A COMPARISON OF THE ROWE AND THE LEE MODELS FOR COMPLEX-VALUED FUNCTIONAL MAGNETIC RESONANCE IMAGING

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ABSTRACT
A COMPARISON OF THE ROWE AND THE LEE MODELS FOR
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In 2005, Rowe published an fMRI magnitude and/or phase activation model that utilized both of the real and imaginary data in each voxel [1]. This model described the magnitude and phase time course as varying linearly with different design matrices where the unknown quantities are estimated voxel-by-voxel. This model follows previous work by Rowe and colleagues [1, 9, 10]. In 2007, Lee as the lead author published a similar model to the Rowe model. Lee (2007) described his model as being equivalent to the Rowe model when the design matrices are the same for the magnitude and phase in addition to having the same contrast matrices [2]. In 2009, Rowe published a letter to the Editor of Magnetic Resonance in Medicine in which he described the Lee (2007) model as elegant and computationally efficient, but Rowe did not completely agree with the properties of the Lee model [3]. In his letter to the Editor, Rowe therefore described four items regarding Lee's model that need to be clarified in addition to describing its relationship to his model. In Rowe's letter to the Editor he specified these four points and provided counter examples to show the non-equivalence of these two models. A response to Rowe's letter to the Editor was published in which Lee agreed with the first of Rowe's items, but disagreed with the remaining three items [4]. In this paper, I will summarize the Rowe model and the Lee model, and discuss the related points in terms of Rowe's letter to the Editor and the response letter from Lee.

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I am really grateful to my academic advisor Dr. Rowe for his infinite help, support and time. He is an associate professor of Marquette University with a great deal of research experience in fMRI. At the beginning of this paper he gave me some papers related to this topic to read. After several talks with him we finally chose fMRI as my essay topic. No matter the difficulties I encountered he always clarified them to me patiently. During the process of writing, I attended a seminar held by Dr. Rowe, and in that seminar I cleared some of my questions that confused me a lot. So I am very appreciative of everything he has done for me.

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1 INTRODUCTION

First I will introduce the Rowe model and then the Lee model.

1.1 The Rowe Model

In each voxel, the complex-valued observation $\boldsymbol{y_t}$ can be represented at time point *t* as a 2×1 vector with phase coupled means instead of as a complex number,

$$\begin{pmatrix} y_{Rt} \\ y_{It} \end{pmatrix} = \begin{pmatrix} \rho_t cos\theta_t \\ \rho_t sin\theta_t \end{pmatrix} + \begin{pmatrix} \eta_{Rt} \\ \eta_{It} \end{pmatrix}, \qquad t = 1, \cdots, n. \tag{1}$$

where

$$\rho_t = x_t'\beta = \beta_0 + \beta_1 x_{1t} + \cdots + \beta_{q1} x_{q_1 t}$$

$$\theta_t = u_t'\gamma = \gamma_0 + \gamma_1 u_{1t} + \cdots + \gamma_{q2} u_{q_1 t}$$

and $(\eta_{Rt}, \eta_{It})' \sim N(0, \sum)$, x_t' is the t^{th} row of an $n \times (q_1 + 1)$ design matrix **X** for the magnitude, u_t' is the t^{th} row of an $n \times (q_2 + 1)$ design matrix **U** for the phase, and $\sum = \sigma^2 I_2$ while β and γ are magnitude and phase regression coefficient vectors, respectively.

1.2 The Lee Model

The Lee model is

$$\begin{pmatrix} y_{Rt} \\ y_{It} \end{pmatrix} = \begin{pmatrix} x_t'\beta_R \\ x_t'\beta_I \end{pmatrix} + \begin{pmatrix} \eta_{Rt} \\ \eta_{It} \end{pmatrix} \tag{2}$$

where β_R and β_I are $(q+1) \times 1$ regression coefficient vectors for the real and imaginary parts of the signal, $q = q_1 = q_2$, and all other variables are as previously defined. Additionally, the Lee model requires that the magnitude and phase design matrices to be the same, $\mathbf{U} = \mathbf{X}$.

