A Full Bayesian Approach to SENSE Image Reconstruction Increases Brain Tissue Contrast and Reduces Noise for More Accurate Statistical Analysis

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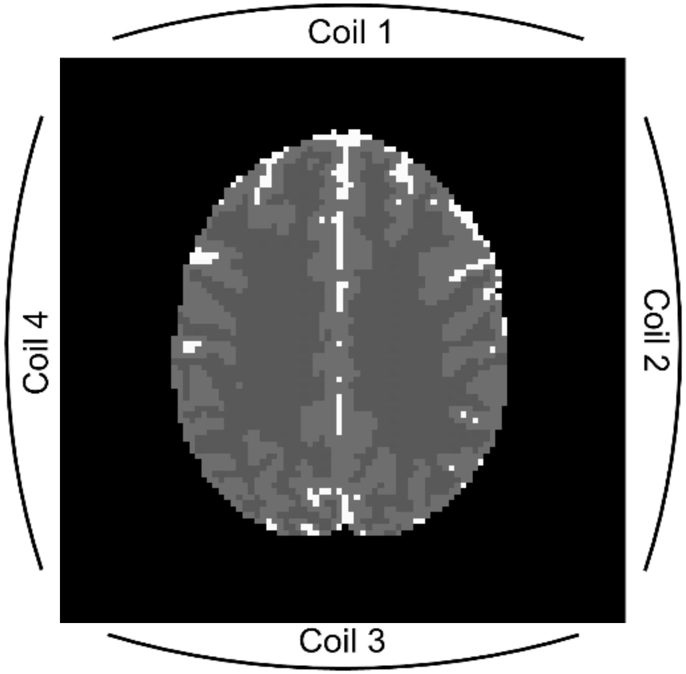
Outline

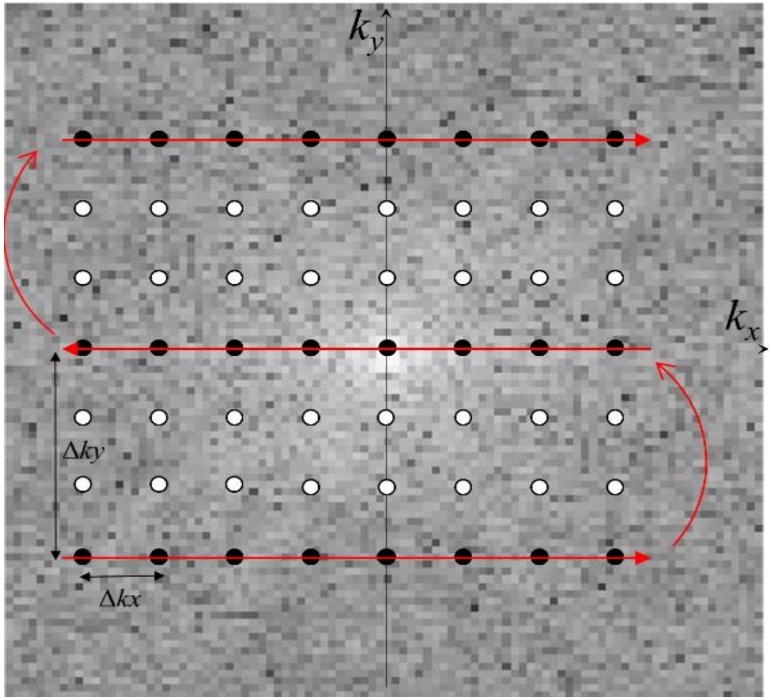
1. Sensitivity Encoding (SENSE)
2. Bayesian Approach to SENSE (BSENSE)
3. Simulated Results
4. Discussion

1. Sensitivity Encoding (SENSE)

Method – Four Coils with Acceleration Factor

- Introducing an acceleration factor $A = 3$
- Measures every third line, essentially cutting the full FOV image into 3 sections and aliasing (overlapping) them

