

# Thresholding Complex Magnetic Resonance Images Using Magnitude and Phase

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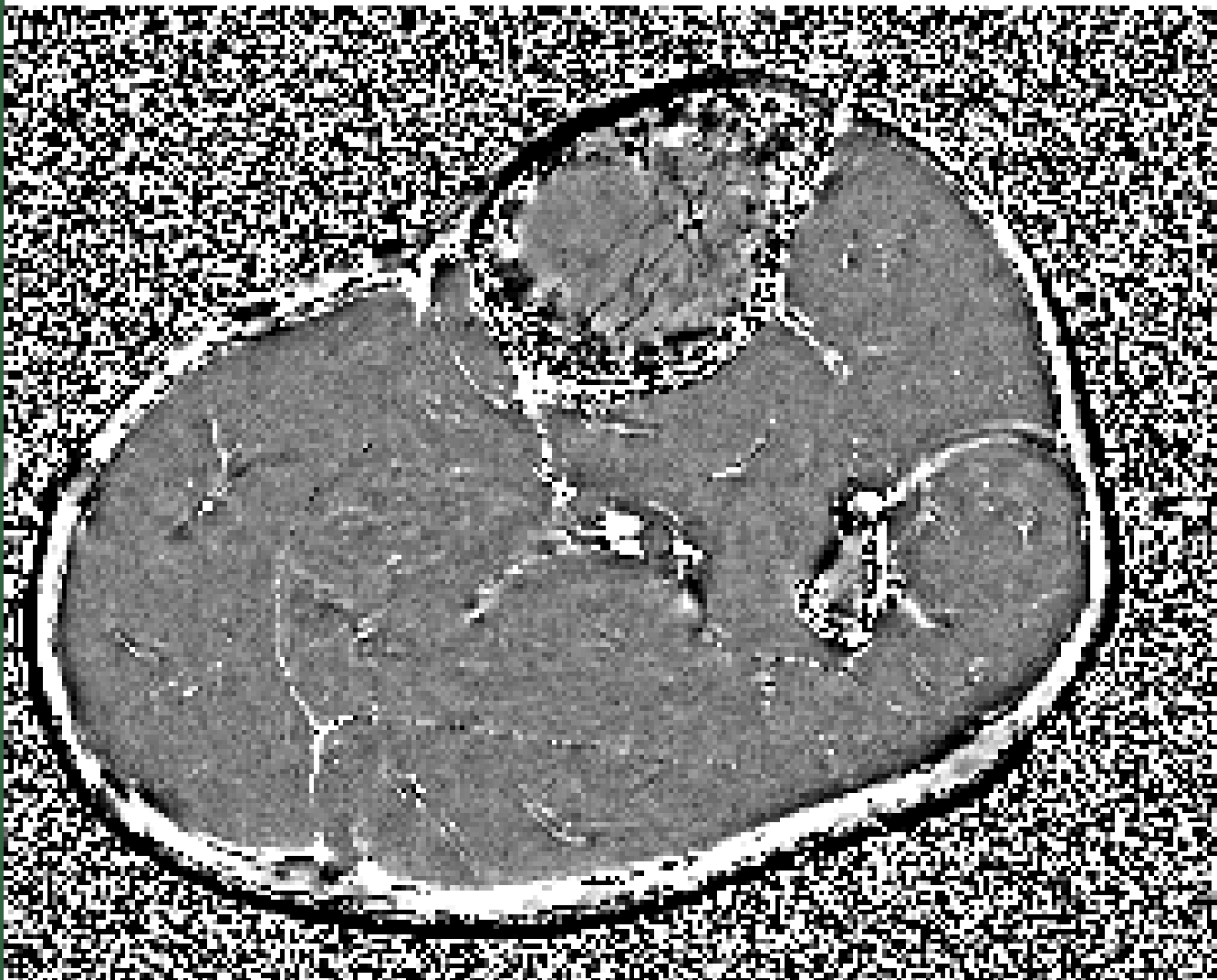
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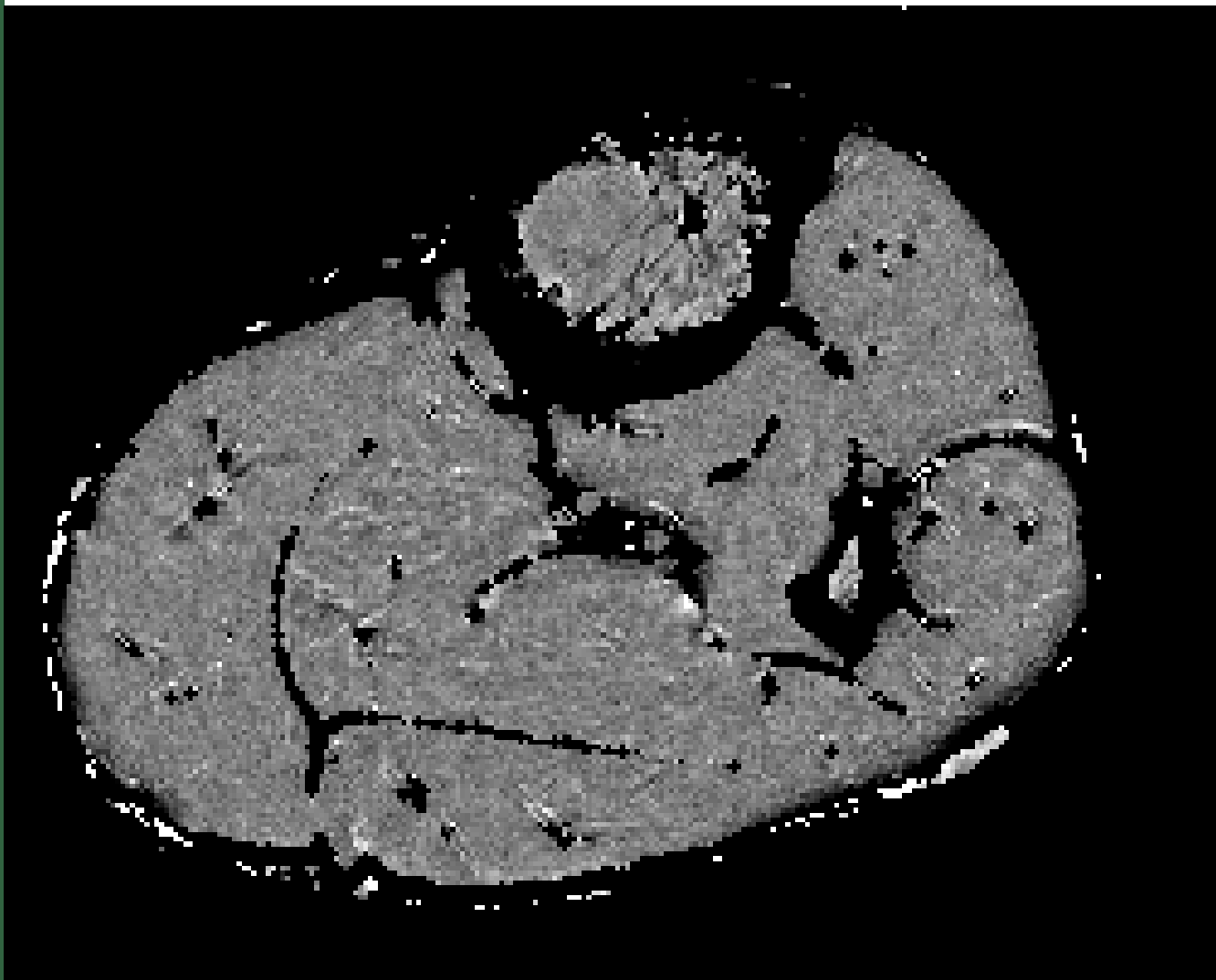
## Motivation

Want to increase image contrast by thresholding noise voxels.