2 THE COMPARISON OF ROWE MODEL AND LEE MODEL

In the letter to Editor of Magnetic Resonance in Medicine by Rowe, he mentioned four items that needed to be clarified regarding the Lee model and its relationship to the Rowe model. The process of a letter to the Editor is as follows. An Author submits a letter to the Editor regarding an issue in a published paper that needs additional commentary. That letter to the Editor goes through the formal peer review process in which it is sent to reviewers, the reviewers return comments to the Editor for the Author with a decision recommendation, the Editor forwards the reviewers comments to the Author with an aggregate decision, this process iterates until it is accepted or rejected. If the Letter to the Editor is accepted, it is then forwarded to the Author of the published paper for a response. The response letter goes through the same peer review process. Both the Author's letter to the Editor and the response from the published paper's Author are published together in the same issue of the journal.

Lee responded to Rowe's letter agreeing with some of Rowe's comments, but reiterated his original viewpoints for the others. After the letter to the Editor by Rowe and the response by Lee were also published in Magnetic Resonance in Medicine in May 2009, Rowe conveyed to me some additional comments on the comparison of the Rowe model and the Lee model. I will summarize the four items by Rowe, the response by Lee, and additional comments by Rowe and my analysis of these two models that will show that Rowe's remarks of the Lee model are accurate.

2.1 Item 1 Mathematical Proof

Rowe's Letter:

A "mathematical proof" is in Appendix B to "show the equivalence" of the

Lee model to the Rowe model (2005) model. This "proof" is a derivation of their test statistic using a likelihood ratio test. This item is stated without proof.

Lee's Response:

Lee agreed with Rowe's comment on Appendix B. However, Lee claimed that the test statistics in Appendix B were correctly derived, it started in Cartesian coordinates, whereas Rowe's model is in polar coordinates. Finally, Lee mentioned that despite their oversight in Appendix B, the rest of the article remained correct.

Additional Comments:

From the above two paragraphs, we can see Lee agreed with the first point of Rowe's letter. In addition, there are two typos needed to be corrected in Lee's Response.

(1) Change "1" subscript to "I" subscript in β_1 which is in the 13th line of the right side of the 1st page of Lee's Response.

(2) The second element of β_I which is also in the 13th line of the right side of the 1st page should be -2.

2.2 Items 2 and 3 Reference Waveform Vectors

In [2], Lee stated: "The complex time-series data are decomposed into real and imaginary axes". To find the mean vector of each state, one structures the design matrix ($\mathbf{X}$) of the GLM by a constant vector ($\mathbf{1} = [1,1,\cdots,1]'$, a real $n \times 1$ vector) and a reference waveform vector($\mathbf{h}$, a real $n \times 1$vector, the convolution of a stimulus pattern and a hemodynamic response function). The following equation shows this modeling:

$$\boldsymbol{y_R} = \mathbf{X}\boldsymbol{\beta_R} + \varepsilon_R = [\mathbf{1}\ \mathbf{h}][\beta_{R1}\ \beta_{R2}]' + \varepsilon_R$$

$$\boldsymbol{y_I} = \mathbf{X}\boldsymbol{\beta_I} + \varepsilon_I = [\mathbf{1}\ \mathbf{h}][\beta_{I1}\ \beta_{I2}]' + \varepsilon_I,$$

where $\boldsymbol{y} = \boldsymbol{y_R} + \boldsymbol{i}\boldsymbol{y_I}$ (a complex $n \times 1$ vector) is the time-series data of one voxel, $\boldsymbol{\beta_R}$ and $\boldsymbol{\beta_I}$ are the parameters of GLM, and ε_R and ε_I are residual error vectors. One can easily incorporate other terms, such as a linear drift, by adding more vectors and parameters into the model (see Appendix B).

Rowe Letter:

Rowe expressed concern about the above point; he thought the description of possible reference waveform vectors was not mathematically correct. A simple example was used to illustrate his viewpoint.

