and Summaries

Given an arbitrary continuous probability density

$f(x|\theta)$, we want to compute quantitative population

summaries of it such as:

population median $\tilde{x}$, $\int_{x=-\infty}^{\tilde{x}} f(x|\theta)dx = \frac{1}{2}$

population mode $\hat{x}$, $\left.\frac{\partial}{\partial x} f(x|\theta)\right|_{\hat{x}} = 0$.

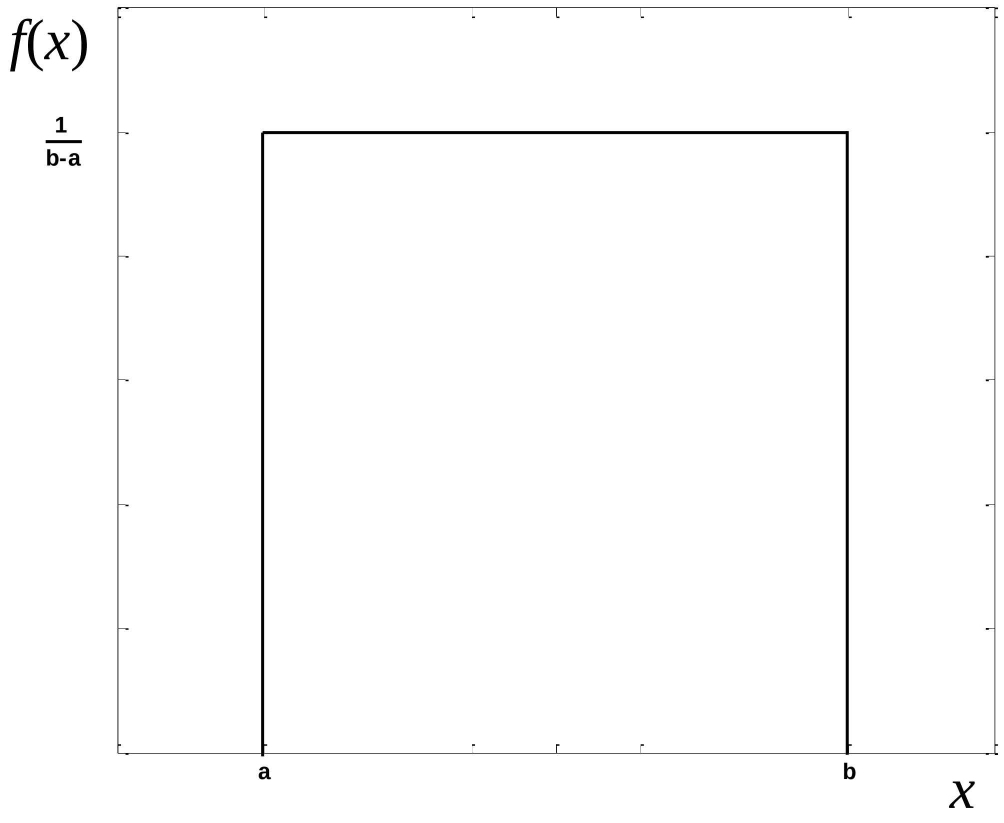
Provided f is differentiable.
Max if 2nd deriivitive neg at point.
Check boundary points for max.
May need to be numerically calculated.

Uniform and Beta PDF

A random variable x has a continuous uniform distribution, x ~uniform(a,b) if

$$f(x|a,b)=\frac{1}{b-a}\quad , \quad x\in[a,b]$$

where, $a<x<b$, $a<b$.



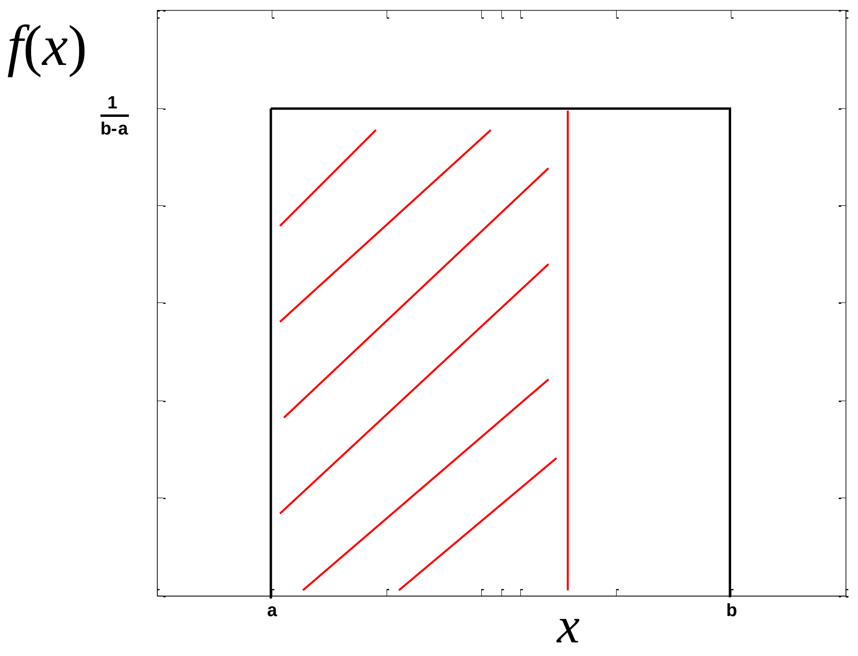
Uniform and Beta PDF

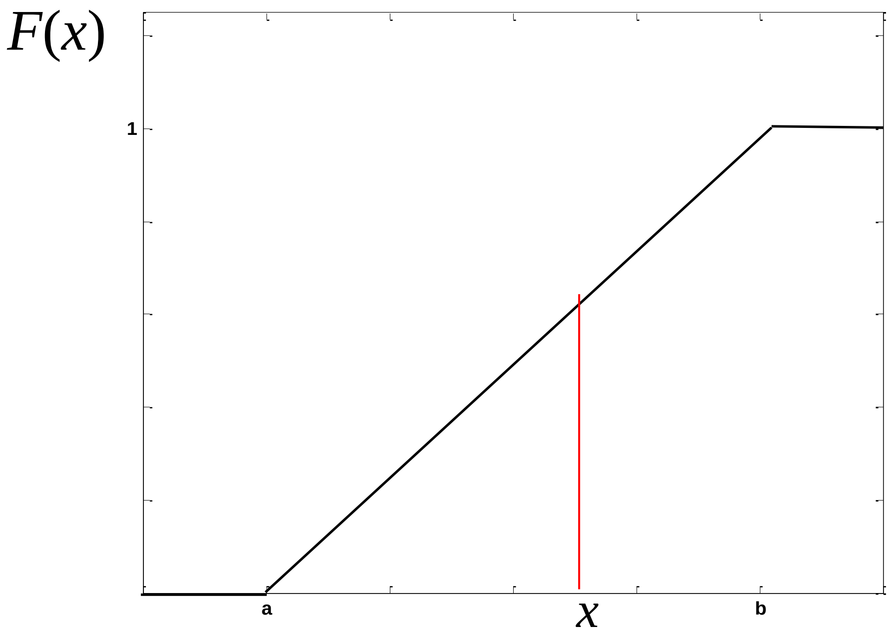
The CDF of the continuous uniform distribution is

$$F(x|\theta) = \int_{t=-\infty}^{x} f(t|\theta)dt$$

$$F(x|a,b) = \int_{t=a}^{x} \frac{1}{b-a}dt$$

$$= \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & x \in [a,b] \\ 1 & x > b \end{cases}$$

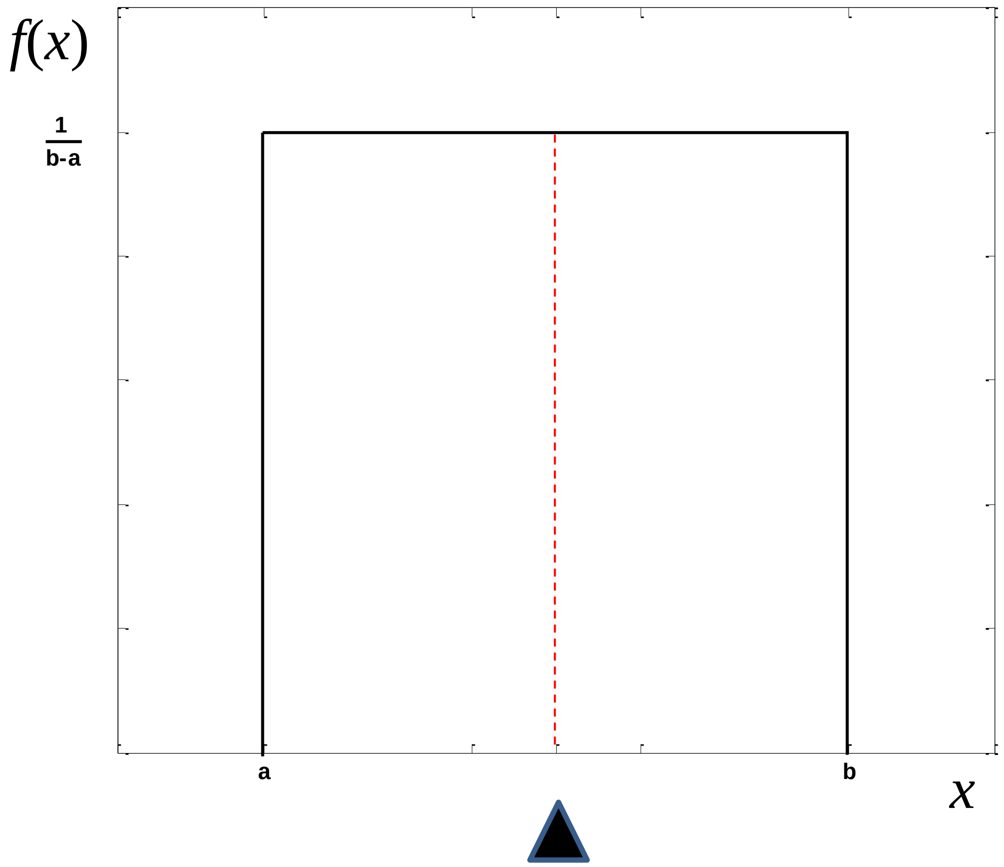




Uniform and Beta PDF

It can be shown that

$$\begin{aligned}\mu &= \int_x x f(x|\theta)dx \\ &= \int_{x=a}^{b} x\frac{1}{b-a}dx \\ &= \frac{b+a}{2}\end{aligned}$$