## Motivation

To get an image like this one.



## Background

In MRI, measurements and images are complex-valued.

Thermal noise manifests as IID  $N(0, \sigma^2)$  noise in the real and imaginary parts of the k-space measurements (Henkelman, 1985; Bernstein, 1989; Macovski, 1996).

A linear relationship exists between complex-valued k-space measurement and complex-valued voxel measurements (Rowe, Nencka, & Hoffmann, 2007).

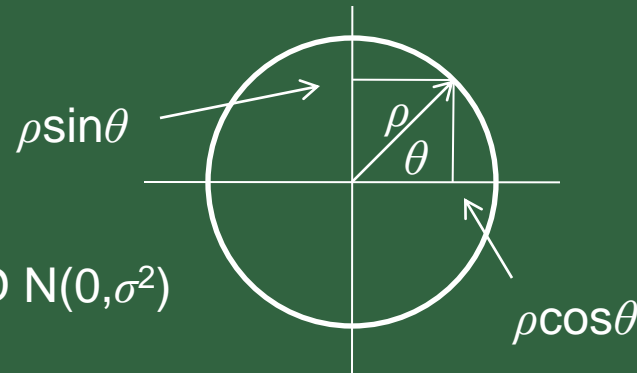
From the above, the voxel measurements are also normally distributed (Henkelman, 1984; Bernstein, 1989).

Voxel measurements can be described as

$$y_R = \rho \cos \theta + \varepsilon_R$$

$$y_I = \rho \sin \theta + \varepsilon_I$$

$\varepsilon_R$  and  $\varepsilon_I$  IID  $N(0, \sigma^2)$



[1]

$y_R$  and  $y_I$  are measurements for the real and imaginary parts

$\varepsilon_R$  and  $\varepsilon_I$  are error terms for the real and imaginary parts

$\rho$  and  $\theta$  are the population magnitude and phase.

Objective is to separate voxels that are pure noise from those that contain signal and noise using both magnitude and phase (Pandya, Ciulla, & Haacke, 2008).

PDF of the voxel's real and imaginary observation ( $y_R, y_I$ ) is

$$p(y_R, y_I | \rho, \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_R - \rho \cos \theta)}{2\sigma^2}\right] \times \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_I - \rho \sin \theta)}{2\sigma^2}\right] \quad [2]$$

PDF of the voxel's magnitude and phase observation ( $m, \varphi$ ) is

$$p(m, \varphi | \rho, \theta, \sigma^2) = \frac{m}{2\pi\sigma^2} \exp\left[-\frac{Q}{2\sigma^2}\right]$$

$$Q = \left[ m^2 + \rho^2 - 2\rho m \cos(\varphi - \theta) \right] \quad [3]$$

We would like to determine if the observed magnitude and phase in a voxel are signal or if they are noise.

## Methods

Voxel measurements  $(m_1, \varphi_1), \dots, (m_n, \varphi_n)$  yield the likelihood

$$L(\rho, \theta, \sigma^2) = (2\pi\sigma^2)^{-n} \left[ \prod_{i=1}^n m_i \right] \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n Q_i \right\} \quad [4]$$

$$Q_i = \left[ m_i^2 + \rho^2 - 2\rho m_i \cos(\varphi_i - \theta) \right]$$

A formal statistic can be derived and a statistical hypothesis test performed on the population magnitude and phase.

$H_0: \rho=0, \theta=0, \sigma^2>0$

vs.

$H_1: \rho>0, \theta \neq 0, \sigma^2>0$

	$H_0$ True	$H_0$ False
Reject $H_0$	Type I Error ( $\alpha$ )	Correct Decision ( $1 - \alpha$ )
Do Not Reject $H_0$	Correct Decision ( $1 - \beta$ )	Type II Error ( $\beta$ )

Table 1: Four outcomes from a hypothesis test.



Under constrained  $H_0: \rho=0, \theta=0, \sigma^2>0$  the MLEs are

$$\tilde{\rho} = 0$$

$$\tilde{\theta} = 0$$

$$\tilde{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^n (y_{Ri}^2 + y_{Ii}^2) \quad [5]$$

Under unconstrained  $H_1: \rho>0, \theta \neq 0, \sigma^2>0$  the MLEs are

$$\hat{\rho} = \left[ (\bar{y}_R)^2 + (\bar{y}_I)^2 \right]^{1/2}$$

$$\hat{\theta} = \tan^{-1} \left[ \frac{\sum_{i=1}^n y_{Ii}}{\sum_{i=1}^n y_{Ri}} \right]$$

$$\hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^n (y_{Ri}^2 + y_{Ii}^2) - \frac{1}{2} \hat{\rho}^2 \quad [6]$$

$\bar{y}_R$  and  $\bar{y}_I$  are the means of the reals and imaginaries.

Insert estimates back into the likelihoods and take the ratio.

$$\lambda = \frac{L(\tilde{\rho}, \tilde{\theta}, \tilde{\sigma}^2)}{L(\hat{\rho}, \hat{\theta}, \hat{\sigma}^2)} \quad [7]$$

$\lambda$  is asymptotically  $\chi^2$  distributed with  $df=2$  in this case.

Algebra can be performed,  $F = 1 - \lambda^{1/n}$ , to arrive at

$$F = \left( \frac{n \left[ (\bar{y}_R)^2 + (\bar{y}_I)^2 \right] / \sigma^2}{2} \right) / \left( \frac{\left[ \sum_{i=1}^n y_{Ri}^2 + \sum_{i=1}^n y_{Ii}^2 \right] / \sigma^2}{2n} \right) \quad [8]$$

The probability distribution of this statistic needs to be found.

The steps through the logic for the derivation of the distribution for the numerator and the denominator of the  $F$  statistic are shown below under  $H_0$ .

Numerator  $\chi^2$  Term

$$y_{Ri} \sim N(0, \sigma^2)$$

$$y_{Ri} / \sigma \sim N(0, 1)$$

$$\sum_{i=1}^n y_{Ri} / \sigma \sim N(0, n)$$

$$\sqrt{n} \bar{y}_R / \sigma \sim N(0, 1)$$

$$\left( \sqrt{n} \bar{y}_R / \sigma \right)^2 \sim \chi^2(1)$$

$$x_1 = \frac{n \left[ (\bar{y}_R)^2 + (\bar{y}_I)^2 \right]}{\sigma^2} \sim \chi^2(2)$$

Denominator  $\chi^2$  Term

$$y_{Ri} \sim N(0, \sigma^2)$$

$$y_{Ri} / \sigma \sim N(0, 1)$$

$$\left( y_{Ri} / \sigma \right)^2 \sim \chi^2(1)$$

$$\sum_{i=1}^n \left( y_{Ri} / \sigma \right)^2 \sim \chi^2(n)$$

$$x_2 = \frac{\left[ \sum_{i=1}^n y_{Ri}^2 + \sum_{i=1}^n y_{Ii}^2 \right]}{\sigma^2} \sim \chi^2(2n)$$

Since the numerator  $x_1 \sim \chi^2(2)$  and denominator  $x_2 \sim \chi^2(2n)$

$$F = \frac{x_1/2}{x_2/2n}$$

should be F distributed under  $H_0$  with 2 and  $2n$  df!

However, this is not true in this case!

These two  $\chi^2$  statistics must be independent for this to be true.

The correlation between these two statistics can be derived.