Consider the example where $L = 2, n = 3,$ and the design matrix $\mathbf{X}$ has the first column $[1,1,1]'$ and second column$[0,1/2,1]'$, thus the design matrix:

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

Upon equating the means of the Rowe model and the Lee model, the real part is

$$\begin{pmatrix} \beta_{R1} \\ \beta_{R1} + 0.5\beta_{R2} \\ \beta_{R1} + \beta_{R2} \end{pmatrix} = \begin{pmatrix} \beta_0 \cos(\gamma_0) \\ (\beta_0 + 0.5\beta_1)\cos(\gamma_0 + 0.5\gamma_1) \\ (\beta_0 + \beta_1)\cos(\gamma_0 + \gamma_1) \end{pmatrix} \quad (3)$$

If we assume $\beta = [10,1]'$ and $\gamma = [\pi/4, \pi/9]'$ of the right side of Equation (3), then one can obtain

$$\begin{cases} \beta_{R1} = 7.0711 \\ \beta_{R1} + 0.5\beta_{R2} = 6.0226 \\ \beta_{R1} + \beta_{R2} = 4.6488 \end{cases}$$

Since

$$\hat{\beta}_R = (X'X)^{-1}X'y_R = [7.1253, -2.4223]'.$$

So, it can be seen that

$$\begin{cases} \hat{\beta}_{R1} = 7.1253 \neq 7.0711 = \beta_{R1} \\ \hat{\beta}_{R1} + 0.5\hat{\beta}_{R2} = 5.9142 \neq 6.0226 = \beta_{R1} + 0.5\beta_{R2} \\ \hat{\beta}_{R1} + \hat{\beta}_{R2} = 4.7030 \neq 4.6488 = \beta_{R1} + \beta_{R2}. \end{cases}$$

Therefore, one can see that the Lee model does not correctly estimate the points. Furthermore, Rowe thought that the Lee model was only mathematically correct with a constant baseline and an on/off (0/1 or -1/1) reference vector. With a 0/1 reference vector, the design matrix is

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Upon equating the means of the Rowe model and the Lee model, the real part is

$$\begin{pmatrix} \beta_{R1} \\ \beta_{R1} + \beta_{R2} \end{pmatrix} = \begin{pmatrix} \beta_0 \cos(\gamma_0) \\ (\beta_0 + \beta_1)\cos(\gamma_0 + \gamma_1) \end{pmatrix}. \quad (4)$$

The imaginary part is:

$$\begin{pmatrix} \beta_{I1} \\ \beta_{I1} + \beta_{I2} \end{pmatrix} = \begin{pmatrix} \beta_0 \sin(\gamma_0) \\ (\beta_0 + \beta_1)\sin(\gamma_0 + \gamma_1) \end{pmatrix}. \quad (5)$$

From Equations (4) and (5) we can get

$$\beta_{R2} = (\beta_0 + \beta_1)\cos(\gamma_0 + \gamma_1) - \beta_0 \cos(\gamma_0)$$

$$\beta_{I2} = (\beta_0 + \beta_1)\sin(\gamma_0 + \gamma_1) - \beta_0 \sin(\gamma_0)$$

Under the null hypothesis of the Lee model H_0: $\mathbf{v}[\boldsymbol{\beta_R}, \boldsymbol{\beta_I}] = \mathbf{0}$,

$$\mathbf{v}[\boldsymbol{\beta_R}, \boldsymbol{\beta_I}] = [0 \quad 1]\begin{bmatrix}\beta_{R1} & \beta_{I1} \\ \beta_{R2} & \beta_{I2}\end{bmatrix} = [\beta_{R2} \quad \beta_{I2}] = [0 \quad 0].$$

Therefore,

$$\begin{cases}(\beta_0 + \beta_1)\cos(\gamma_0 + \gamma_1) - \beta_0 \cos(\gamma_0) = 0 \\ (\beta_0 + \beta_1)\sin(\gamma_0 + \gamma_1) - \beta_0 \sin(\gamma_0) = 0\end{cases} \tag{6}$$

The above equation (6) indirectly implying $\beta_1 = 0$ and $\gamma_1 = 0$.