Uniform and Beta PDF

It can be shown that

median

$$\int_{x=a}^{\tilde{x}} f(x|\theta)dx = \frac{1}{2}$$

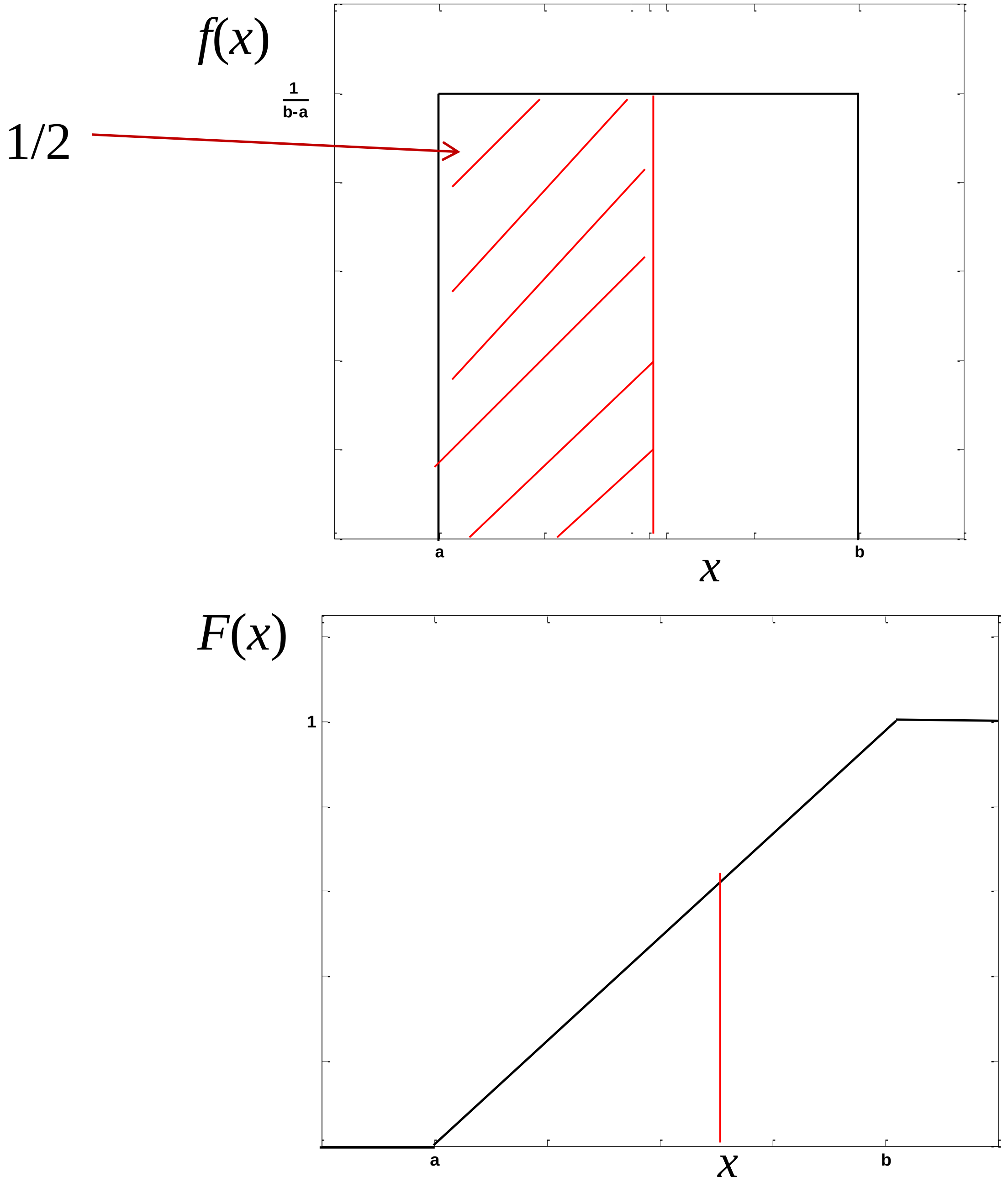
$$\tilde{x} = \frac{b+a}{2}$$

mode

$$\left.\frac{\partial}{\partial x} f(x|\theta)\right|_{\hat{x}} = 0$$

$\hat{x} =$ any value $\in [a,b]$

flat distribution



Uniform and Beta PDF

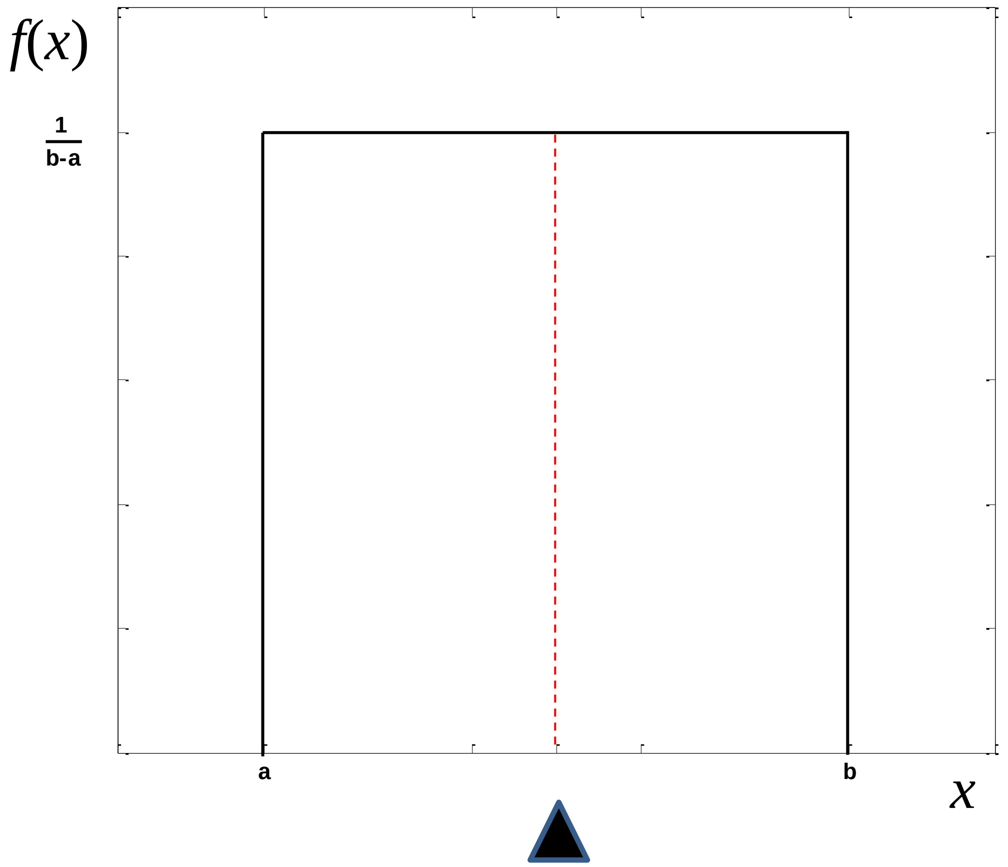
that

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=a}^{b} (x-\mu)^2 \frac{1}{b-a} dx$$

$$= \frac{(b-a)^2}{12}$$

$$\sigma = \frac{(b-a)}{\sqrt{12}}$$



Uniform and Beta PDF

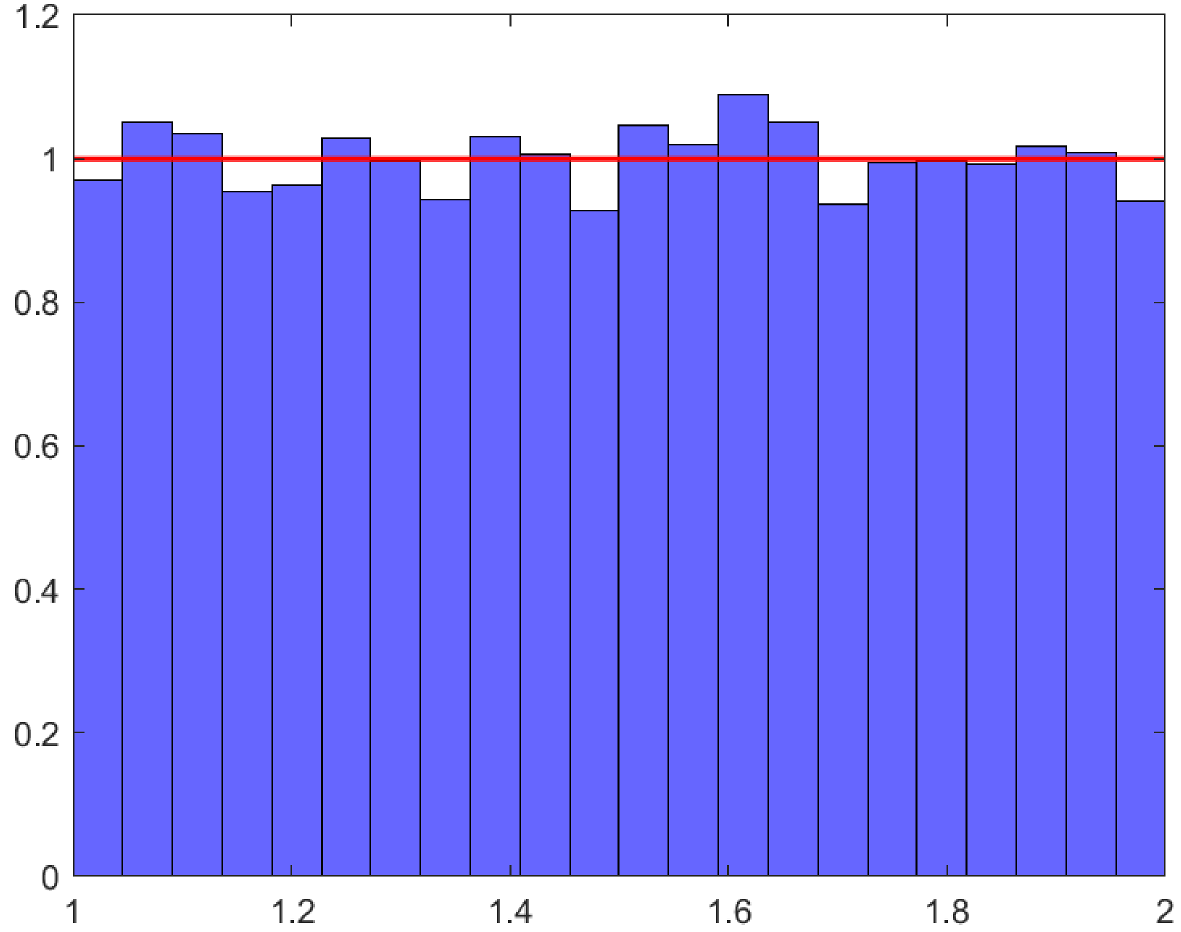
```
rng('default'), a=1;,b=2;,num=10^4;
x=a+(b-a)*rand(num,1);
figure;
histogram(x,'BinLimits',[1,2],'normalization','pdf','FaceColor','blue')
hold on
line([1,2],[1,1],'Color','red','LineWidth',1.5)
xlim([1,2])
```

$$f(x|a,b)=\frac{1}{b-a}$$

	True	Simulated
μ	1.5	1.4996
σ^2	0.0833	0.0829

Can also find and plot ECDF.