Since these are  $\chi^2$  distributed:  $E(x_1)=2$ ,  $E(x_2)=2n$ ,  
 $var(x_1)=4$ ,  $var(x_2)=4n$ ,  $cov(x_1, x_2)=E(x_1 \cdot x_2)-E(x_1)E(x_2)$

$$\begin{aligned}
 E(x_1 \cdot x_2) &= E \left\{ \frac{1}{n\sigma^2} \left[ \left( \sum_{i=1}^n y_{Ri} \right)^2 + \left( \sum_{i=1}^n y_{Li} \right)^2 \right] \cdot \frac{1}{\sigma^2} \left[ \sum_{i=1}^n y_{Ri}^2 + \sum_{i=1}^n y_{Li}^2 \right] \right\} \\
 &= \frac{1}{n\sigma^2} E \left\{ \left( \sum_{i=1}^n y_{Ri} \right)^2 \sum_{i=1}^n y_{Ri}^2 + \left( \sum_{i=1}^n y_{Ri} \right)^2 \sum_{i=1}^n y_{Li}^2 \right. \\
 &\quad \left. + \left( \sum_{i=1}^n y_{Li} \right)^2 \sum_{i=1}^n y_{Ri}^2 + \left( \sum_{i=1}^n y_{Li} \right)^2 \sum_{i=1}^n y_{Li}^2 \right\} \\
 &= \frac{1}{n\sigma^2} E \left\{ \left( \sum_{k=1}^n y_{Rk} \sum_{j=1}^n y_{Rj} \right) \sum_{i=1}^n y_{Ri}^2 + \left( \sum_{k=1}^n y_{Rk} \sum_{j=1}^n y_{Rj} \right) \sum_{i=1}^n y_{Li}^2 \right. \\
 &\quad \left. + \left( \sum_{k=1}^n y_{Lk} \sum_{j=1}^n y_{Lj} \right) \sum_{i=1}^n y_{Ri}^2 + \left( \sum_{k=1}^n y_{Lk} \sum_{j=1}^n y_{Lj} \right) \sum_{i=1}^n y_{Li}^2 \right\} \\
 &= \{ (n+2) + n + n + (n+2) \} = 4n + 4.
 \end{aligned}$$

[9]

The correlation between  $x_1$  and  $x_2$  is  $1/\sqrt{n}$ .

This correlation tends to zero in large samples and the  $F$  statistic becomes  $F$  distributed.  $F$  critical values can be used.

However, critical values for small  $n$  can be achieved by way of Monte Carlo simulation.

For a given level of significance (Type I error rate  $\alpha$ ), we

reject  $H_0$  (do not threshold voxel) if the test statistic  $F$  is larger than the critical value  $F_{\alpha}(2,2n)$  and

do not reject (threshold voxel) if  $F$  is smaller than the critical value  $F_{\alpha}(2,2n)$ .

To examine the convergence,  $10^6$  simulated data sets created under  $H_0$  ( $\rho=0$  and  $\theta=0$ ) for  $n=5, 10, 25, 50, 100,$  and  $250$ .

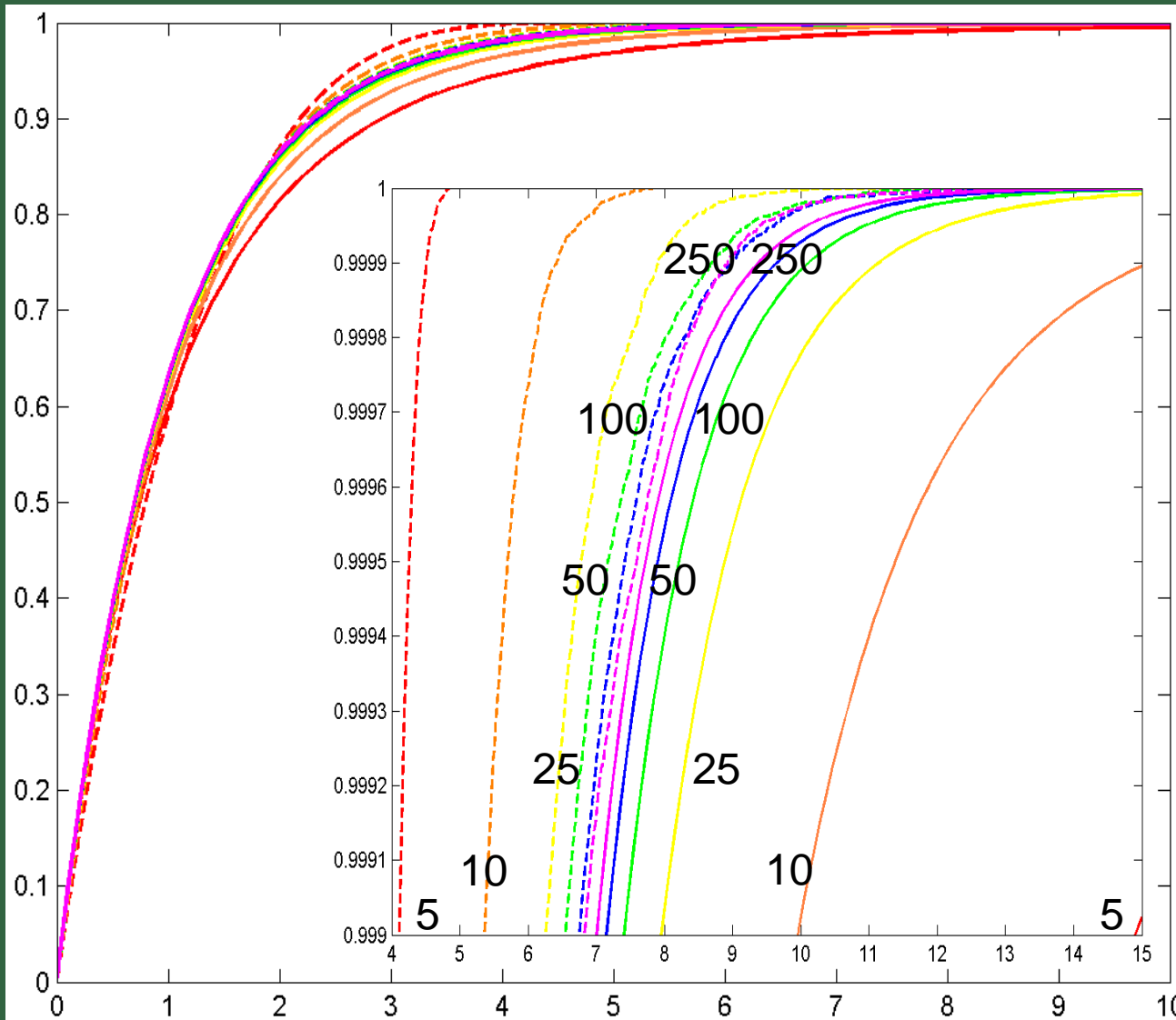
IID  $N(0, \sigma^2=1)$  variates generated for reals and imaginaries.

Regardless of the sample size  $n$ ,  $x_1$  and  $x_2$  are  $\chi^2$  distributed.

The correlation between the numerator and denominator  $\chi^2$  statistics is  $1/\sqrt{n}$ .

For each sample size  $n$ , the  $F$  statistic was computed for each of the data sets.

CDFs are made for each sample size and compared to  $F$  CDF.



Note disparity between the MC and the asymptotic F CDF.

It takes a large  $n$  for disparity to decrease.

Therefore for small  $n$ , MC critical values need to be used.

**Figure 1:** CDF from Monte Carlo simulation (dashed) and F distribution (solid) for  $n = 5$  (red),  $10$  (orange),  $25$  (yellow),  $50$  (green),  $100$  (blue),  $250$  (violet).



Will be using the test statistic denoted by  $F$  for  $n=5$ .

The Type I error rate was examined in Figure 1 for  $n=5$ .

To examine the Type II error rate, generated  $10^6$  data sets for  $n=5$  under  $H_1$  ( $\rho \neq 0$  and  $\theta \neq 0$ ) with  $\rho=(0, 1, 2, 3, 5)$  &  $\theta=0^\circ$ .

IID  $N(0, \sigma^2=1)$  variates generated for reals and imaginaries.