Lee Response:

In the example of the Lee model on this point, he used the same design matrix $\mathbf{X}$, and then they assumed the observation made was $\boldsymbol{y_R} = [4, \quad 5, \quad 6]'$ and $\boldsymbol{y_I} = [8, \quad 7, \quad 6]'$. One can easily see that this is a noiseless observation in Cartesian coordinates because a contrast and a linear regressor are assumed.

If $\boldsymbol{\beta_R} = [4 \quad 2]'$and $\boldsymbol{\beta_R} = [8 \quad -2]'$, then

$$\boldsymbol{X}[\boldsymbol{\beta_R}, \boldsymbol{\beta_I}] = \begin{bmatrix}1 & 0 \\ 1 & \frac{1}{2} \\ 1 & 1\end{bmatrix}\begin{bmatrix}4 & 8 \\ 2 & -2\end{bmatrix} = \begin{bmatrix}4 & 8 \\ 5 & 7 \\ 6 & 6\end{bmatrix}.$$

Hence, Lee claimed his model correctly produced a noiseless estimation.

Using the same observations, let us apply them to the Rowe model.

$$\begin{pmatrix} x_1'\beta cos(u_1'\gamma) \\ x_1'\beta sin(u_1'\gamma)\end{pmatrix} = \begin{pmatrix}\beta_1 cos\gamma_1 \\ \beta_1 sin\gamma_1\end{pmatrix} = \begin{pmatrix}4 \\ 8\end{pmatrix} \tag{7}$$

$$\begin{pmatrix} x_2'\beta cos(u_2'\gamma) \\ x_2'\beta sin(u_2'\gamma) \end{pmatrix} = \begin{pmatrix} (\beta_1 + 0.5\beta_2)cos(\gamma_1 + 0.5\gamma_2) \\ (\beta_1 + 0.5\beta_2)sin(\gamma_1 + 0.5\gamma_2) \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \tag{8}$$

The solution of Equations (7) and (8) are

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 8.94427 \\ -0.68389 \\ 1.10715 \\ -0.3132 \end{pmatrix}.$$

Then we insert the numbers to the third equation

$$\begin{pmatrix} x_3'\beta cos(u_3'\gamma) \\ x_3'\beta sin(u_3'\gamma) \end{pmatrix} = \begin{pmatrix} (\beta_1 + \beta_2)cos(\gamma_1 + \gamma_2) \\ (\beta_1 + \beta_2)sin(\gamma_1 + \gamma_2) \end{pmatrix} = \begin{pmatrix} 5.79081 \\ 5.89071 \end{pmatrix} \neq \begin{pmatrix} 6 \\ 6 \end{pmatrix}. \tag{9}$$

Hence, the Rowe model does not estimate the parameters correctly. This result leads to exactly the opposite conclusion from Rowe's letter.

Additional Comments:

When the design matrix consists of a first column of ones, and a second column that is a graded reference waveform, the Lee model can only correctly produce the correct noiseless estimated parameter values or data values when the data values lie perfectly on a line. If the data are changed to $\boldsymbol{y_R} = [4, \quad 5, \quad 6]'$ and $\boldsymbol{y_I} = [8, \quad 6, \quad 7]'$from $\boldsymbol{y_R} = [4, \quad 5, \quad 6]'$ and $\boldsymbol{y_I} = [8, \quad 7, \quad 6]'$, we can see that the Lee model cannot estimate the data correctly.

Since

$$\widehat{\boldsymbol{\beta}}_{\mathbf{R}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y_R} = [4 \quad 2]'$$

$$\widehat{\boldsymbol{\beta}}_{\mathbf{I}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y_I} = [7.5 \quad -1]'.$$

So,

$$[\hat{\boldsymbol{y}}_R \quad \hat{\boldsymbol{y}}_I] = \boldsymbol{X}[\boldsymbol{\beta}_R, \boldsymbol{\beta}_I] = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7.5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 7.5 \\ 5 & 7 \\ 6 & 6.5 \end{bmatrix}.$$

From the above estimation, we can see that the Lee model does not estimate the data correctly.