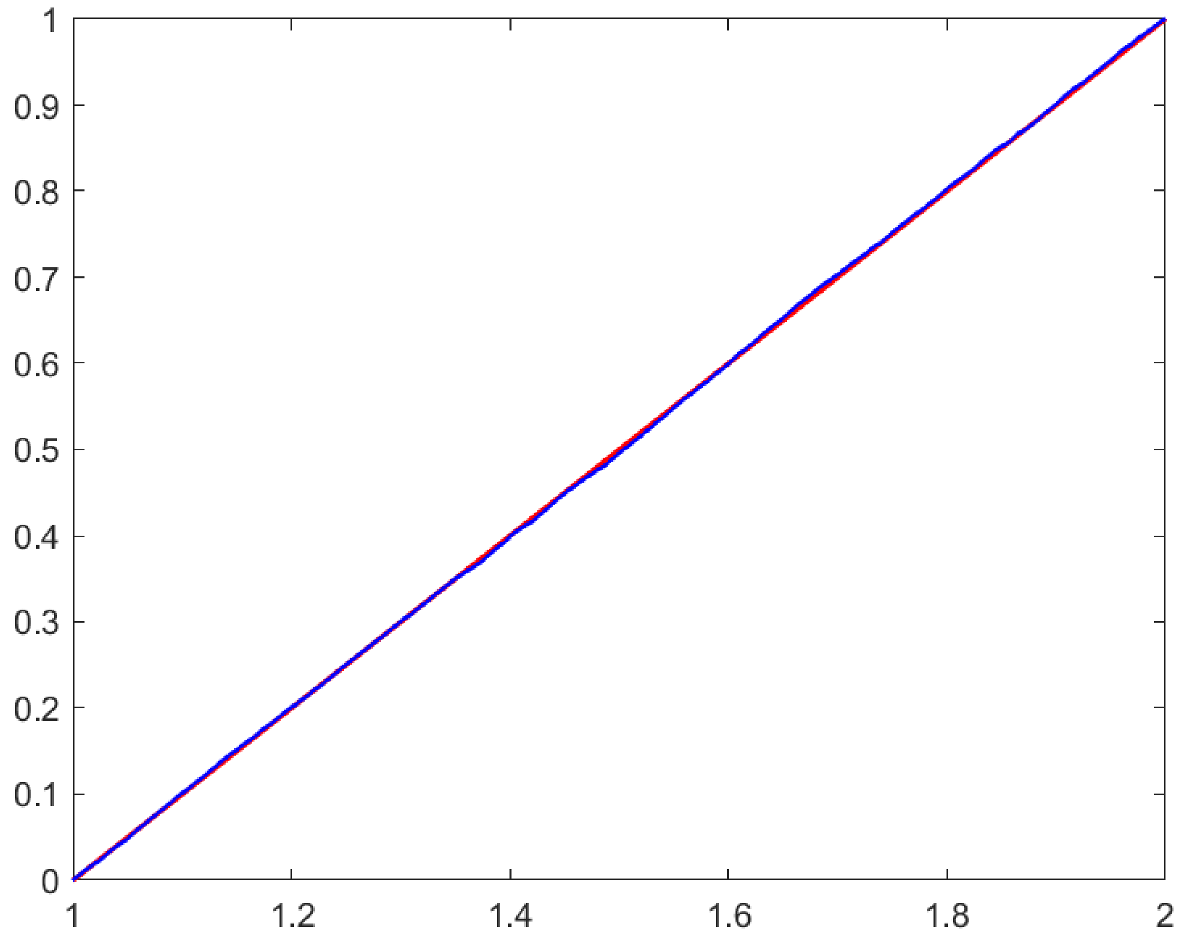
```
themean=(b-a)/2
thevar=(b-a)^2/12
[mean(x),var(x)]
```



Uniform and Beta PDF

$$F(x|a,b)=\frac{1}{b-a}x$$

```
y =cdf('unif',(a:.01:b),a,b);
figure;
plot((1:.01:2),y,'Color','red','LineWidth',1.5)
[F,xx]=ecdf(x);
hold on
stairs(xx,F,'Color','blue','LineWidth',1.5)
```



Uniform and Beta PDF

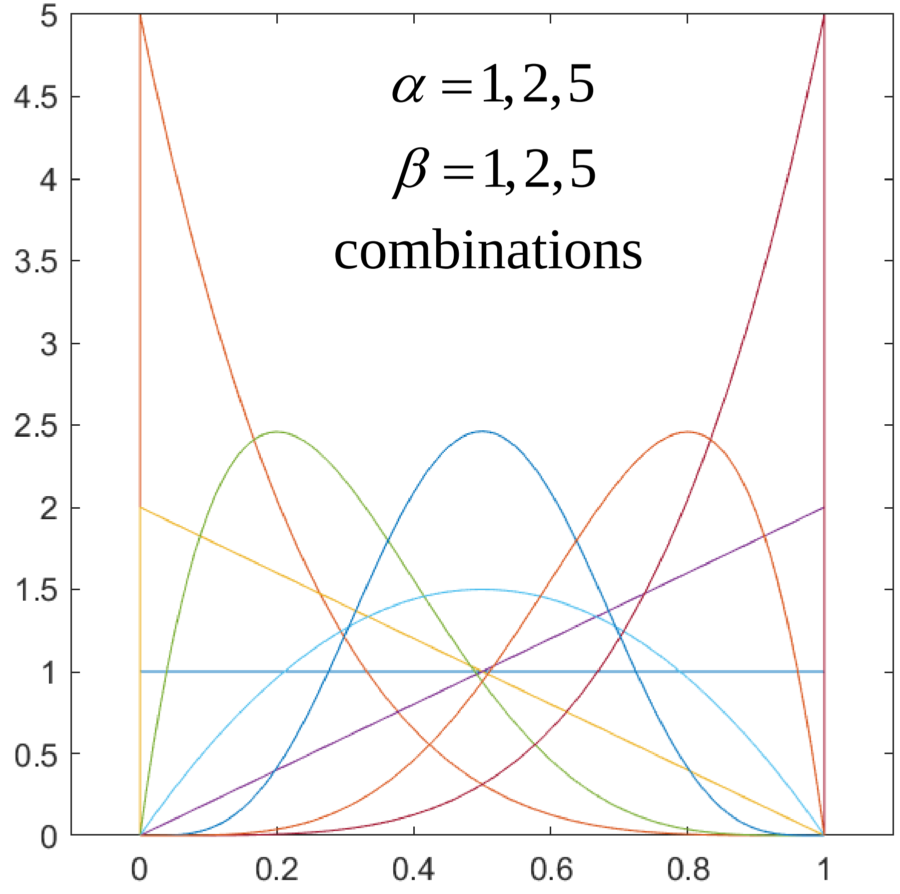
A random variable x has a continuous Beta distribution, x~Beta(α,β) if

$$f(x|\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},$$

where, $\alpha,\beta>0$, $x\in[a,b]$.

And can take on many shapes.

$\Gamma(a)=(a-1)!$ for integer a



Uniform and Beta PDF

There is no closed form analytic formula for the CDF $F(x|\alpha,\beta)$.

It can be shown that the mean is

$$\begin{aligned}
\mu &= \int_x x f(x|\theta)dx \\
&= \int_{x=0}^{1} x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

Uniform and Beta PDF

It can be shown that the variance is

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=0}^{1} (x-\mu)^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$$

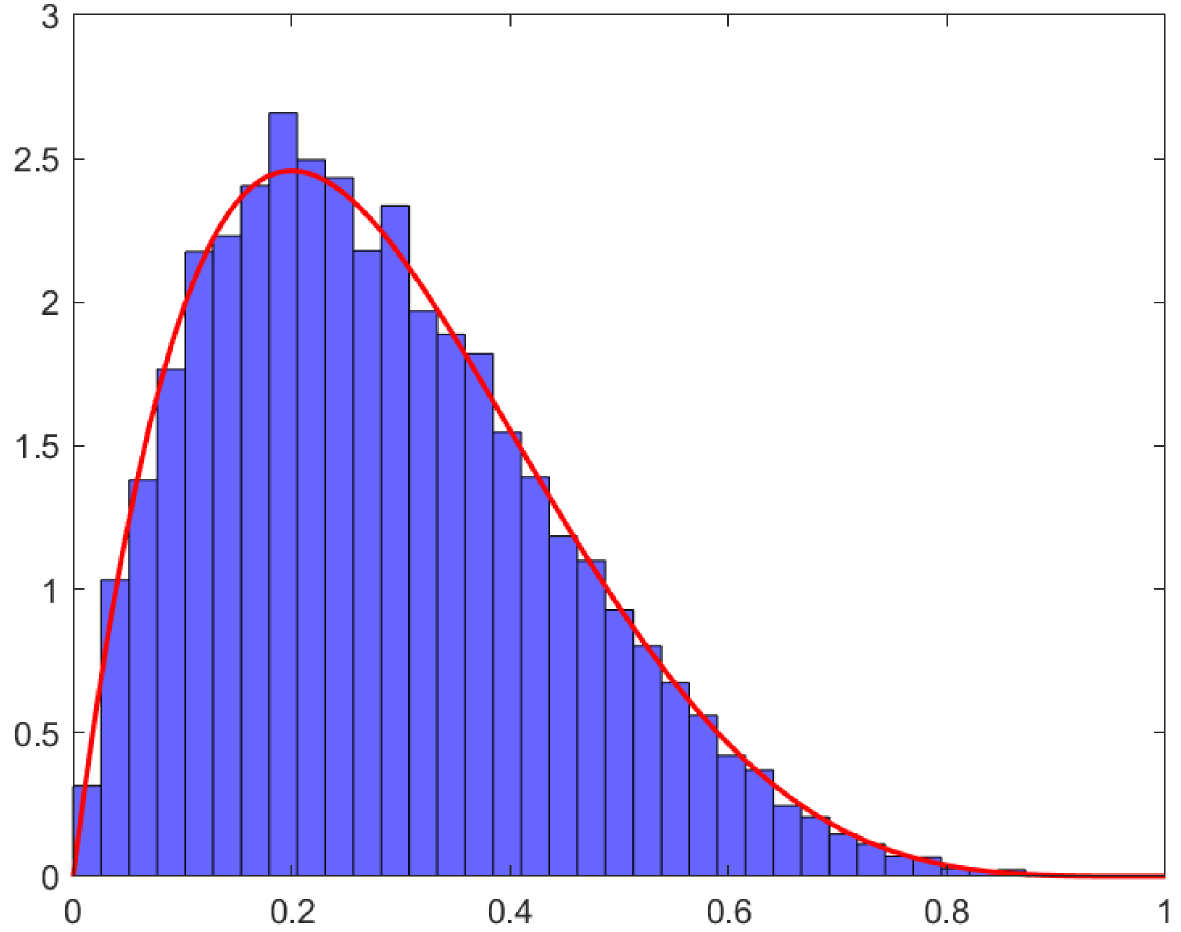
$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Uniform and Beta PDF

```
rng('default'), a=2; b=5; num=10^4;
x=betarnd(a,b,[num,1]);
figure;
histogram(x,'BinLimits',[0,1],'normalization','pdf','FaceColor','blue')
hold on
fx = @(x) factorial(a+b-1)/factorial(a-1)/factorial(b-1)...
.*x.^(a-1).*(1-x).^(b-1);
fplot(fx,[0,1],'r','LineWidth',1.5)
```