Histograms for the  $10^6$  data sets when  $\rho=0$  and  $\rho=2$ .

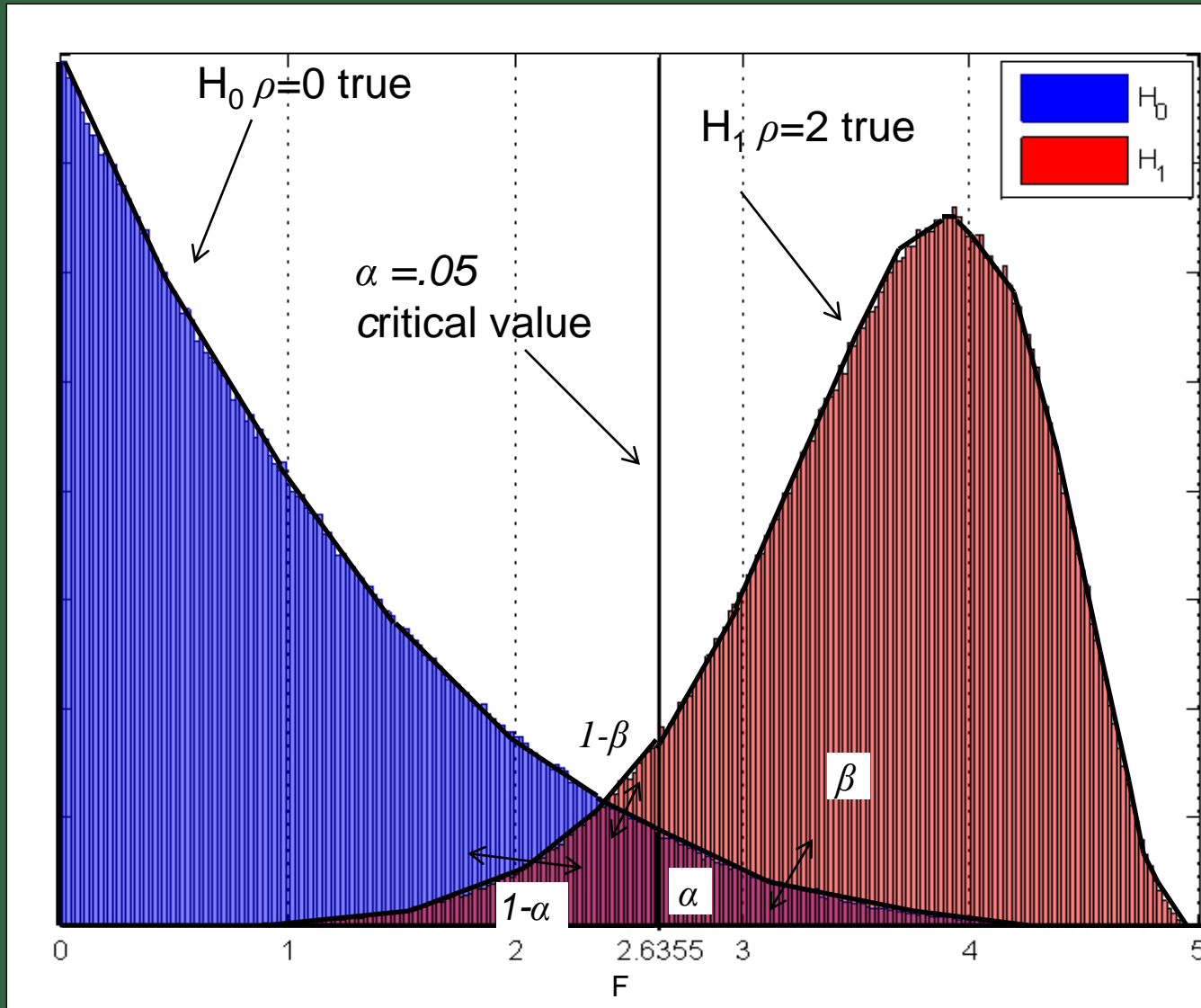
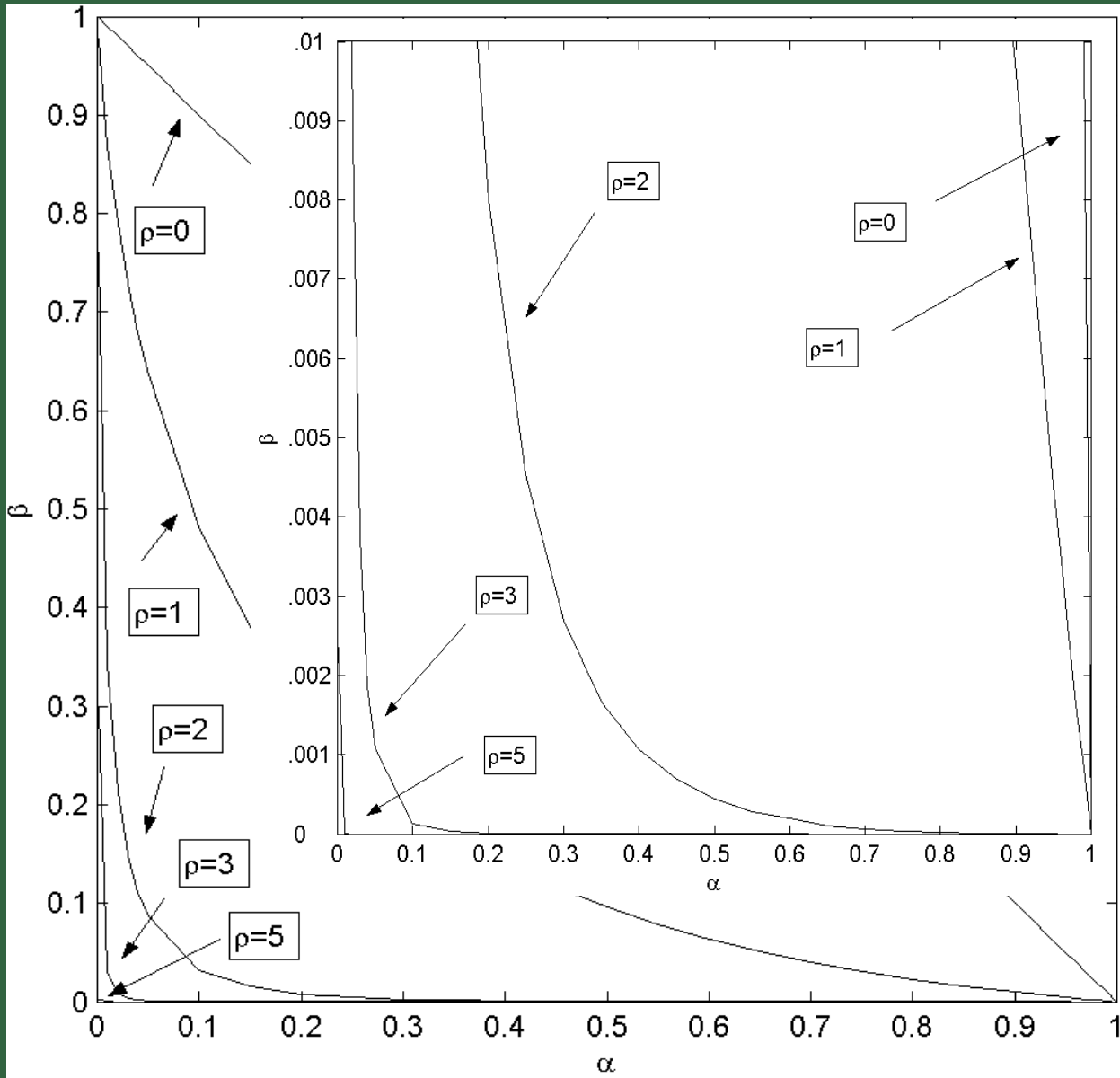


Figure 2: Histogram when  $H_0$  is true and when  $H_1$  is true.