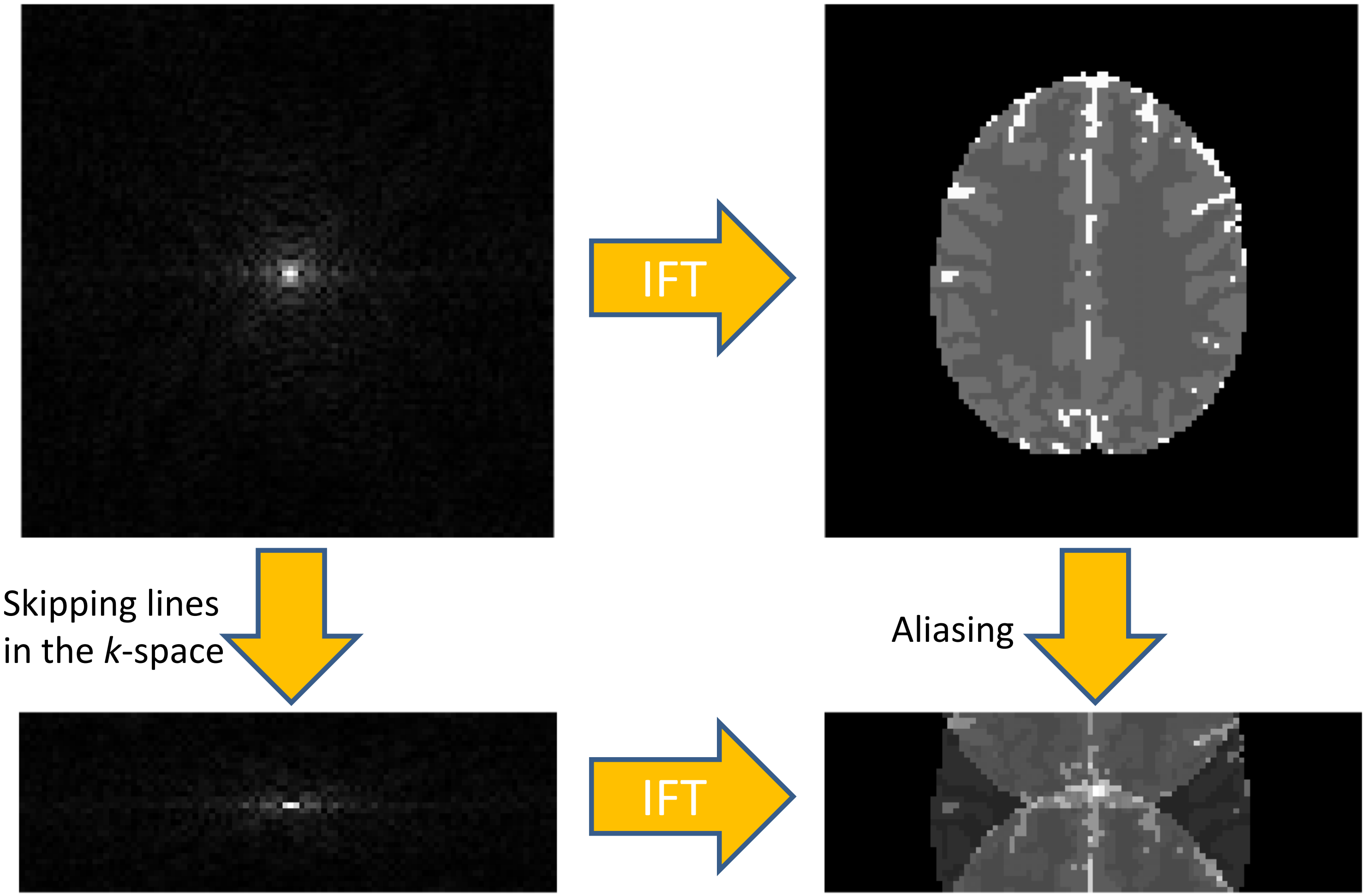




1. Sensitivity Encoding (SENSE)

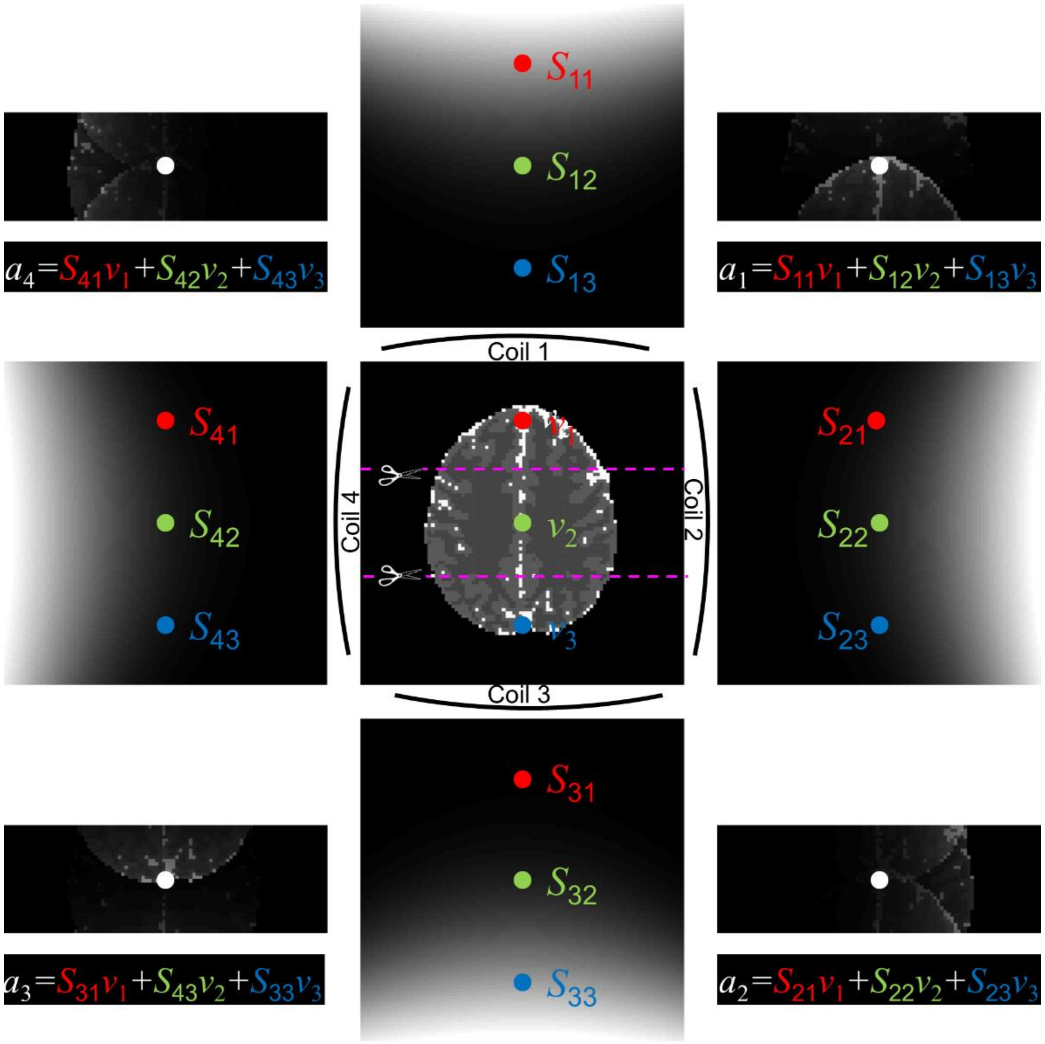
Acceleration Factor

$n_A = acceleration\ factor (3)$



1. Sensitivity Encoding (SENSE)