$$f(x\,|\,\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

	True	Simulated
μ	0.2857	0.2858
σ^2	0.0255	0.0251

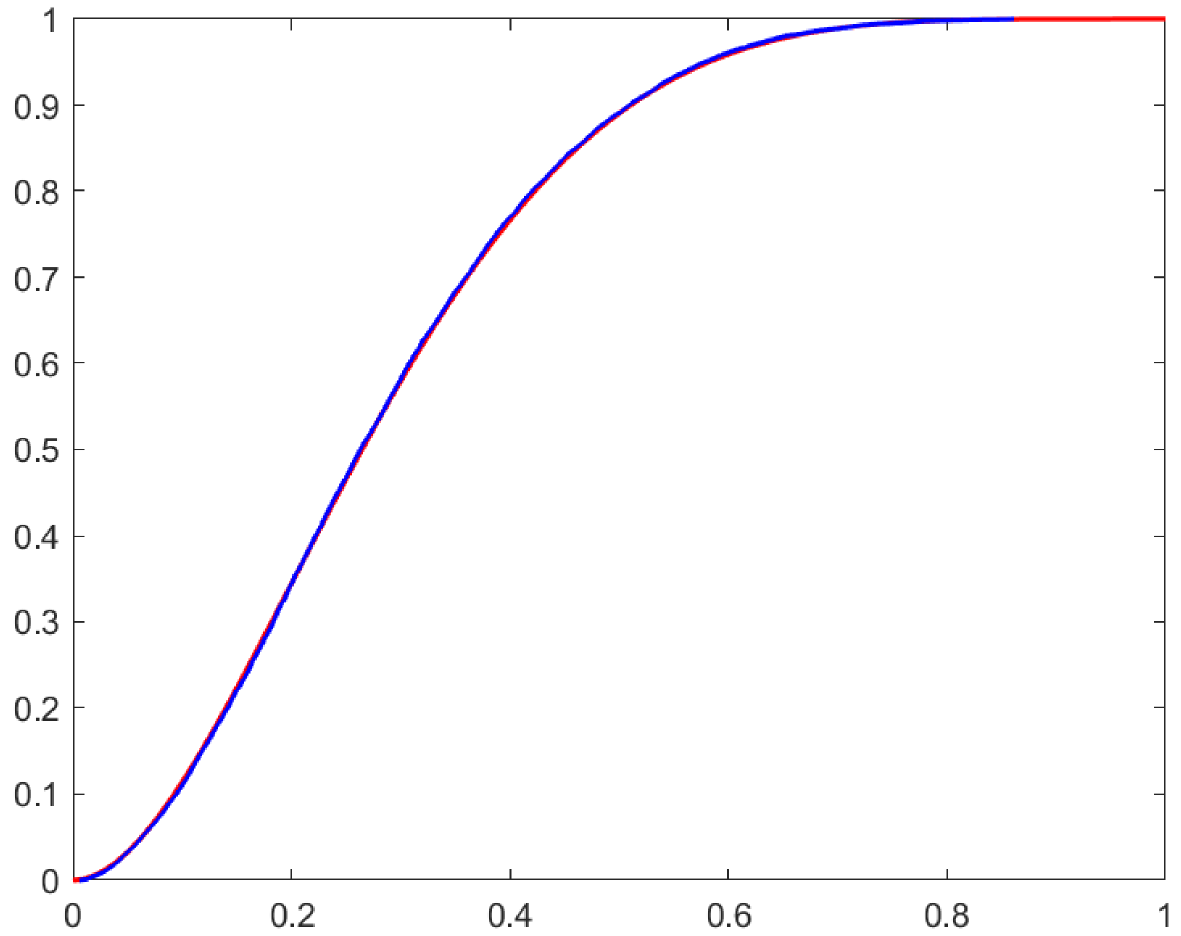


Can also find and plot ECDF.

```
themean=a/(a+b)
thevar=(a*b)/((a+b)^2*(a+b+1))
[mean(x),var(x)]
```

Uniform and Beta PDF

```
y =cdf('beta',(0:.01:1),a,b);
figure;
plot((0:.01:1),y,'Color','red','LineWidth',1.5)
[F,xx]=ecdf(x);
hold on
stairs(xx,F,'Color','blue','LineWidth',1.5)
```

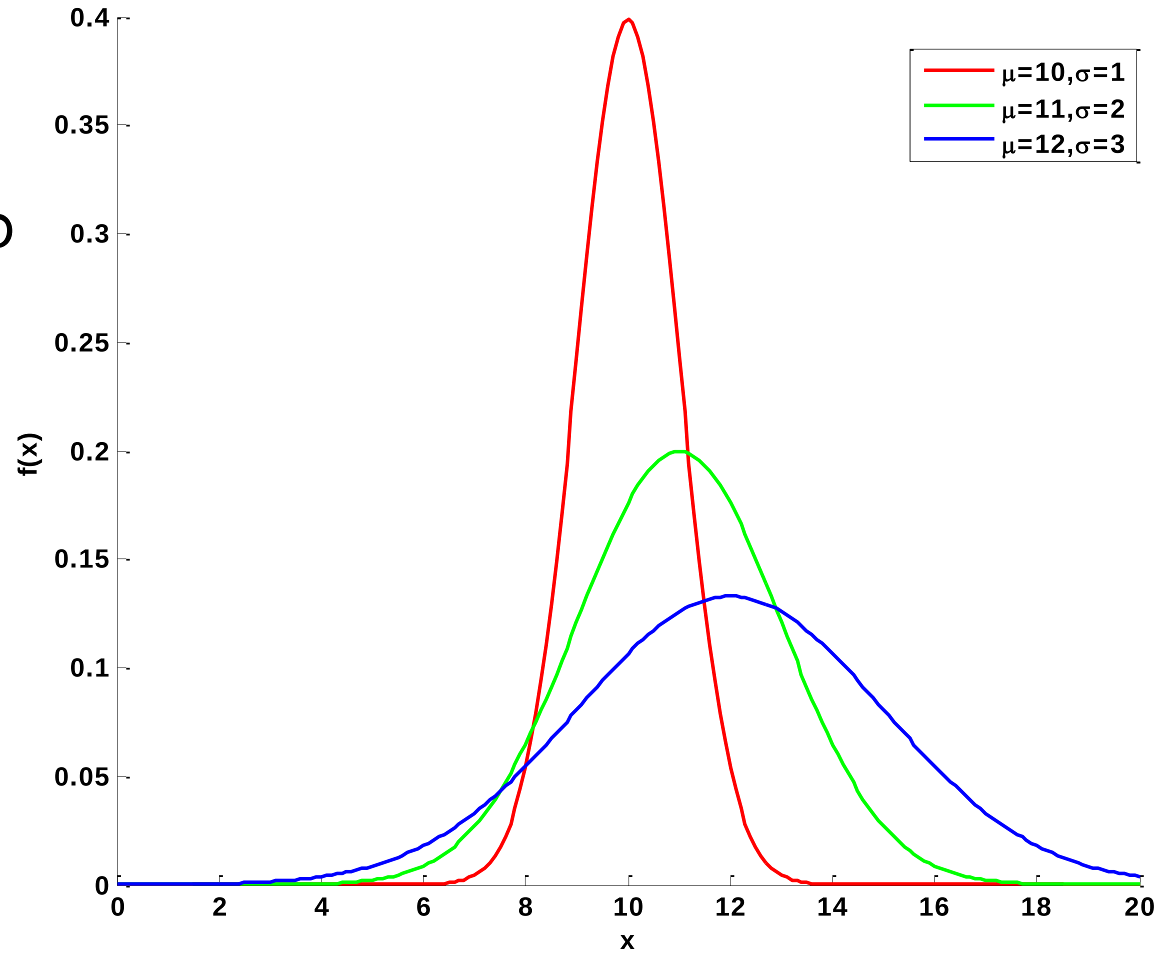


Normal PDF

A random variable x has a continuous normal distribution, x~normal(μ,σ^2) if

$$f(x\,|\,\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \quad , \quad -\infty<x<+\infty$$

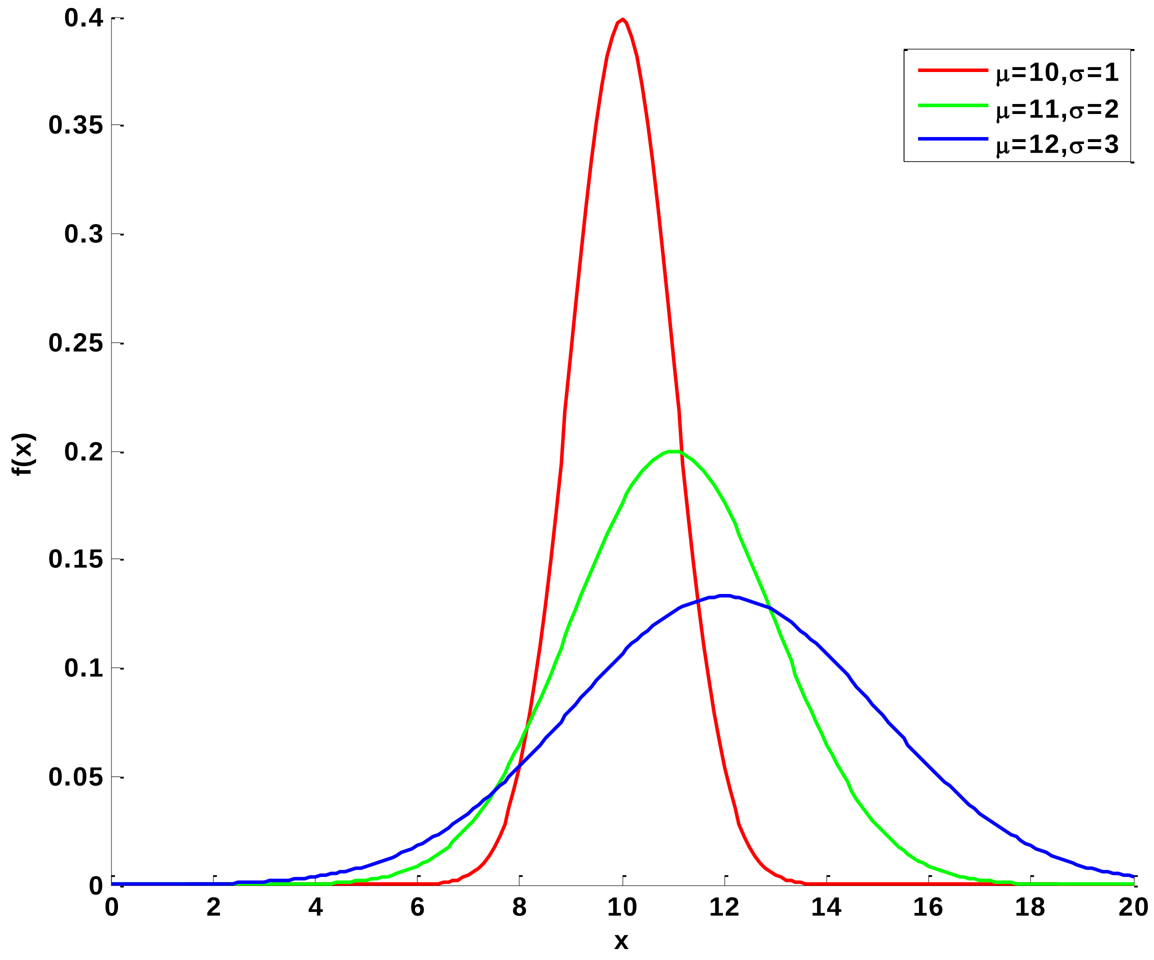
where, $-\infty<\mu<+\infty$ and $\sigma>0$.