Slide vertical line L-R.

Find  $\beta$  for given  $\alpha$ .

Plot of  $\alpha$  vs.  $\beta$  called ROC curve (Haacke et al., 1999).

Curve for each  $(\rho, \theta)$  combination.

Figure 3: ROC curves from Monte Carlo simulation for  $\rho=(0,1,2,3,5)$  and  $\theta=0^\circ$ .

For more accurate CVs in the upper tail of  $F$  statistic  $5 \times 10^7$  data sets were generated under  $H_0$ ,  $F$  statistic computed for each set and a histogram of these  $F$  statistics made.

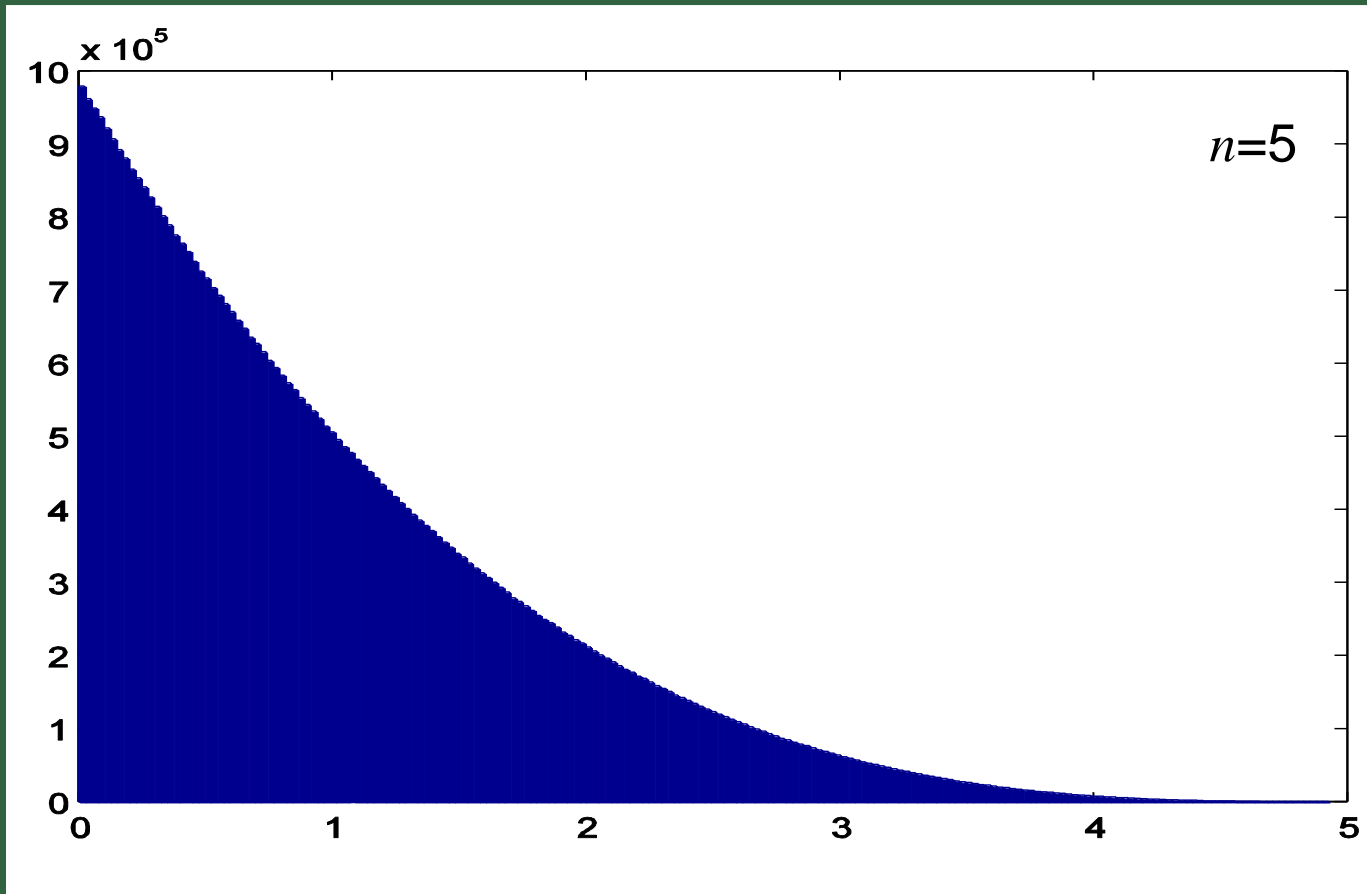


Figure 4: Histogram of  $F$  statistic for  $5 \times 10^7$  data sets under  $H_0$  for critical values.

$F$  statistics were ordered and percentiles determined.

For example, the  $.95 \times (5 \times 10^7)^{\text{th}}$  largest value is 95<sup>th</sup> percentile.

$n=5$									
$\alpha$	.05	.01	.01	.001	.0001	.00001	.05/256/256	.05/512/384	.05/512/512
$F_\alpha$	2.6355	3.4189	4.1104	4.4992	4.7162	4.8445	4.8617	4.8858	4.8874
$n=9$									
$\alpha$	.05	.01	.001	.0001	.00001	.00001	.05/256/256	.05/512/384	.05/512/512
$F_\alpha$	2.8102	3.9377	5.1991	6.1512	6.8678	7.3911	7.5627	7.7051	7.7575

**Table 2:** Critical  $F$  statistic values for  $n=5$  and  $n=9$ .

Critical values will be used for thresholding the magnitude and phase of voxels.

Additional critical values can reliably be interpolated.

We would like to have  $n$  repeated images.

Have high-resolution anatomical images where replicates are rarely available.

Use observed  $r$  and  $\varphi$  values in each voxel with 4 neighbors ( $n=5$ ) to estimate each voxel's  $\rho$  and  $\theta$  then compute  $F$  statistic.

Minor local  $F$  statistics correlation does not affect global image threshold as previous reports suggest (Logan & Rowe, 2004).

$F$  statistic map is thresholded with the critical values in Table 2.

A 0/1 mask from thresholded  $F$  map applied to  $r$  and  $\varphi$  images.

## Results

Data:

Susceptibility weighted imaging (SWI) (Haacke et al., 2004)  
MRI data is used to test the noise removal procedure in  
magnitude and phase.

SWI leg data was collected on a 3T Siemens Trio:  
in-plane resolution of 512 x 384 (0.5 x 0.5 mm<sup>2</sup>),  
TR/TE= 20/10 msec,  
flip angle (FA) = 15°,  
FOV = 256 mm x 192 mm  
(Haacke et al., 1999).