Method – Four Coils with Acceleration Factor (cont.)



**All Parameters are complex-valued

$$a = Sv$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

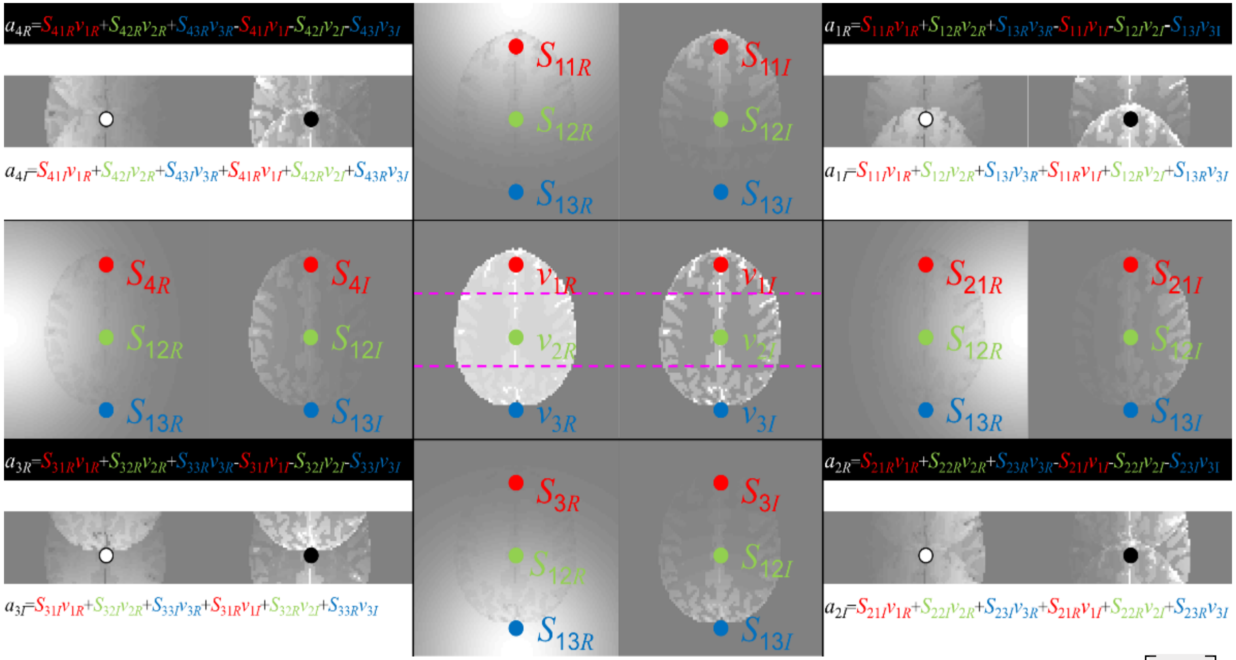
$$\hat{v} = \left(\hat{S}'\hat{S}\right)^{-1}\hat{S}'a$$