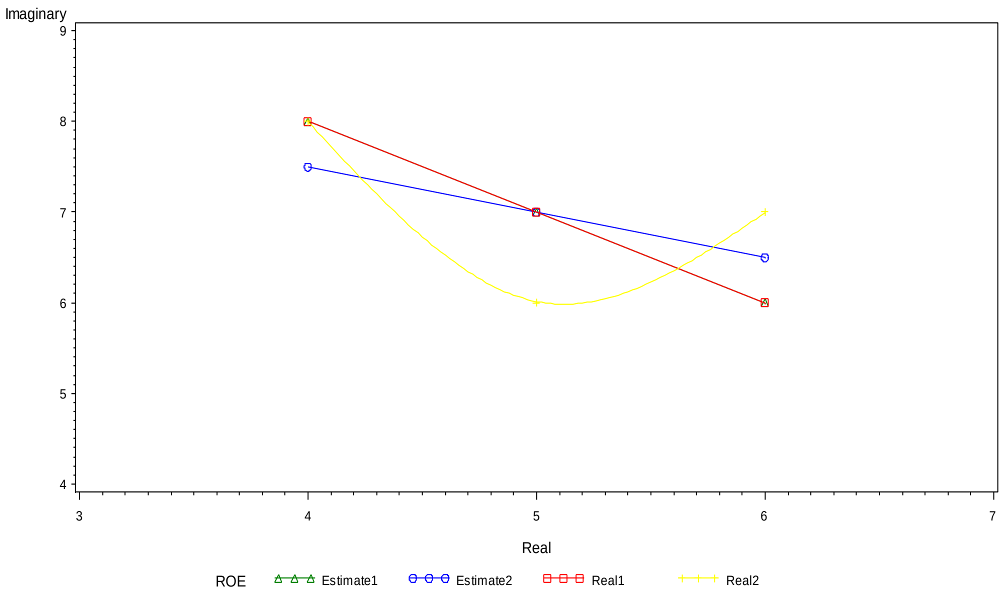

Figure1

From Figure 1, one can see more easily that the Lee model can correctly estimate data values if and only if the data values lie perfectly on a line. The red rectangles represent the real data that lie perfectly on a straight line and the green triangles represent the estimated data values using the Lee model. From these two lines we can see that the Lee model can correctly estimate data values if the data values lie perfectly on a line. But if the data do not lie perfectly on a straight line, what will happen? We can see that from the yellow pluses and the blue circles. The yellow pluses represent the real data that do not lie on a straight line and the blue circles represent the estimated data values using the Lee model. From the above graph we can see that the Lee model does not estimate the data correctly if the original data values do not lie perfectly on a straight line. In other words, if the data is not noiseless, the Lee model cannot be used.

Using the same observations, Lee claimed that the Rowe model cannot correctly estimate the data, either. However, Lee made a subtle but critical error in that he did not estimate the parameters using multiple iterations which was a fundamental criteria of the Rowe model. This is counter to his description in [2] that the Rowe "method requires multiple iterations in each voxel for optimum parameter values." In the following, this point will be explained in detail.

Lee concluded that the Rowe model did not estimate the parameters correctly. But this conclusion does not make sense because they did not use the Rowe model in the right way. If one wants to use the Rowe model to estimate the parameters, the multiple iterations should be used. The right way is as following:

One solves for β_1 and $\gamma_1$from the real and imaginary equation for time point one. From Equation (7), one obtains $\hat{\beta}_1 = 8.94427$ and $\hat{\gamma}_1 = 1.10715$. Inserting these values back into Equation (7) above yields $\hat{y}_{R1} = 4$ and $\hat{y}_{I1} = 8$ which are the correct values. If one solves for β_2 and γ_2 form the real and imaginary equations for time point two, from Equation (8), one obtains $\hat{\beta}_2 = -0.6839$ and $\hat{\gamma}_2 = -0.3132$. Inserting these values back into Equation (8) above yields $\hat{y}_{R2} = 5$ and $\hat{y}_{I2} = 7$ which are the correct values. Instead of using estimated $\hat{\beta}_1$, $\hat{\gamma}_1$, $\hat{\beta}_2$ and $\hat{\gamma}_2$from time points one and two inserted into Equation (9), if one solves for β_2 and γ_2 from the real and imaginary equations for time point three:

$$\begin{pmatrix} (\beta_1 + \beta_2)cos(\gamma_1 + \gamma_2) \\ (\beta_1 + \beta_2)sin(\gamma_1 + \gamma_2) \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{10}$$

where

$$\begin{pmatrix}\beta_1\\ \gamma_1\end{pmatrix} = \begin{pmatrix}8.94427\\ 1.10715\end{pmatrix}$$

Then

$$\begin{pmatrix}\hat{\beta}_2\\ \hat{\gamma}_2\end{pmatrix} = \begin{pmatrix}-0.4590\\ -0.3218\end{pmatrix}$$

If one inserts these values back to Equation (10) above yields $\hat{y}_{R3} = 6$ and$\hat{y}_{I3} = 6$ which are the correct values. If one mistakenly inserted $\hat{\beta}_1$, $\hat{\gamma}_1$, $\hat{\beta}_2$ and $\hat{\gamma}_2$ from time points one and two into Equation (10), one obtains $\hat{y}_{R3} = 5.7908$ and $\hat{y}_{I3} = 5.8907$. These estimations did not use the Rowe model that requires multiple iterations.

Thus, we can conclude that the Lee model is only mathematically correct with a constant baseline and an on/off (0/1 or -1/1) reference vector; the Lee model can correctly estimate data values with a constant baseline and graded reference waveform if and only if the data values lie perfectly on a line, which means the data are noiseless data; Rowe's model correctly estimates the data values if multiple iterations are used in the estimation.

2.4 Item 4 Test Statistic and Critical Value

In [2], Lee stated: "Similarly to the Student's t-test $t = (\bar{x} - \mu)/(\hat{\sigma}/\sqrt{n})$, (where n is the number of samples, $\bar{x}$ is the sample mean, μ is the hypothetical population mean, and $\hat{\sigma}$ is the sample standard deviation), the Hotelling's T^2-test is defined as,

$$T^2 = (\bar{\boldsymbol{x}} - \boldsymbol{\mu})'(\widehat{\boldsymbol{S}}/\boldsymbol{n})^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})$$

where *n* is the number of the samples, $\bar{\boldsymbol{x}}$ is the sample mean vector, $\boldsymbol{\mu}$ is the hypothetical population mean vector, and $\widehat{\boldsymbol{S}}$ is the sample covariance matrix". When the sample has two variates (for example, real and imaginary), $\bar{\boldsymbol{x}}$ and $\boldsymbol{\mu}$ become 2×1 vectors, whereas $\widehat{\boldsymbol{S}}$ becomes a 2×2 matirx.

If all of the variates (in this case, both the real and imaginary parts of the data) are distributed according to the Gaussian (normal) distribution, the T^2-test will be proportional to an F-distribution with *m* and *n-m* degrees of freedom (where *m* is the number of variates in the sample (i.e., *m=2* in the complex data) and *n* is the number of samples). This T^2 statistic can be derived from a likelihood ratio test (a full derivation can be found in Ref. 5). For a given significance level α, the null hypothesis is rejected when

$$T^2 > \left[\frac{(n-1)m}{n-m}\right] F_{m,n-m}(\alpha).$$

Rowe Letter:

However, Rowe did not agree with this point, and he claimed that this test statistic and critical value equation is not mathematically correct. The likelihood ratio statistic λ when *m*=2 and *L*=2 can be rewritten as

$$F = \frac{(2n-4)}{2}\left(1 - \lambda^{-\frac{1}{n}}\right) = \frac{X_1/2}{X_2/(2n-4)} \quad (11)$$

where

$$X_1 = \frac{\left(\hat{\beta}_R - \tilde{\beta}_R\right)'(X'X)\left(\hat{\beta}_R - \tilde{\beta}_R\right)}{\sigma^2} + \frac{\left(\hat{\beta}_R - \tilde{\beta}_I\right)'(X'X)\left(\hat{\beta}_R - \tilde{\beta}_I\right)}{\sigma^2}$$

$$X_2 = (\mathrm{y_R} - \mathrm{X}\hat{\beta}_\mathrm{R})'(\mathrm{y_R} - \mathrm{X}\hat{\beta}_\mathrm{R}) + (\mathrm{y_I} - \mathrm{X}\tilde{\beta}_\mathrm{I})'(\mathrm{y_I} - \mathrm{X}\tilde{\beta}_\mathrm{I}).$$

And $X_1 \sim \chi^2(2)$, $X_2 \sim \chi^2(2\mathrm{n}-4)$, then the ratio which is given in Equation (11) is F distribution with 2 and 2n-4 degrees of freedom. The proper Lee (2007) test statistic and critical value that it should be compared to are

$$\mathrm{F} = \frac{(2n-4)}{2}T^2 > \mathrm{F}_{2,2n-4}(\alpha).$$

Lee Response:

The degrees of freedom and scaling factor (the value in front of the F-distribution) are correctly written in our article. A detailed derivation of theses parameters can be found in [11]. Dr. Rowe's conclusion in his point 4 originates from the assumption in his article that residual errors are independent. In our article, this was not assumed in the theory section. Only in Appendix B (page 916), where we mentioned "Assuming the case in which noise in the real and imaginary axes is independent, as assumed in Ref. 9 (which is Dr. Rowe's article)." Hence, the degrees of freedom should be *m, n-m*. In general, the residual errors could be correlated because of physiologic noise, because of hardware imperfection, and when phased-array coils are used. When the independence of the residual errors is assumed, the correct scaling factor is 2(n-1)/ (2n-4), which is still different from what Dr. Rowe has suggested in his comment.

Additional Comments

In order to illustrate Rowe's point of view, a detailed proof was supplied as following:

Specifying the case in which noise in the real and imaginary axes is independent, as assumed in [1], then $\mathrm{COV}(\hat{\varepsilon}_R, \hat{\varepsilon}_I) = \hat{\sigma}^2\mathbf{I}$ and $\hat{\sigma}^2$ can be defined as the mean of the real and imaginary variances:

$$2\hat{\sigma}^2 = \mathrm{var}(\hat{\varepsilon}_R) + \mathrm{var}(\hat{\varepsilon}_I)$$

$$= \frac{1}{n - L}\left(y_R - X\hat{\beta}_R\right)'\left(y_R - X\hat{\beta}_R\right) + \frac{1}{n - L}\left(y_I - X\hat{\beta}_I\right)'\left(y_I - X\hat{\beta}_I\right)$$

In the Appendix B of [2], it was mentioned that under the alternative hypothesis (H_a: $\mathbf{v}\beta_R \neq 0, \mathbf{v}\beta_I \neq 0$), the estimated log PDF of the signal is given by:

$$\mathrm{LLa} = \log P(y|H_a)$$

$$= -n\log\left(2\pi\sigma_{H_a}^2\right)$$

$$- \frac{1}{2\sigma_{H_a}^2}\left[\left(y_R - X\hat{\beta}_R\right)'\left(y_R - X\hat{\beta}_R\right) + \left(y_I - X\hat{\beta}_I\right)'\left(y_I - X\hat{\beta}_I\right)\right].$$

One can find the maximum likelihood estimates of the parameters by equating

$$\frac{\partial \mathrm{LLa}}{\partial \beta_R} = 0, \frac{\partial \mathrm{LLa}}{\partial \beta_I} = 0, \text{and } \frac{\partial \mathrm{LLa}}{\partial \sigma_{H_a}^2} = 0.$$