Normal PDF

```
x=(0:.1:20)';
mu=[10,11,12];, sigma=[1,2,3];
figure(1)
hold on
for count=1:length(sigma)
    y = normpdf(x,mu(count),sigma(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    end
end
xlim([0 20])
```

$$f(x\mid\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$



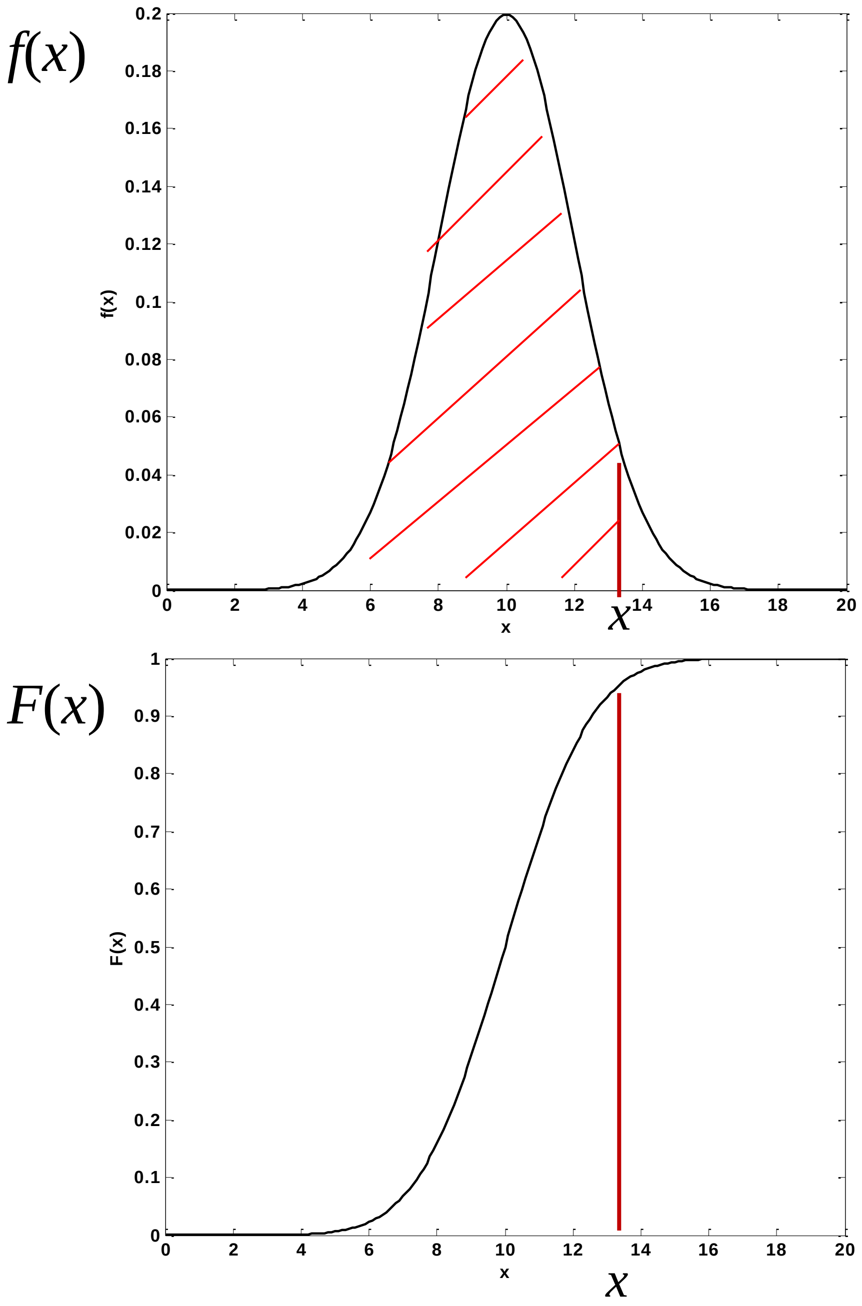
Normal PDF

The CDF of the continuous normal distribution is

$$F(x\,|\,\theta) \quad = \quad \int_{t=-\infty}^{x} f(t\,|\,\theta)\,dt$$

$$F(x\,|\,\mu,\sigma^2) \quad = \quad \int_{t=-\infty}^{x} \frac{e^{-\frac{(t-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dt$$

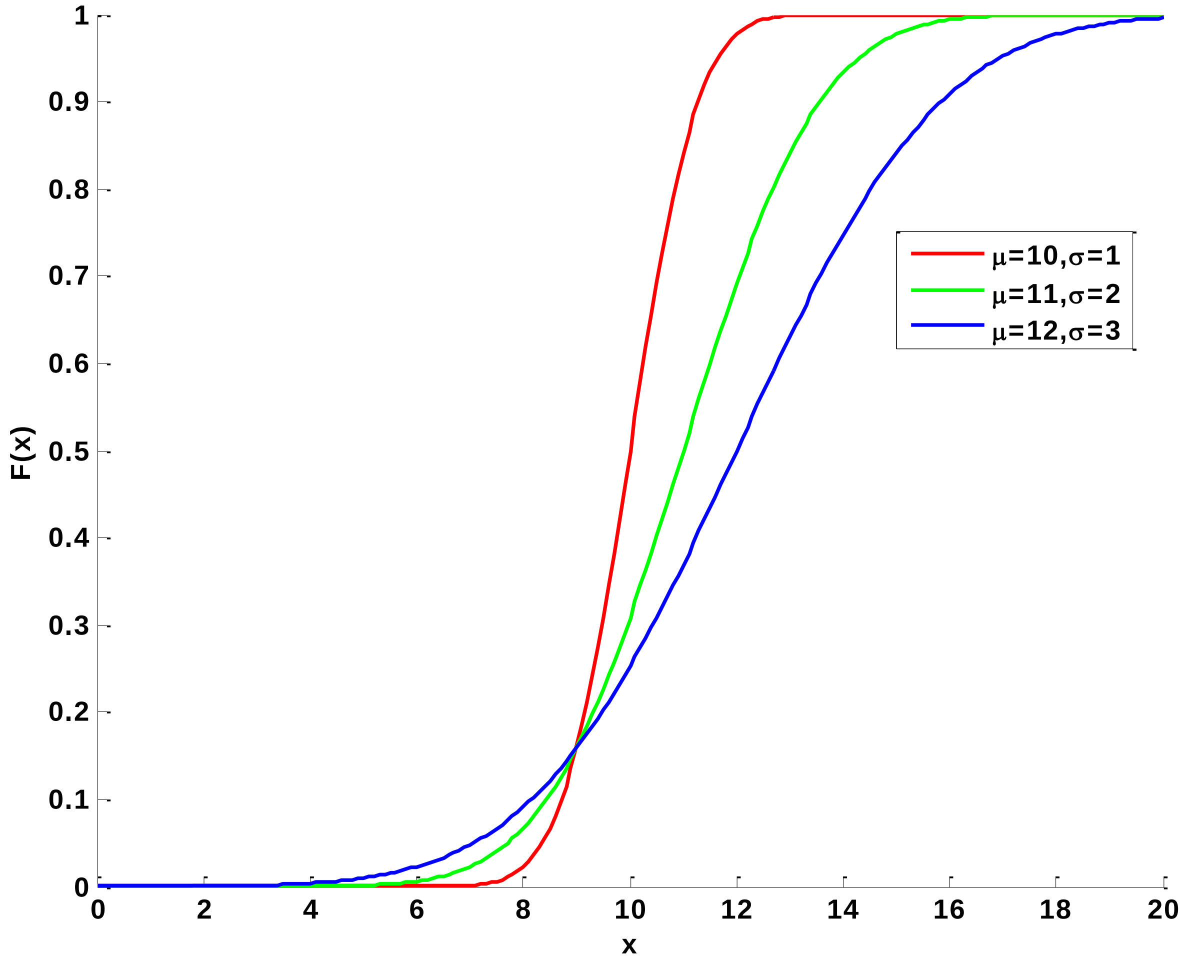
No closed form analytic solution.