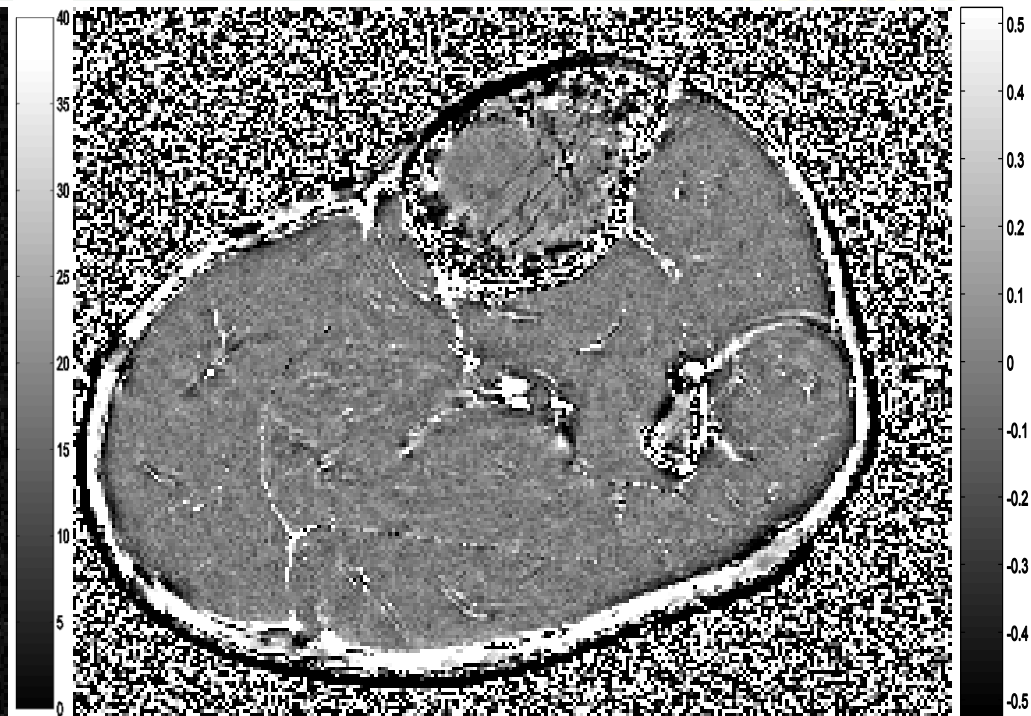


Figure 5a: Observed Magnitudes

Figure 5b: Observed Phases

Images were cropped to show vertical voxels 160 to 360 of 512  
and horizontal voxels 100 to 350 of 384



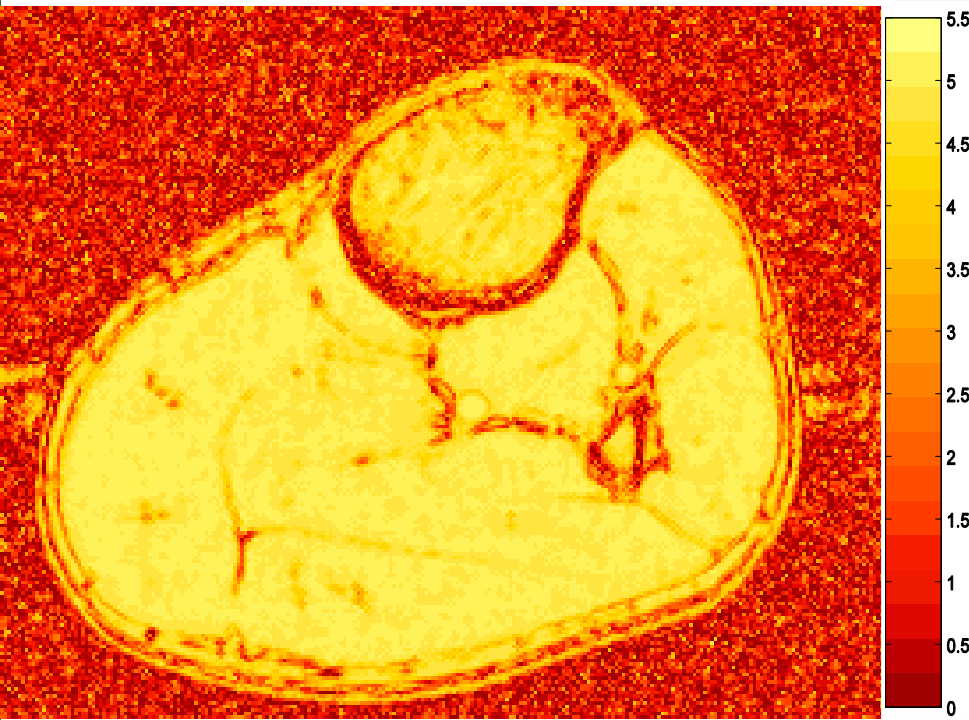
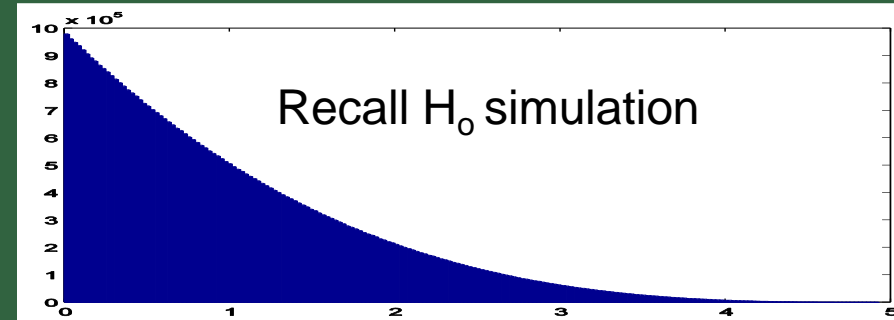
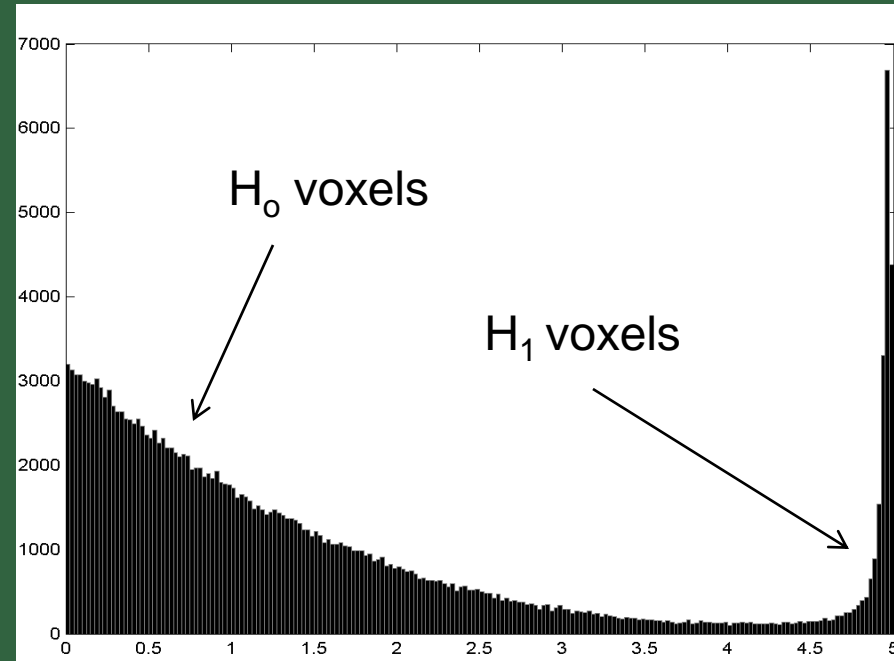