These equations lead to

$$\hat{\beta}_R = (X'X)^{-1}X'y_R$$

$$\hat{\beta}_I = (X'X)^{-1}X'y_I.$$

One can show as on Page 271 of [5] that

$$2(n - L)\hat{\sigma}^2 \sim \chi^2(2n - 2L) \tag{12}$$

Since

$$\mathrm{E}(y_R) = \mathrm{E}(X\beta_R + \varepsilon_R) = X\beta_R$$

$$\mathrm{E}(y_I) = \mathrm{E}(\mathrm{X}\beta_I + \varepsilon_\mathrm{I}) = \mathrm{X}\beta_I$$

$$\mathrm{var}(y_R) = \mathrm{var}(\mathrm{X}\beta_R + \varepsilon_\mathrm{R}) = \sigma^2\mathrm{I}$$

$$\mathrm{var}(y_I) = \mathrm{var}(\mathrm{X}\beta_I + \varepsilon_\mathrm{I}) = \sigma^2\mathrm{I}.$$

So,

$$\left(\hat{\beta}_\mathrm{R} - \tilde{\beta}_\mathrm{R}\right)'(\mathrm{X}'\mathrm{X})\left(\hat{\beta}_\mathrm{R} - \tilde{\beta}_\mathrm{R}\right)/\sigma^2 \sim \chi^2(1)$$

$$\frac{\left(\hat{\beta}_\mathrm{I} - \tilde{\beta}_\mathrm{I}\right)'(\mathrm{X}'\mathrm{X})\left(\hat{\beta}_\mathrm{I} - \tilde{\beta}_\mathrm{I}\right)}{\sigma^2} \sim \chi^2(1).$$

Thus,

$$X_1 = \frac{\left(\hat{\beta}_\mathrm{R} - \tilde{\beta}_\mathrm{R}\right)'(\mathrm{X}'\mathrm{X})\left(\hat{\beta}_\mathrm{R} - \tilde{\beta}_\mathrm{R}\right)}{\sigma^2} + \frac{\left(\hat{\beta}_\mathrm{I} - \tilde{\beta}_\mathrm{I}\right)'(\mathrm{X}'\mathrm{X})\left(\hat{\beta}_\mathrm{I} - \tilde{\beta}_\mathrm{I}\right)}{\sigma^2} \sim \chi^2(2). \qquad (13)$$

From Equations (12) and (13), we can conclude that

$$\mathrm{F} = \frac{X_1/2}{X_2/(2n-4)} \sim \mathrm{F}_{2,2n-4.}$$

Hence, the proper Lee model test statistic and critical value that it should be compared to are

$$\mathrm{F} = \frac{(2n-4)}{2} T^2 > \mathrm{F}_{2,2n-4}(\alpha).$$

Thus, one can make a conclusion that the test statistics and critical value that Lee provided in [2] are incorrect. The degrees of freedom should be changed to $2, 2(n-2)$ from $2, \mathrm{n}-2$ and the scaling factor should be changed to $2(\mathrm{n}-2)/2$ from $(\mathrm{n}-2)/2(\mathrm{n}-1)$ if $m = 2$. In general, the degrees of freedom is $\mathrm{m}, 2(\mathrm{n} -$

L) and the scaling factor is $2(n - L)/m$, where m is the number of contrasts, n is the number of observations and L is the number of the parameters. This conclusion can be derived from page 273 of [5].

3 CONCLUSION

After reviewing these papers on fMRI, and the deep discussion on [1, 2, 3, 4], one can make a final conclusion that there is no problem with the Rowe model as long as it is used the correct way to estimate the parameters and data values. Rowe introduced a general complex fMRI model that can describe both the magnitude and phase which is different from the previous models that can only describe the magnitude, but ignore the phase information which contains biological information regarding the vasculature contained within voxels [7, 8]. This model has already been widely used in the field of fMRI. Despite some inaccuracies, the Lee model is elegant and is recommended when the magnitude and phase design matrices are identical with a column of ones for a constant baseline and a column with on/off (0/1 or -1/+1) elements for the reference waveform vector or in ideal conditions without much fluctuations or noise

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This is to certify that we have examined this copy of the thesis by

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