a observed aliased
v, S unobserved

**$\hat{S}'\hat{S}$ is not generally positive definite

1. Sensitivity Encoding (SENSE)

Complex-valued Nature



$$a = Sv \quad \Rightarrow \quad \begin{bmatrix} a_{1R}+ia_{1I} \\ a_{2R}+ia_{2I} \\ a_{3R}+ia_{3I} \\ a_{4R}+ia_{4I} \end{bmatrix} = \begin{bmatrix} S_{11R}+iS_{11I} & S_{12R}+iS_{12I} & S_{13R}+iS_{13I} \\ S_{21R}+iS_{21I} & S_{22R}+iS_{22I} & S_{23R}+iS_{23I} \\ S_{31R}+iS_{3I} & S_{32R}+iS_{32I} & S_{33R}+iS_{33I} \\ S_{41R}+iS_{41I} & S_{42R}+iS_{42I} & S_{43R}+iS_{43I} \end{bmatrix} \begin{bmatrix} v_{1R}+iv_{1I} \\ v_{2R}+iv_{2I} \\ v_{3R}+iv_{3I} \end{bmatrix}$$

$$\hat{v} = \left(\hat{S}'\hat{S}\right)^{-1}\hat{S}'a$$

** $\hat{S}'\hat{S}$ is not generally positive definite

Real-valued Isomorphism Representation:

$$\Rightarrow \quad \begin{bmatrix} a_{1R} \\ a_{2R} \\ a_{3R} \\ a_{4R} \\ a_{1I} \\ a_{2I} \\ a_{3I} \\ a_{4I} \end{bmatrix} = \begin{bmatrix} S_{11R} & S_{12R} & S_{13R} & -S_{11I} & -S_{12I} & -S_{13I} \\ S_{21R} & S_{22R} & S_{23R} & -S_{21I} & -S_{22I} & -S_{23I} \\ S_{31R} & S_{32R} & S_{33R} & -S_{31I} & -S_{32I} & -S_{33I} \\ S_{41R} & S_{42R} & S_{43R} & -S_{41I} & -S_{42I} & -S_{43I} \\ S_{11I} & S_{12I} & S_{13I} & S_{11R} & S_{12R} & S_{13R} \\ S_{21I} & S_{22I} & S_{23I} & S_{21R} & S_{22R} & S_{23R} \\ S_{31I} & S_{32I} & S_{33I} & S_{31R} & S_{32R} & S_{33R} \\ S_{41I} & S_{42I} & S_{43I} & S_{41R} & S_{42R} & S_{43R} \end{bmatrix} \begin{bmatrix} v_{1R} \\ v_{2R} \\ v_{3R} \\ v_{1I} \\ v_{2I} \\ v_{3I} \end{bmatrix}$$

2. Bayesian Approach to SENSE (BSENSE)

Introduction

- Instead of creating a “design” matrix from the a priori calibration images, will be used to form a joint conjugate multivariate normal-inverse gamma prior distribution for the unobserved coil sensitivities and the reconstructed images
- The prior is combined with the data likelihood to form a posterior distribution
- Technique for parameter (image) estimation:
    - Maximum A Posteriori (MAP) estimate using the Iterated Conditional Modes (ICM) algorithm
    - MCMC Gibbs sampler implemented with posterior conditional distributions to form a chain of reconstructed posterior conditional images