Normal PDF

```
x=(0:.1:20)';
mu=[10,11,12];, sigma=[1,2,3];
figure(1)
hold on
for count=1:length(sigma)
    y = normcdf(x,mu(count),sigma(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    end
end
xlim([0 20])
```

$$f(x\,|\,\mu,\sigma^2) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

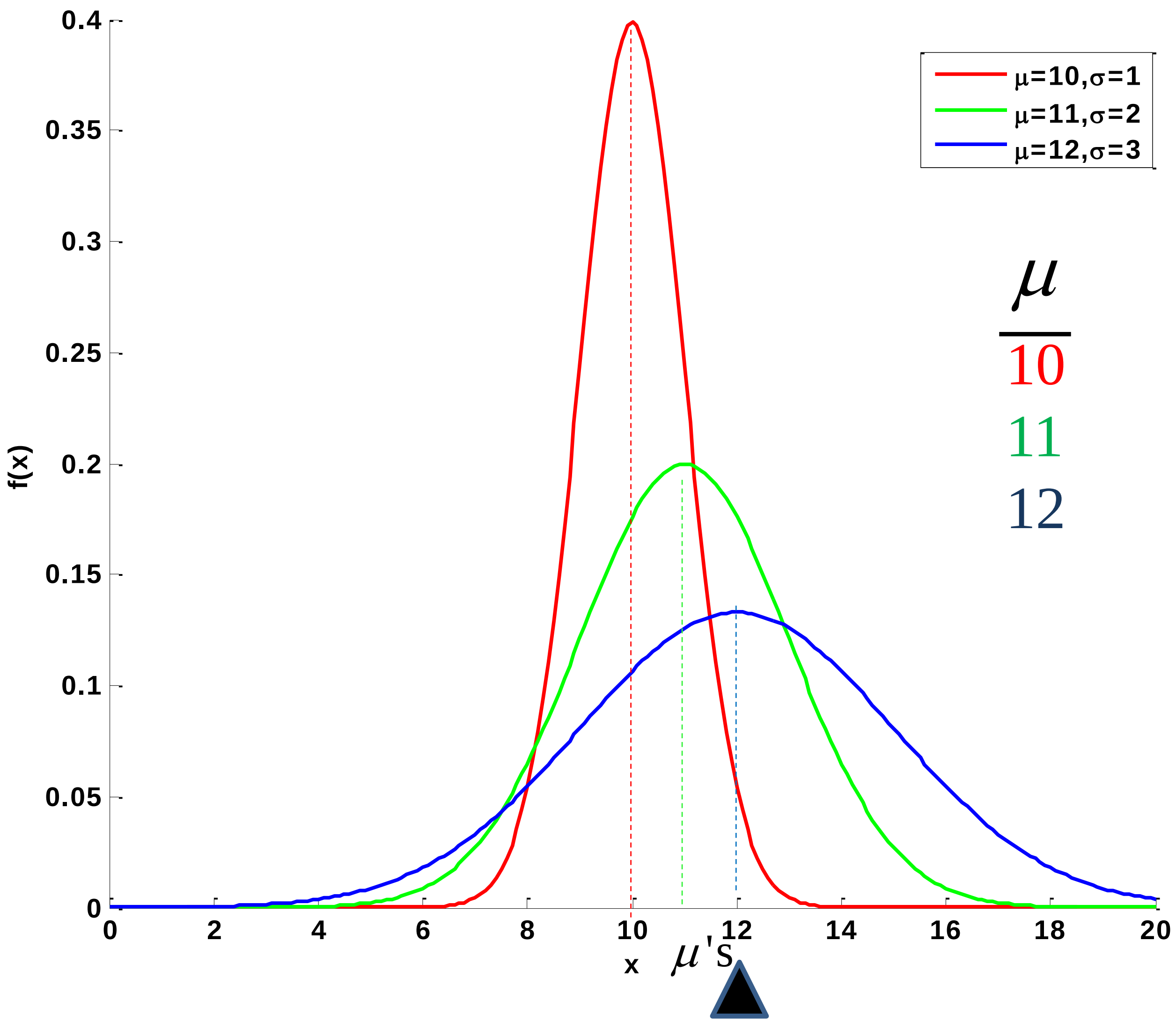


Normal PDF

It can be shown that

$$\mu = \int_x x f(x|\theta) dx$$

$$= \int_{x=-\infty}^{\infty} x \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx$$



Normal PDF

It can be shown that

median

$$\int_{x=-\infty}^{\tilde{x}} f(x|\theta)dx = \frac{1}{2}$$

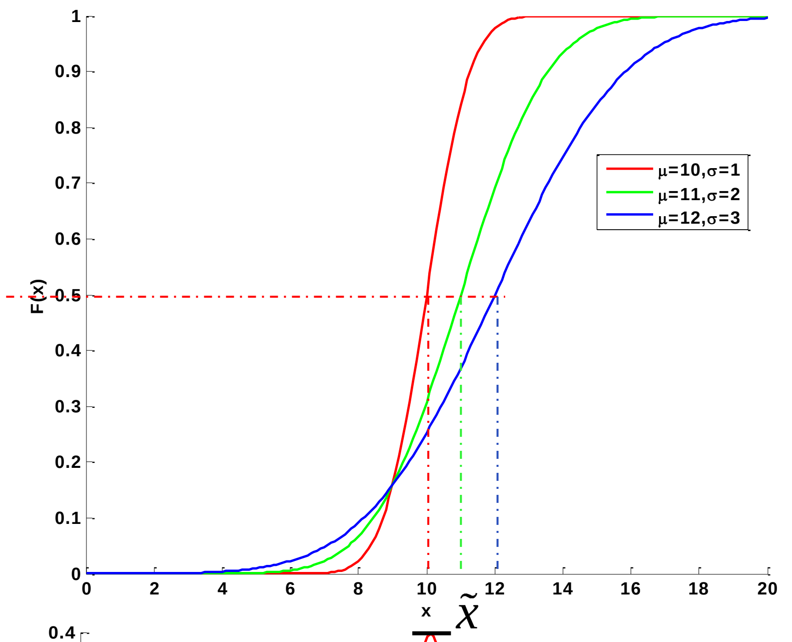
$$\tilde{x} = \mu$$

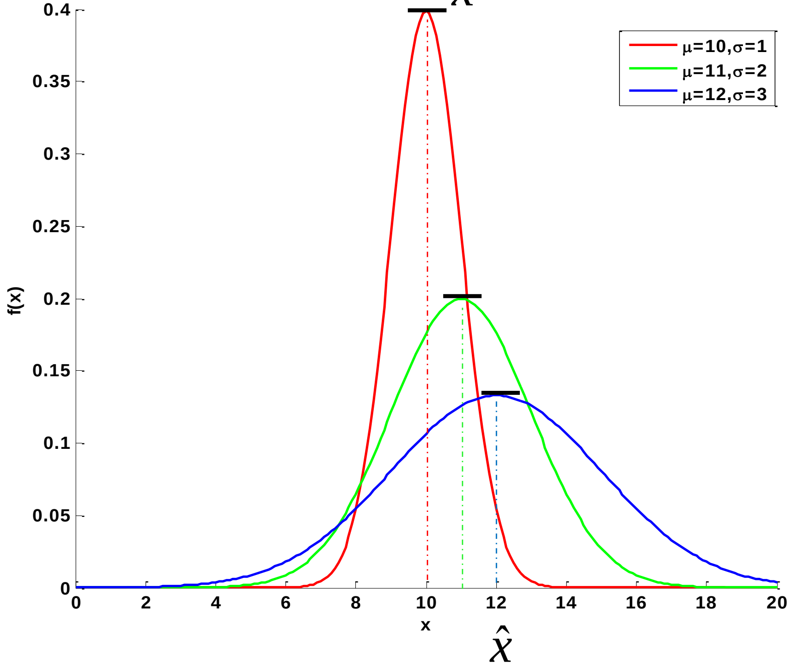
mode

$$\left.\frac{\partial}{\partial x} f(x|\theta)\right|_{\hat{x}} = 0$$

$$\hat{x} = \mu$$

1/2





$\tilde{x}$
10
11
12

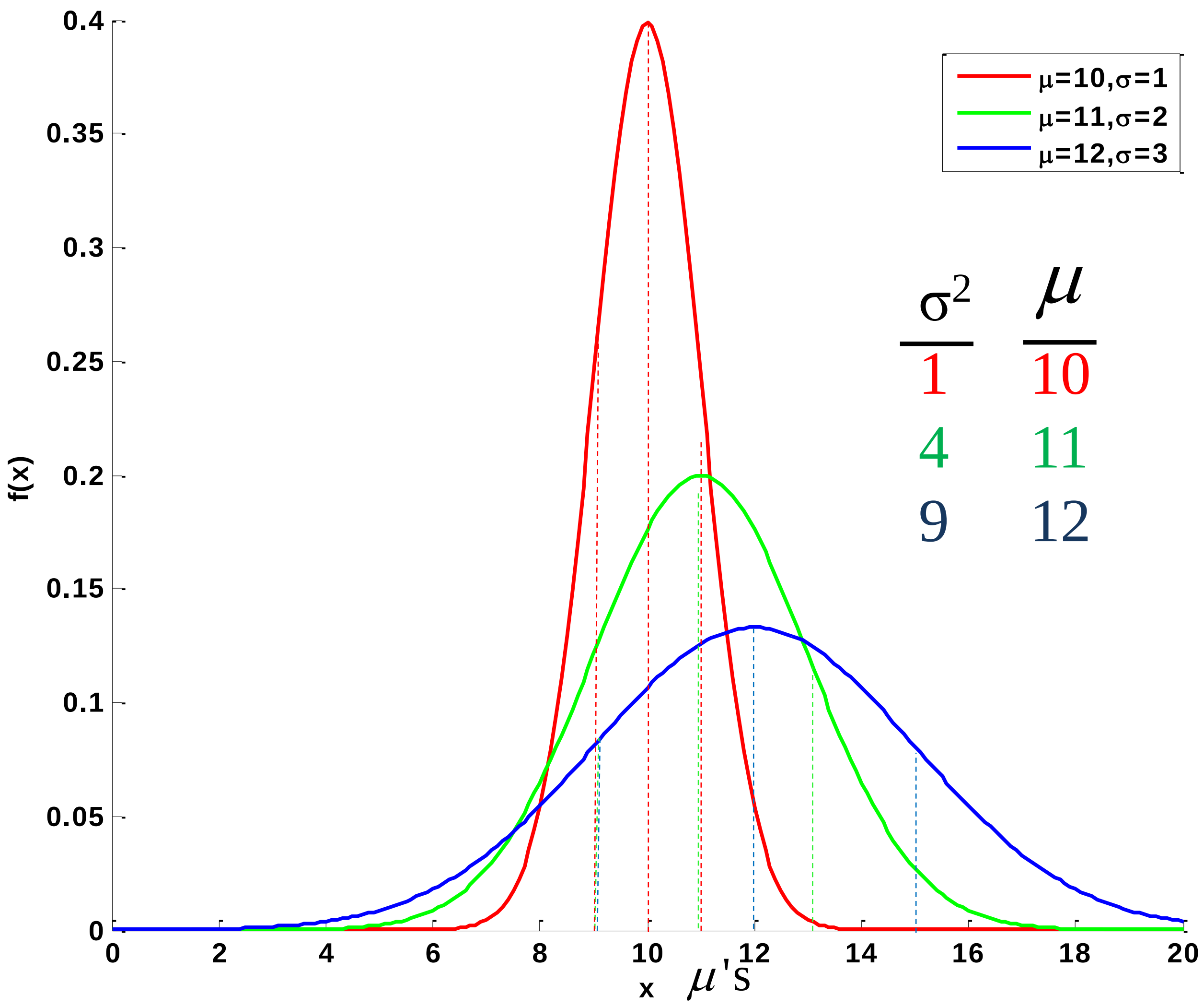
$\hat{x}$
10
11
12

Normal PDF

that

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=-\infty}^{\infty} (x-\mu)^2 \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx$$