Figure 5c: Computed  $F$ 's

Figure 5d: Histogram of Computed  $F$ 's



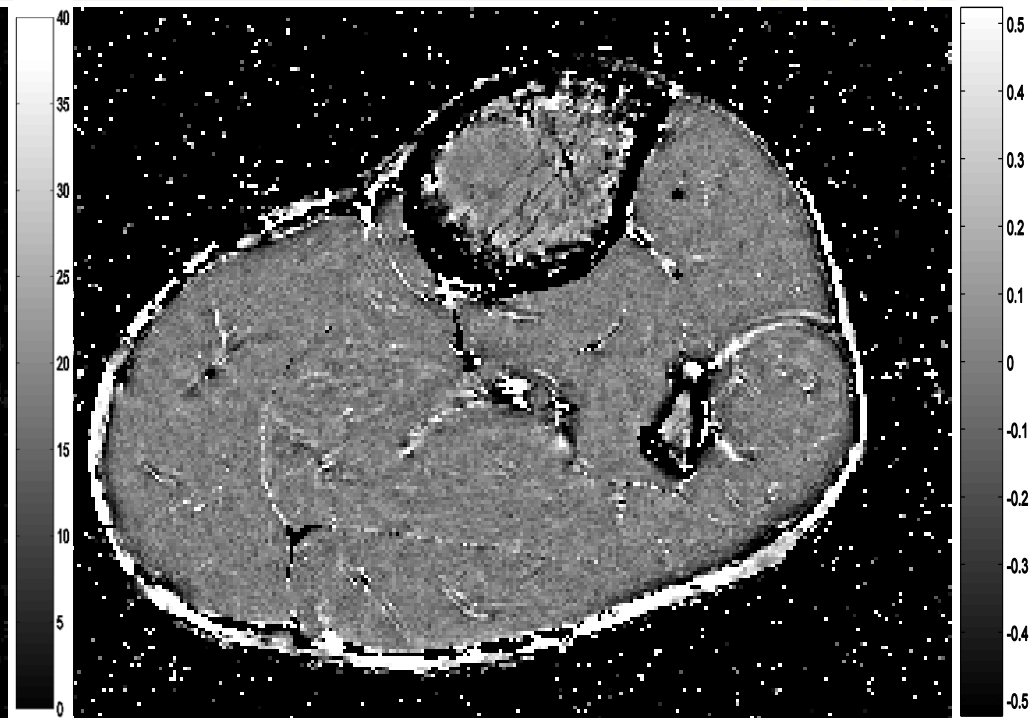
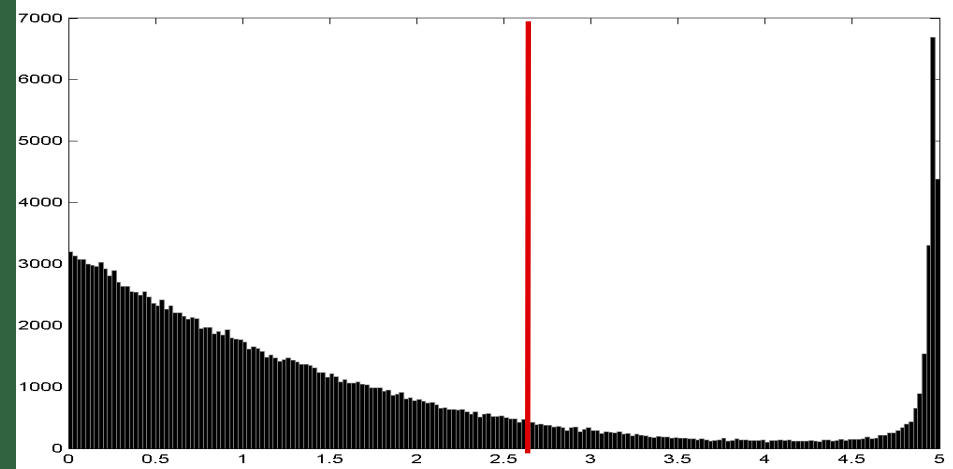


Figure 6a: Thresholded Observed Magnitudes.

Figure 6b: Thresholded Observed Phases.

$$\alpha=0.05 \text{ or } F_{\text{crit}} = 2.6355$$



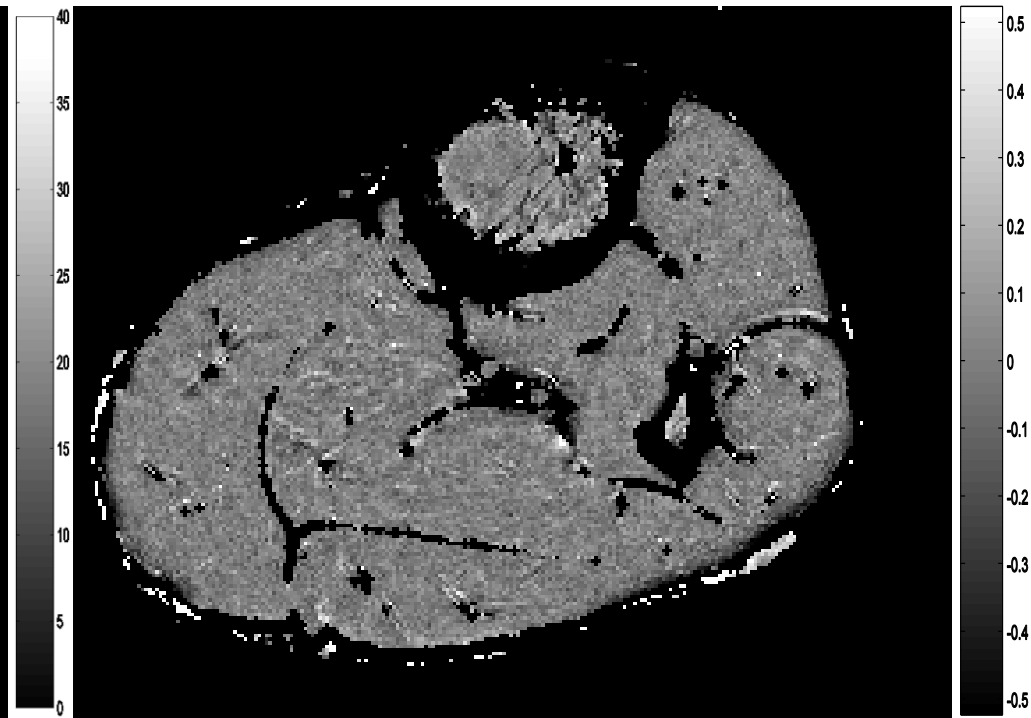
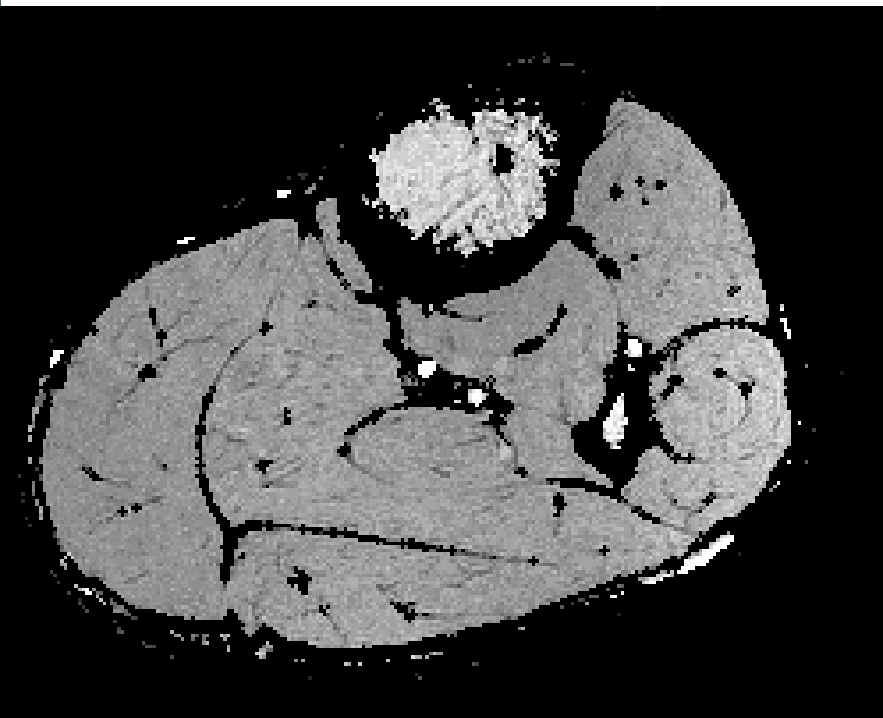
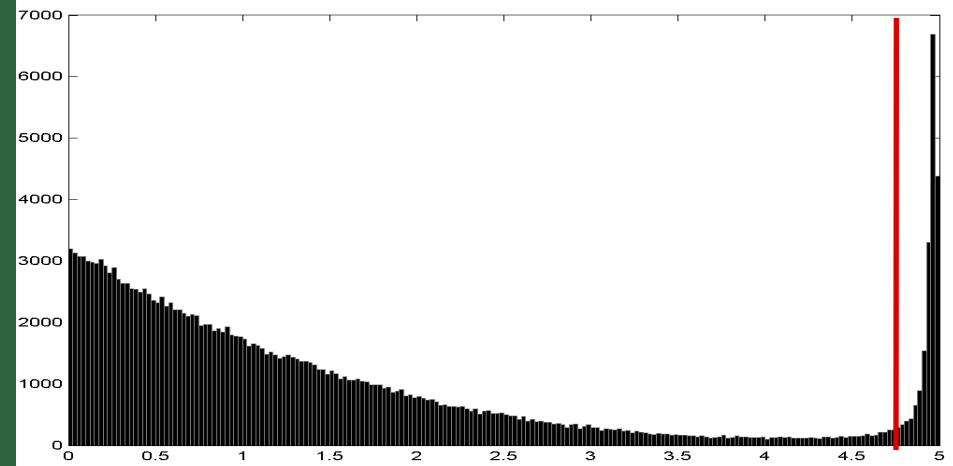