2. Bayesian Approach to SENSE (BSENSE)

$n_C = number\ of\ coils(4)$
$n_A = acceleration\ factor(3)$

Independent Model

- Aliased voxel measurements are observed with random error
  - $a = Sv + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 I_{n_c})$
- Data Likelihood
  - $P(a \mid S, v, \sigma^2) \propto (\sigma^2)^{-\frac{2n_C}{2}} \exp\left[-\frac{1}{2\sigma^2}(a - Sv)'(a - Sv)\right]$
- Priors:
  - $P(H \mid n_S, H_0, \sigma^2) \propto (\sigma^2)^{-\frac{2n_C n_A}{2}} \exp\left[-\frac{n_S}{2\sigma^2} tr[(H - H_0)'(H - H_0)]\right]$
  - $P(v \mid n_v, v_0, \sigma^2) \propto (\sigma^2)^{-\frac{2n_A}{2}} \exp\left[-\frac{n_v}{2\sigma^2}(v - v_0)'(v - v_0)\right]$
  - $P(\sigma^2 \mid \alpha, \beta) \propto (\sigma^2)^{-(\alpha+1)} \exp\left[-\frac{\beta}{\sigma^2}\right]$
- Assessed Hyperparameters: n_S, H_0, n_v,v_0, α, and β
- Posterior
  - $P(H, v, \sigma^2 \mid a) \propto P(a \mid S, v, \sigma^2) P(H \mid n_S, H_0, \sigma^2) P(v \mid n_v, v_0, \sigma^2) P(\sigma^2 \mid \alpha, \beta)$

$$\underset{2n_C \times 1}{a} = \begin{bmatrix} a_R \\ a_I \end{bmatrix}$$

$$\underset{2n_c \times 2n_A}{S} = \begin{bmatrix} S_R & -S_I \\ S_I & S_R \end{bmatrix}$$

$$\underset{n_c \times 2n_A}{H} = \begin{bmatrix} S_R & S_I \end{bmatrix}$$

$$\underset{2n_A \times 1}{v} = \begin{bmatrix} v_R \\ v_I \end{bmatrix}$$

2. Bayesian Approach to SENSE (BSENSE)

Independent Model - Parameter Estimation

$n_C = number\ of\ coils (4)$
$n_A = acceleration\ factor (3)$

- Maximum A Posteriori (MAP) estimate using the Iterated Conditional Modes (ICM) algorithm
  - $\hat{H} = (YC' + n_S H_0)(CC' + n_S I_{2n_A})^{-1}$
  - $\hat{v} = (S'S + n_v I_{2n_A})^{-1}(S'a + n_v v_0)$
  - $\hat{\sigma}^2 = \dfrac{(a - Sv)'(a - Sv) + n_v (v - v_0)'(v - v_0) + \alpha\beta + n_S tr[(H - H_0)(H - H_0)]}{2(2n_c + 2n_A + \alpha + 2n_c n_A + 1)}$
- Assessed Hyperparameters: n_S, H_0, n_v, v_0, α, and β

Given: $\hat{v}_{(0)}$ → Solve: $\hat{H}_{(1)}$

Given: $\hat{H}_{(1)}$ → Solve: $\hat{v}_{(1)}$

Given: $\hat{H}_{(1)}, \hat{v}_{(1)}$ → Solve: $\hat{\sigma}^2_{(1)}$

***Repeated *n* number of times**

$$\underset{n_c \times 2n_A}{\hat{H}} = \begin{bmatrix} \hat{S}_R & \hat{S}_I \end{bmatrix}$$

$$\underset{2n_A \times 1}{\hat{v}} = \begin{bmatrix} \hat{v}_R \\ \hat{v}_I \end{bmatrix}$$

$$\underset{n_c \times 2n_A}{H_0} = \begin{bmatrix} S_{0R} & S_{0I} \end{bmatrix}$$

$$\underset{2n_A \times 1}{v_0} = \begin{bmatrix} v_{0R} \\ v_{0I} \end{bmatrix}$$

2. Bayesian Approach to SENSE (BSENSE)

Independent Model - Parameter Estimation (cont.)

$n_C = number\ of\ coils (4)$
$n_A = acceleration\ factor (3)$

- MCMC Gibbs sampler implemented with posterior conditional distributions to form a chain of reconstructed posterior conditional images
  - $H \mid v, \sigma^2, a \sim N(\hat{H}, \Sigma_S = I_{n_c} \otimes \sigma^2 (CC' + n_S I_{2n_A})^{-1})$
    - $\hat{H} = (YC' + n_S H_0)(CC' + n_S I_{2n_A})^{-1}$
  - $v \mid S, \sigma^2, a \sim N(\hat{v}, \Sigma_v = \sigma^2 (S'S + n_v I_{2n_A})^{-1})$
    - $\hat{v} = (S'S + n_v I_{2n_A})^{-1}(S'a + n_v v_0)$
  - $\sigma^2 \mid v, S, a \sim IG(\alpha_*, \beta_*)$
    - $\alpha_* = n_C n_A + n_C + n_A + \alpha$
    - $\beta_* = \frac{1}{2}\left[(a - Sv)'(a - Sv) + n_v (v - v_0)'(v - v_0) + n_S tr(H - H_0)(H - H_0)' + 2\beta\right]$