Normal PDF

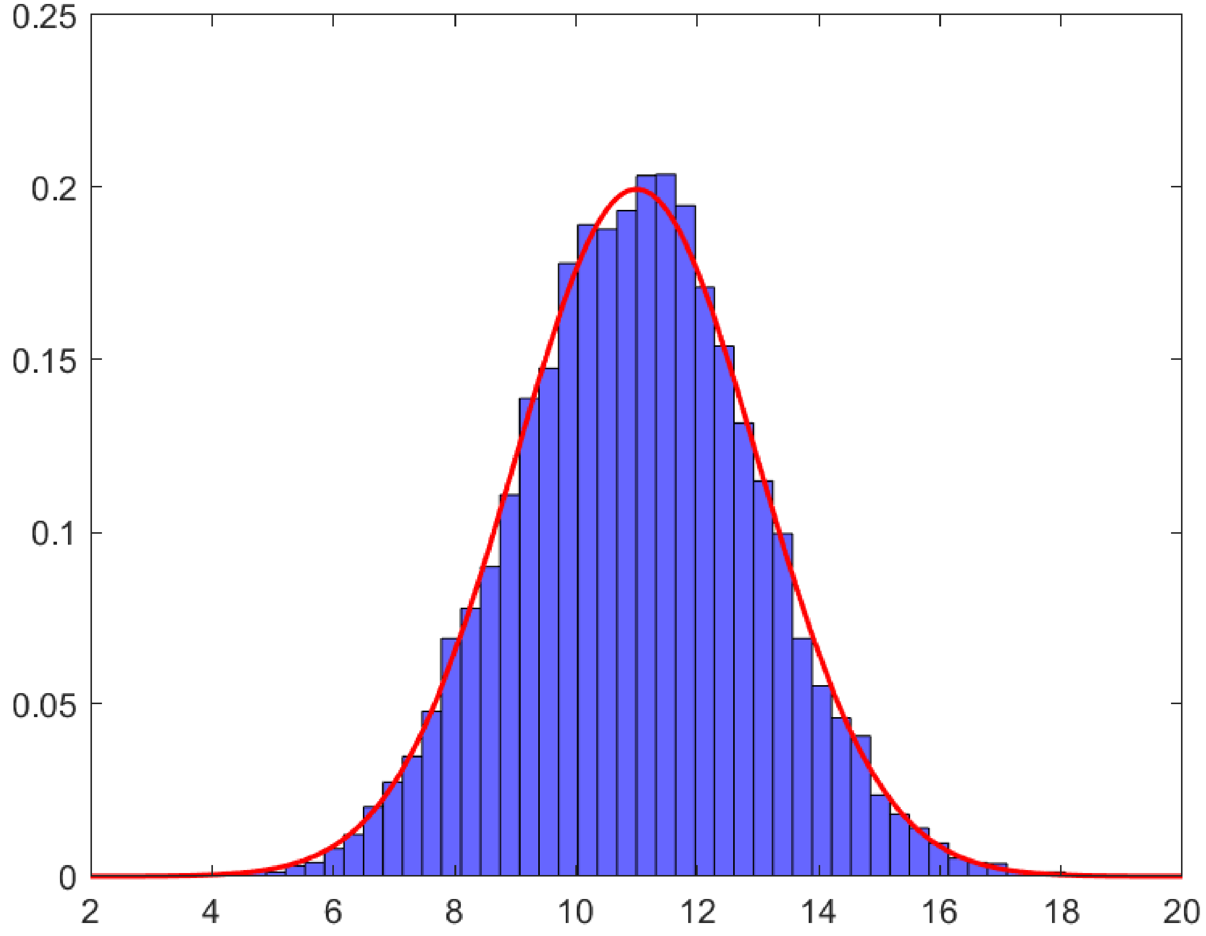
```
rng('default’), mu=11;,sigma=2;,num=10^4;
x=sigma*randn(num,1)+mu;
figure;
histogram(x,'BinLimits',[2,20],'normalization','pdf','FaceColor','blue')
hold on
fx = @(z) 1/sqrt(2*pi*sigma^2).*exp(-(z-mu).^2/(2*sigma^2));
fplot(fx,[0,22],'r','LineWidth',1.5)
xlim([2,20])
```

$$f(x\,|\,\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

	True	Simulated
μ	11	11.0033
σ^2	4	3.9321

Can also find and plot ECDF.

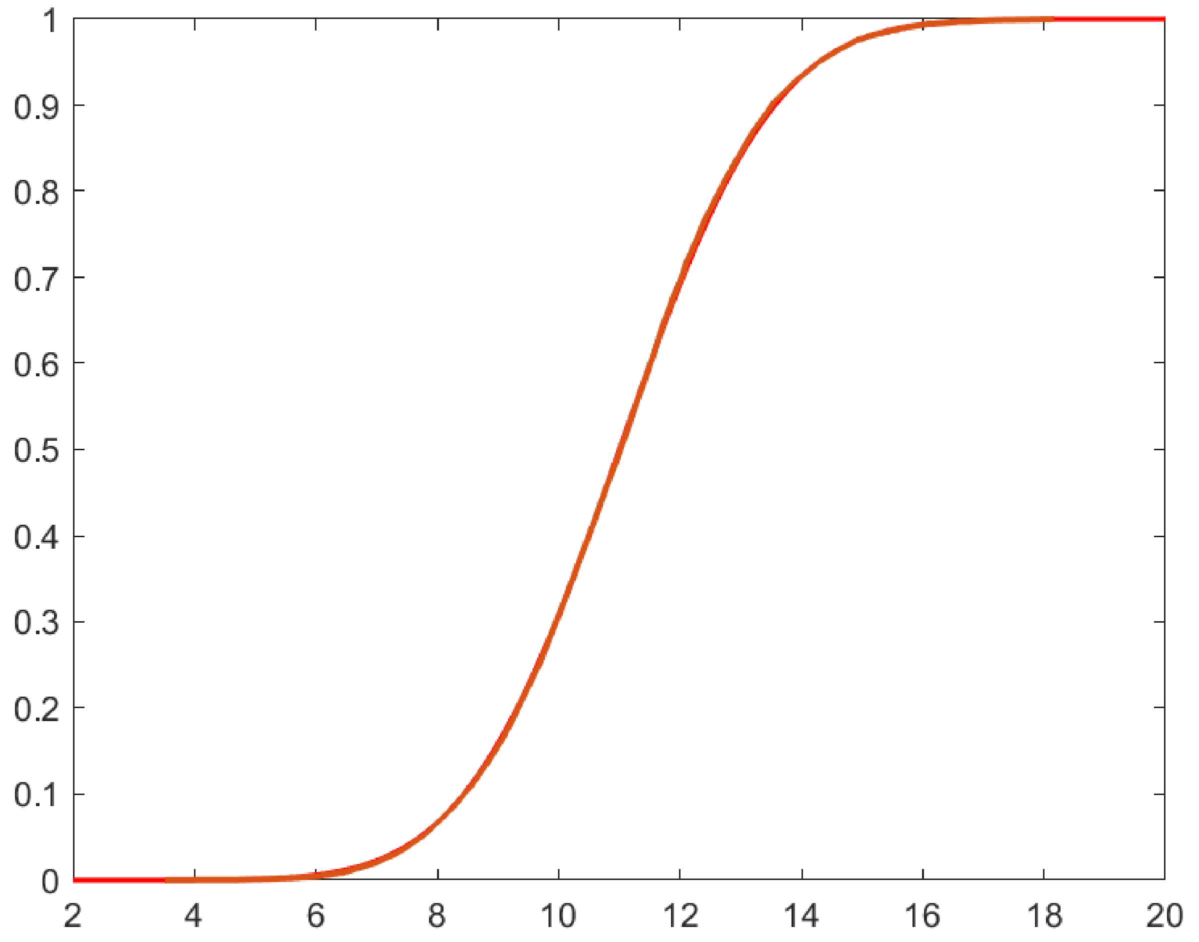
```
themean=mu
thevar=sigma^2
[mean(x),var(x)]
```



Normal PDF

$$f(x|\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

```
y =cdf('Normal',(2:.01:20),mu,sigma);
figure;
plot((2:.01:20),y,'Color','red','LineWidth',1.5)
[F,xx]=ecdf(x);
hold on
stairs(xx,F,'Color','blue','LineWidth',1.5)
```



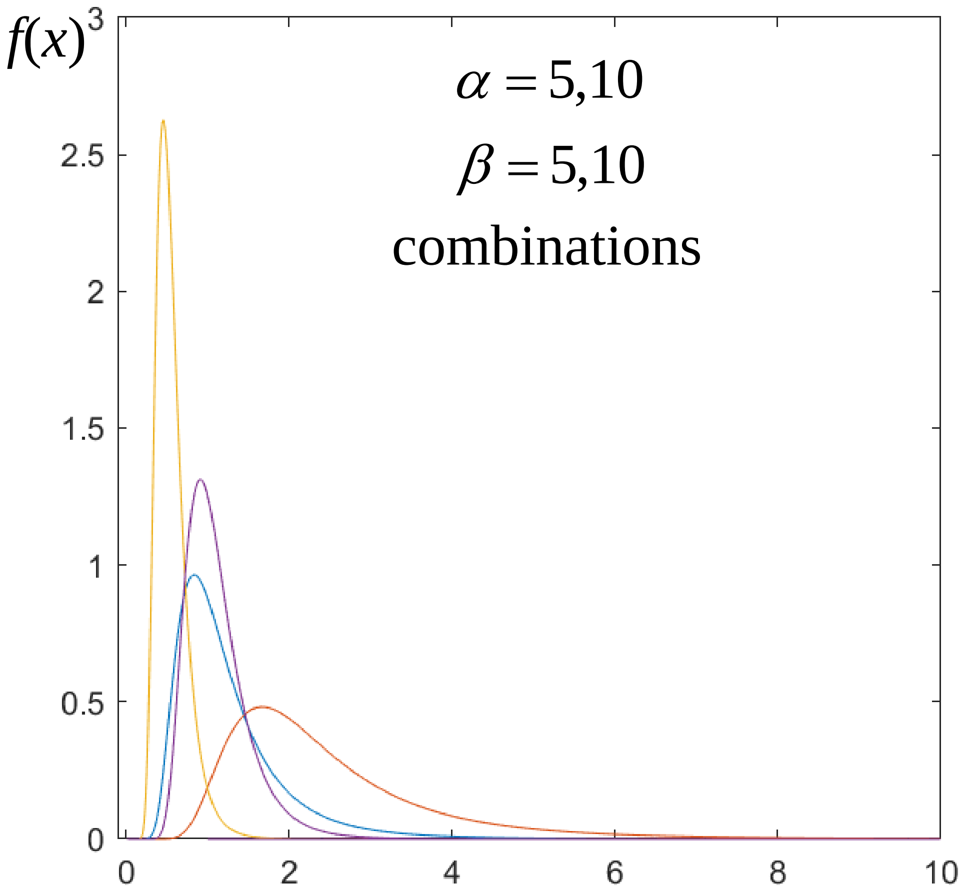
Inverse Gamma PDF

A random variable x has a continuous Inverse Gamma distribution, $x \sim \mathrm{Inv}\Gamma(\alpha,\beta)$ if

$$f(x|\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\beta/x} \quad ,$$

where, $\alpha,\beta>0$, $0<x<\infty$.