Figure 7a: Thresholded Observed Magnitudes.

Figure 7b: Thresholded Observed Phases.

$$\alpha=0.0001 \text{ or } F_{\text{crit}} = 4.7162$$



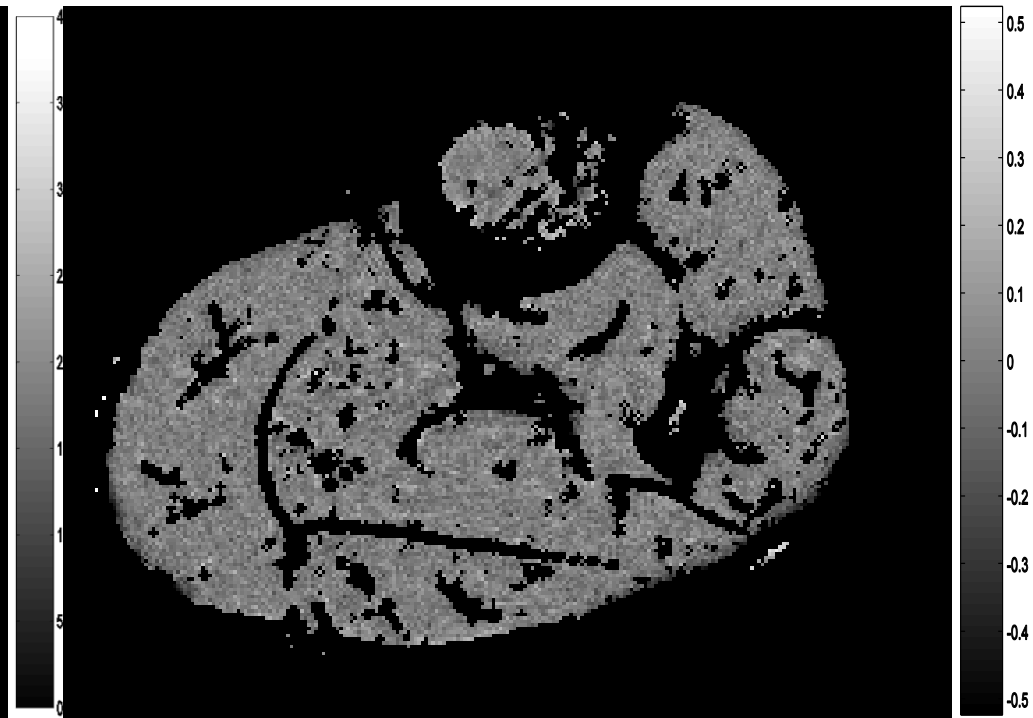
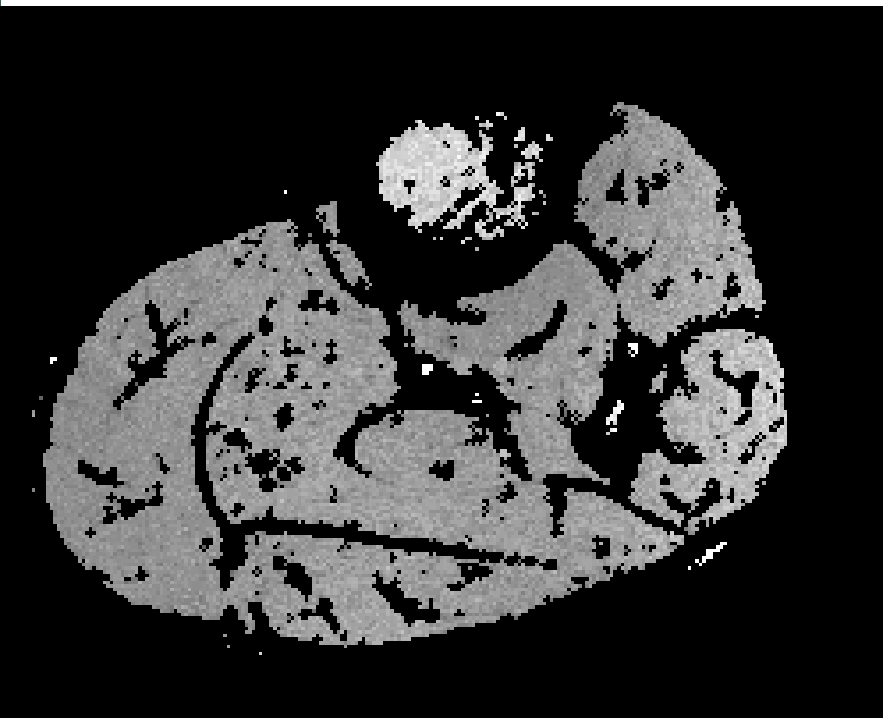
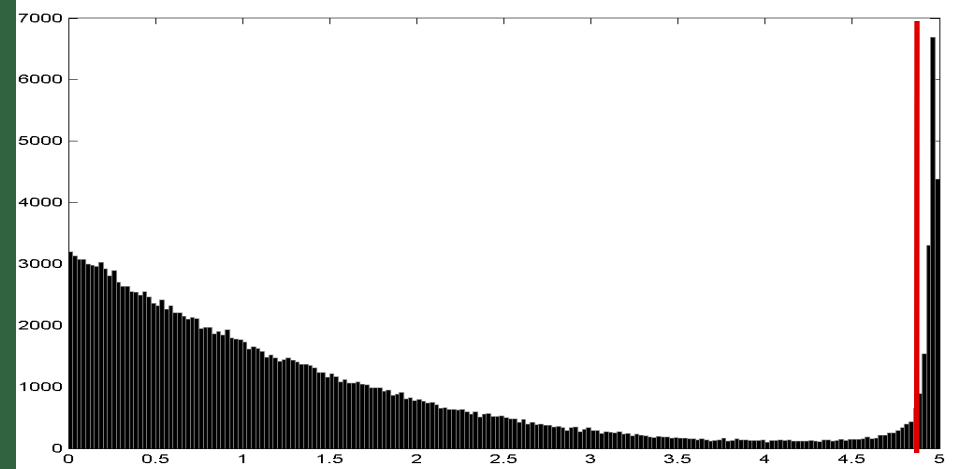


Figure 8a: Thresholded Observed Magnitudes.

Figure 8b: Thresholded Observed Phases.

$$\alpha = .05/512/384 \text{ or } F_{\text{crit}} = 4.8858$$



## Discussion

A magnitude and phase statistical thresholding procedure based upon a likelihood ratio test was presented.

It was shown through Monte Carlo simulation that that this method operates according to its theoretical statistical properties in terms of both false positives and false negatives.

This thresholding method was successfully applied to real human SWI data and shown to increase image contrast.

**Thank You**

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