$$\underset{2n_A \times 2}{C} = \begin{bmatrix} v_R & v_I \\ -v_I & v_R \end{bmatrix}$$

$$\underset{n_C \times 2}{Y} = \begin{bmatrix} a_R & a_I \end{bmatrix}$$

$$\underset{2n_c \times 2n_A}{S} = \begin{bmatrix} S_R & -S_I \\ S_I & S_R \end{bmatrix}$$

$$\underset{n_c \times 2n_A}{H} = \begin{bmatrix} S_R & S_I \end{bmatrix}$$

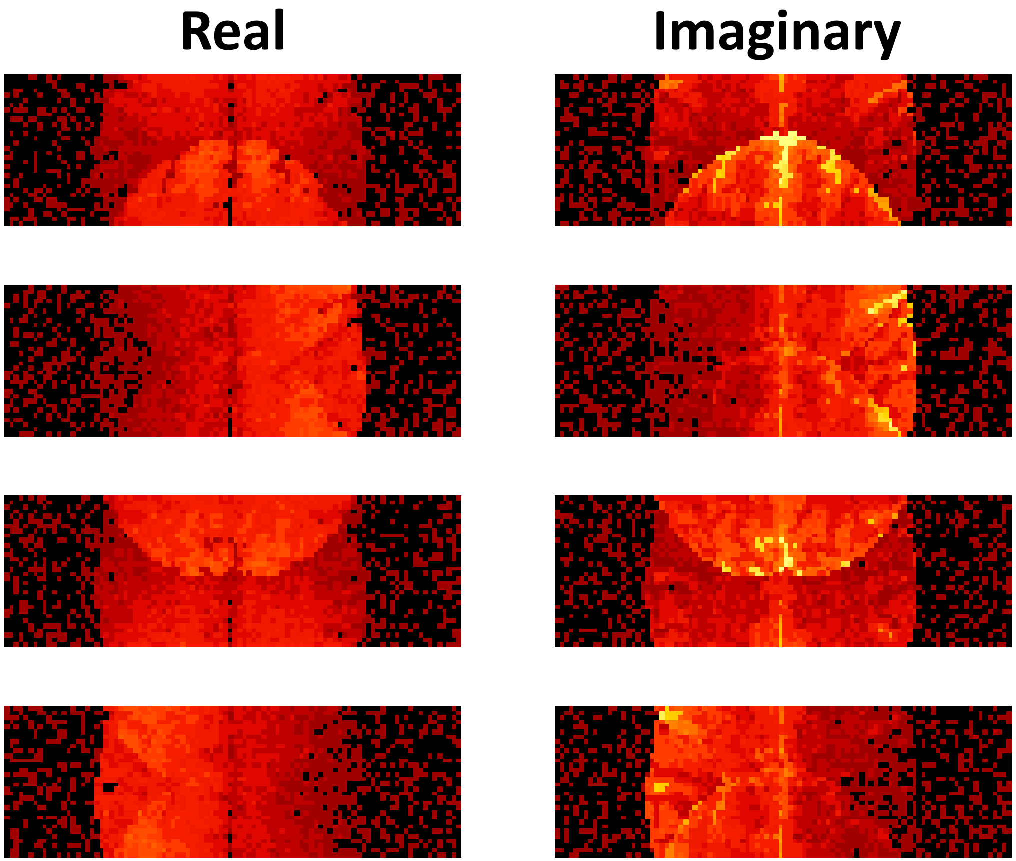
Given: $H_{(0)}, v_{(0)}$ → Generate: $\sigma^2_{(1)}$ → Given: $\sigma^2_{(1)}, H_{(0)}$ → Generate: $v_{(1)}$ → Given: $\sigma^2_{(1)}, v_{(1)}$ → Generate: $H_{(1)}$