And can take on many shapes.



$\Gamma(a)=(a-1)!$ for integer a

Inverse Gamma PDF

There is no closed form analytic formula for the CDF $F(x|\ \alpha,\beta)$.

It can be shown that the mean is

$$\begin{aligned}\mu &= \int_x xf(x\,|\,\theta)dx \\ &= \int_{x=0}^{\infty} x\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-\alpha-1}\mathrm{e}^{-\beta/x}\,dx \\ &= \frac{\beta}{\alpha-1}\end{aligned}$$

Inverse Gamma PDF

It can be shown that the variance is

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=0}^{\infty} (x-\mu)^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x} dx$$

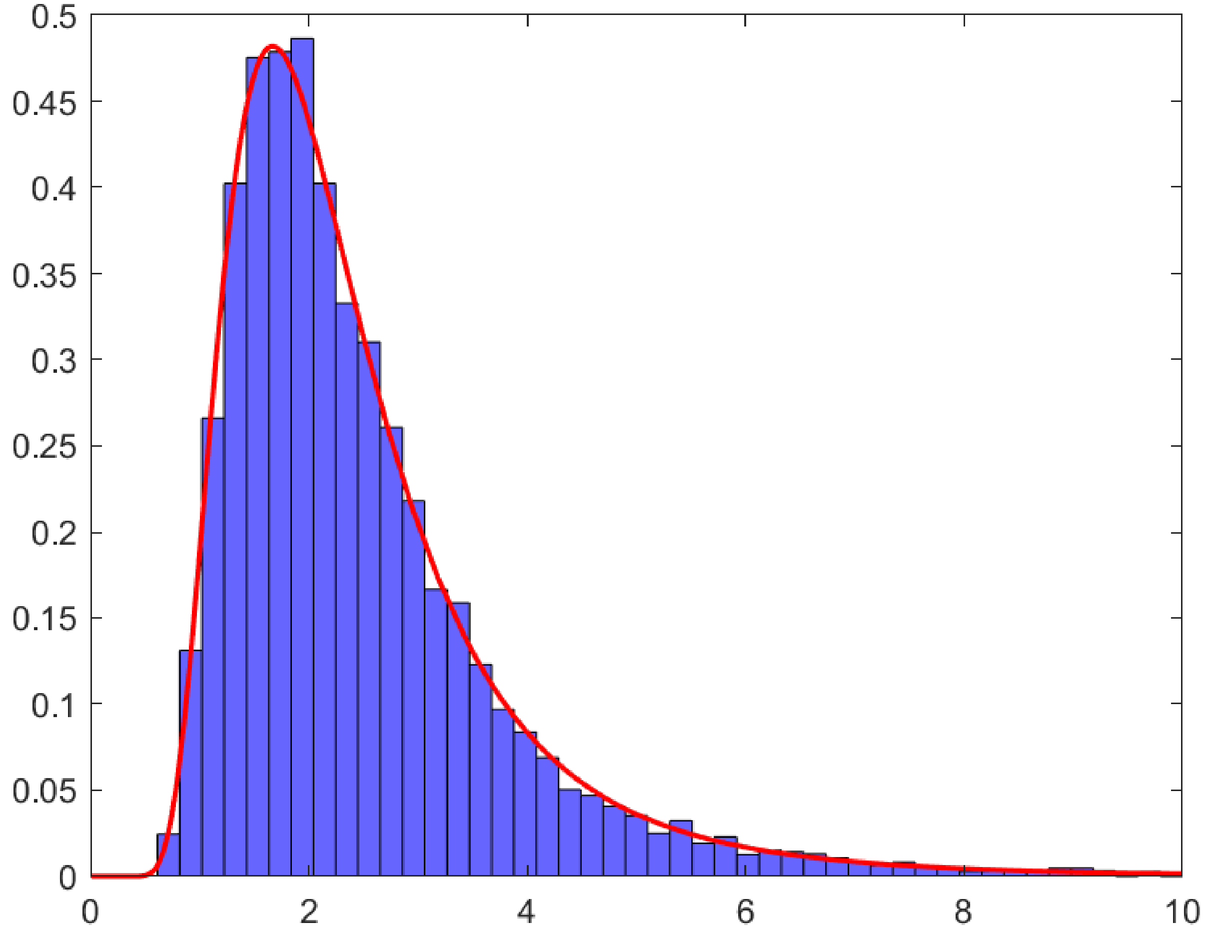
$$= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$$

Inverse Gamma PDF

```
rng('default'), a=5; b=10; c=1/b; num=10^4;
x=1./gamrnd(a,c,[num,1]);
figure;
histogram(x,'BinLimits',[0,10],'normalization','pdf','FaceColor','blue')
hold on
fx = @(z) b.^a/factorial(a-1).*z.^(-a-1).*exp(-b./z);
fplot(fx,[0,10],'r','LineWidth',1.5)
xlim([0,10])
```

$$f(x|\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-\alpha-1}\mathrm{e}^{-\beta/x}$$

	True	Simulated
μ	2.5000	2.4877
σ^2	2.0833	1.9690

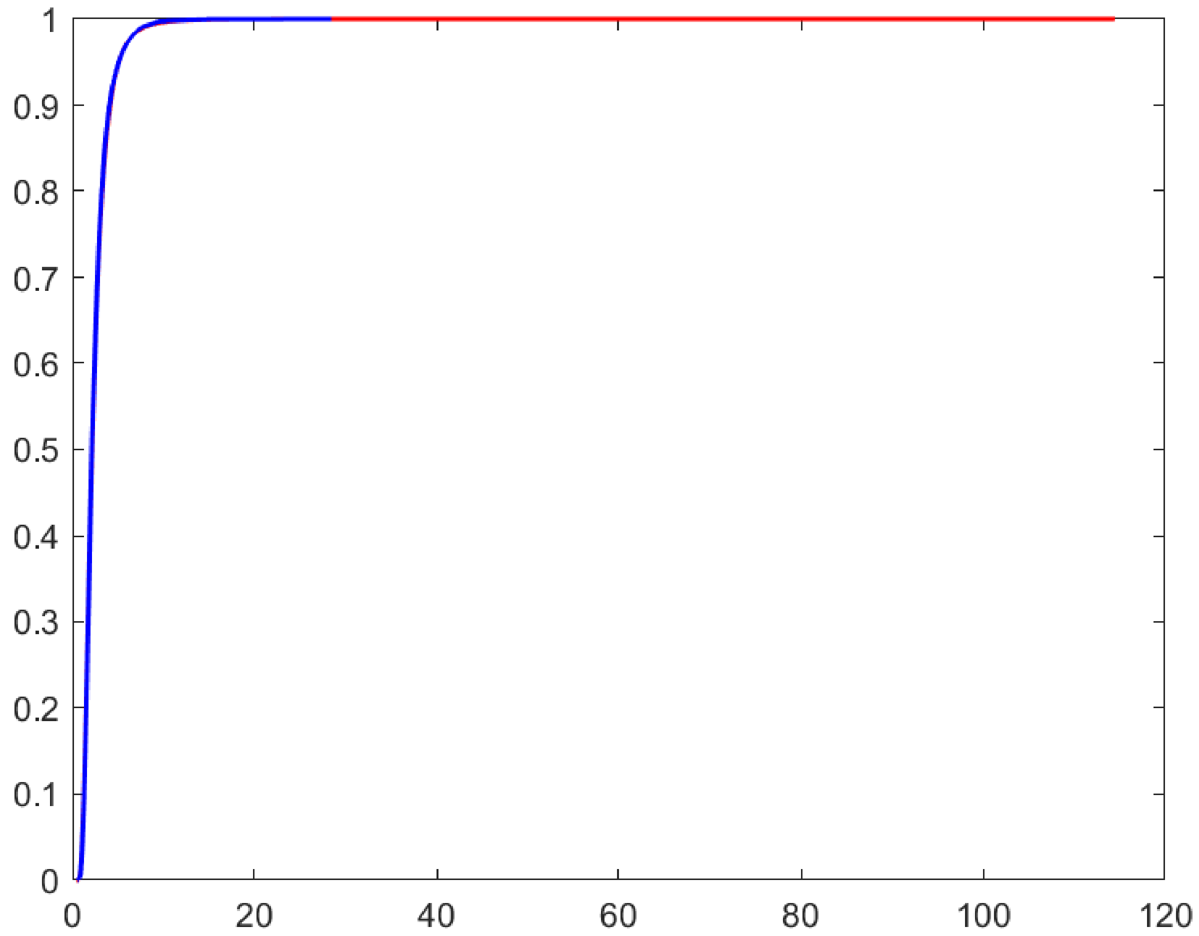


Can also find and plot ECDF.

```
themean=b/(a-1)
thevar=(b^2)/((a-1)^2*(a-2))
[mean(x),var(x)]
```

Inverse Gamma PDF

```
y=1./gamrnd(a,c,[5*10^6,1]);
[Fy,yy]=ecdf(y);
figure;
stairs(yy,Fy,'Color','red','LineWidth',1.5)
[F,xx]=ecdf(x);
hold on
stairs(xx,F,'Color','blue','LineWidth',1.5)
```



Discussion

Questions?

Homework 3

$$f(x|\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

1. Use Matlab to generate n=1000 observations from the Beta PDF with α=3 and β=7. Change α,β to observe shapes.

a. Calculate the sample mean.

b. Calculate the sample variance.

c. Calculate the median.

d. Make a histogram.

e. Make an empirical CDF and calculate the 95$^{\text{th}}$ percentile

Homework 3

$$f(x|\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

2. Use Matlab to generate n=1000 observations from the normal PDF with μ=100 and σ^2=5.

a. Calculate the sample mean.

b. Calculate the sample variance.

c. Calculate the median.

d. Make a histogram.

e. Make an empirical CDF and calculate the 95^{th} percentile.

Homework 3

$$f(x|\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}$$

3. Assume y has a gamma distribution with parameters with $\alpha=3$, $\beta=7$, and $c=1/\beta$. Change α,β to observe shapes.

a. Generate $n=1000$ random y observations.
Calculate the sample mean and variance.
Make a histogram. Compare to the Gamma PDF.

b*. Using pencil and paper transform to x as x=1/y.

c. Take $x=1/y$ for each x random observation in part a.

d. Repeat a. but now for y's and compare to inverse gamma PDF.

*For students in MSSC 5790