***Repeated L number of times (remove burn)**

3. Simulated Results

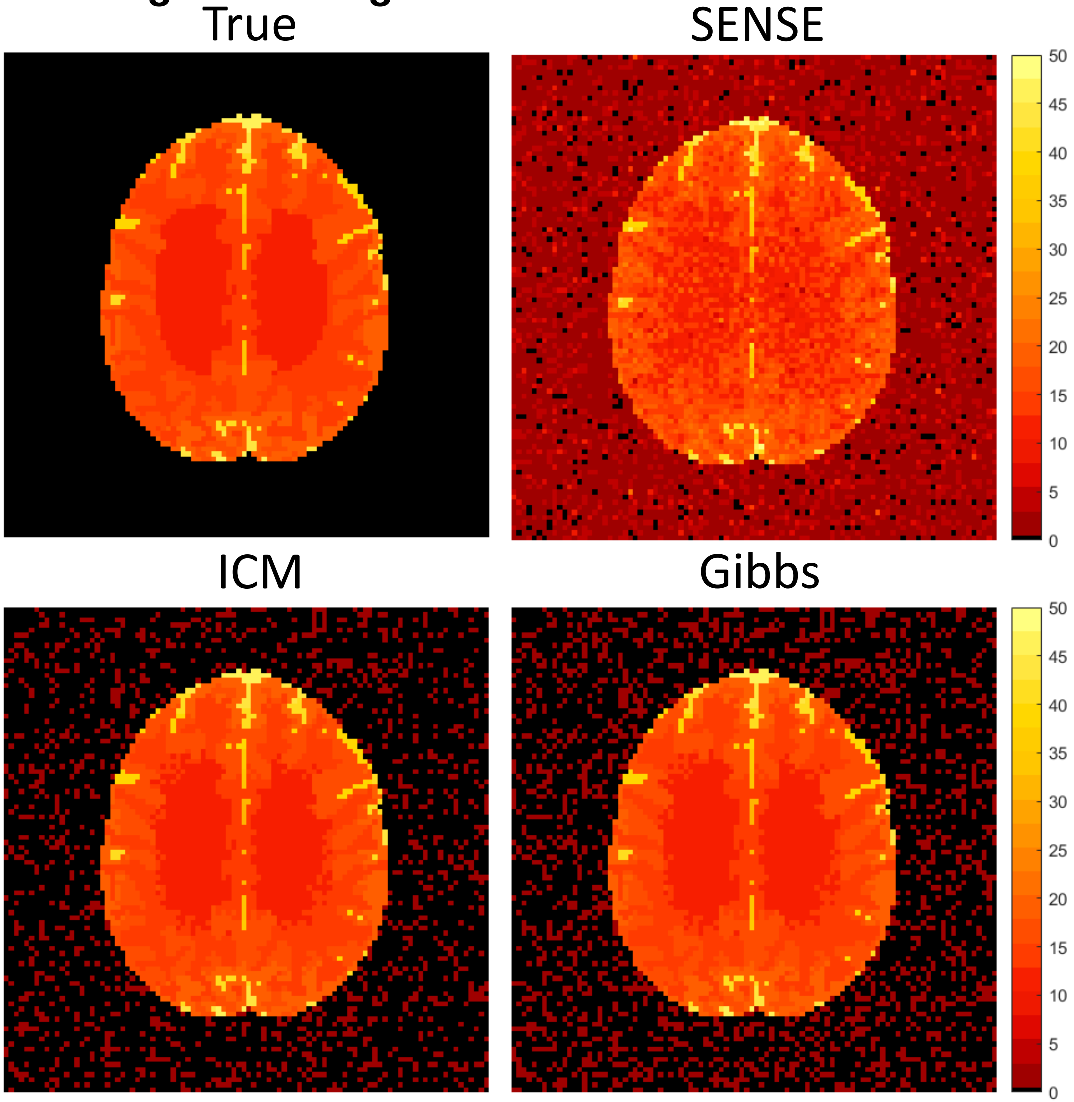
Aliased Coil Measurements

- Single Slice, One Timepoint



3. Simulated Results

Reconstructed Images for Single Slice



3. Simulated Results

Task Activation

- With non-task images, we are given a baseline estimate β_0 for the voxel values of the reconstructed images
  - $Y = \beta_0$, where Y is the magnitude of the estimated voxel value
- By adding task activation, we introduce a β_1 to this regression model giving us:
  - $Y = \beta_0 + X\beta_1$
  - Written as $Y = X\beta$, where X is the *n*×*2* design matrix
    - *n* = number of images in the series
    - First column is a column of ones, and the second column is a column of zeros and ones (zero for the non-task images in the series and one for task images in the series)
- Objective: to detect the task activation
  - Use a simple t-test to check for which voxels in the 96×96 reconstructed image have statistically significant β_1's

3. Simulated Results

Task Activation (cont.)

β_0 (non-task) $\beta_1 = 0.5$ Task

$$SNR = \frac{\rho}{\sigma} \qquad CNR = \frac{\beta_1}{\sigma}$$

3. Simulated Results

Task Activation (cont.)

- We have 500 images in the series
  - First 250 are non-task images, last 250 are task images
- For each voxel of the 96x96 reconstructed images, we have the regression model:
  - $Y = X\beta$, where Y is the 500 estimated magnitude voxel values, $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ are the parameters to estimate for statistical significance, and X is the design matrix as shown to the right
- To find β, we use the equation $\beta = (X'X)^{-1}X'Y$
  - This is repeated for every voxel in the 96x96 image
- The following hypothesis is run:
  - $H_0\colon\ \beta_1 \leq 0, H_a\colon\ \beta_1 > 0$
  - This is a simple one-tailed t-test with $\alpha = 0.05$ and n equal to the number of images in the series (500 for this example)

$$\frac{\beta_1 - 0}{SE(\beta_1)}$$

$$X = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{250 zeros} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{250 ones} \end{matrix}$$

3. Simulated Results

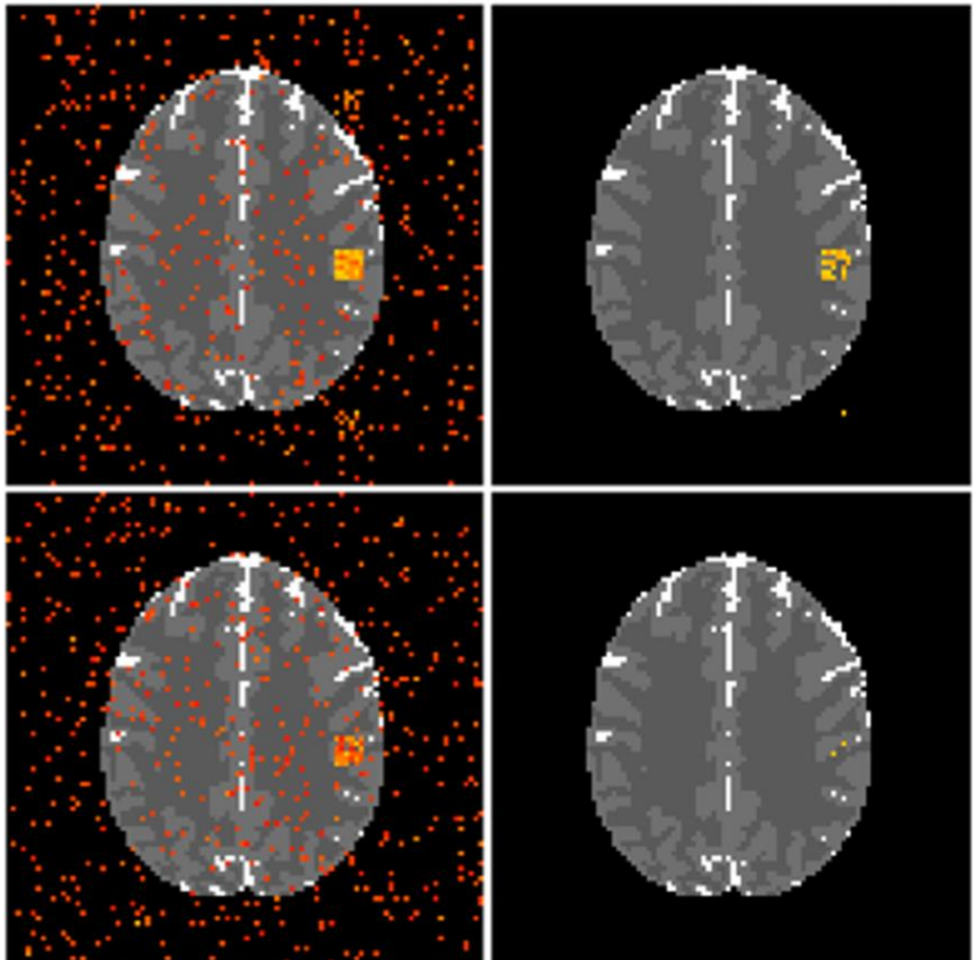
Threshold = 0.05

Detecting Task Activation

PCE | BH FDR

BSENSE MAP

SENSE



$crit.value = 1.645$ | $crit.value = 3.6568$

4. Discussion

Overview

- Advantages of BSENSE over SENSE
  - Utilizes more available prior information from the calibration images
  - Clearer reconstructed image
  - Better performance when detecting task
  - Allows more flexibility (able to access entire posterior or just a single point)
- Future work
  - Apply this method to experimental data
  - Introduce the correlated model
    - With this independent model, we work under the assumption of no coil covariance or aliased voxel covariance

Thank You!

Questions?