Signal and Noise in Complex-Valued SENSE MR Image Reconstruction

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OUTLINE

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Motivation

In MRI it is not voxel values that are measured.

The actual measurements are spatial frequencies (k-space).

The k-space measurements are not acquired instantaneously.

In parallel imaging, k-space is subsampled and measured in parallel then combined to form a single image.

Image and volume measurement time is decreased at the expense of increased image reconstruction difficulty and time.

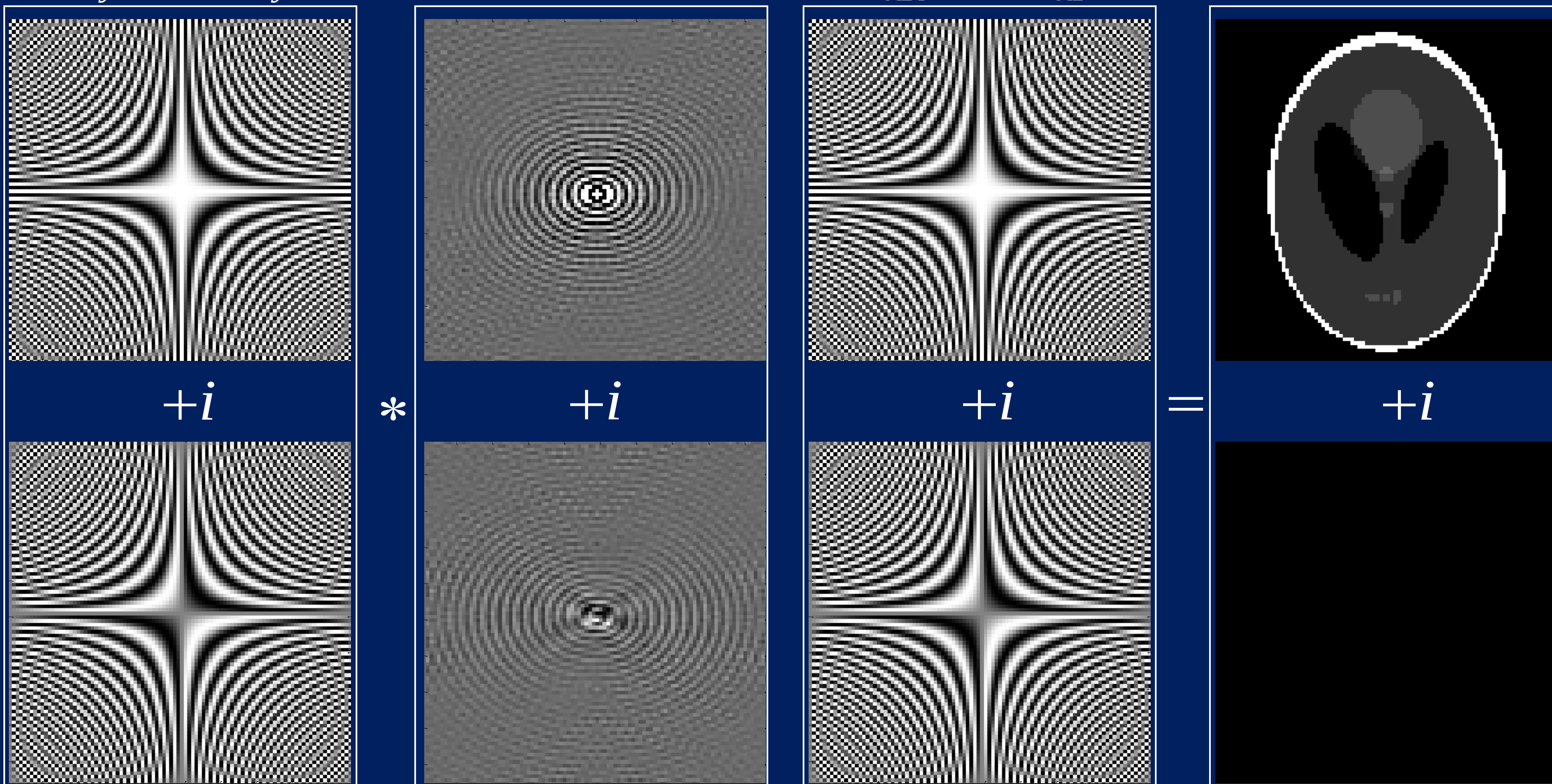
One popular parallel imaging method is SENSE.

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.

Background

Image inverse Fourier Reconstruction for single coil.

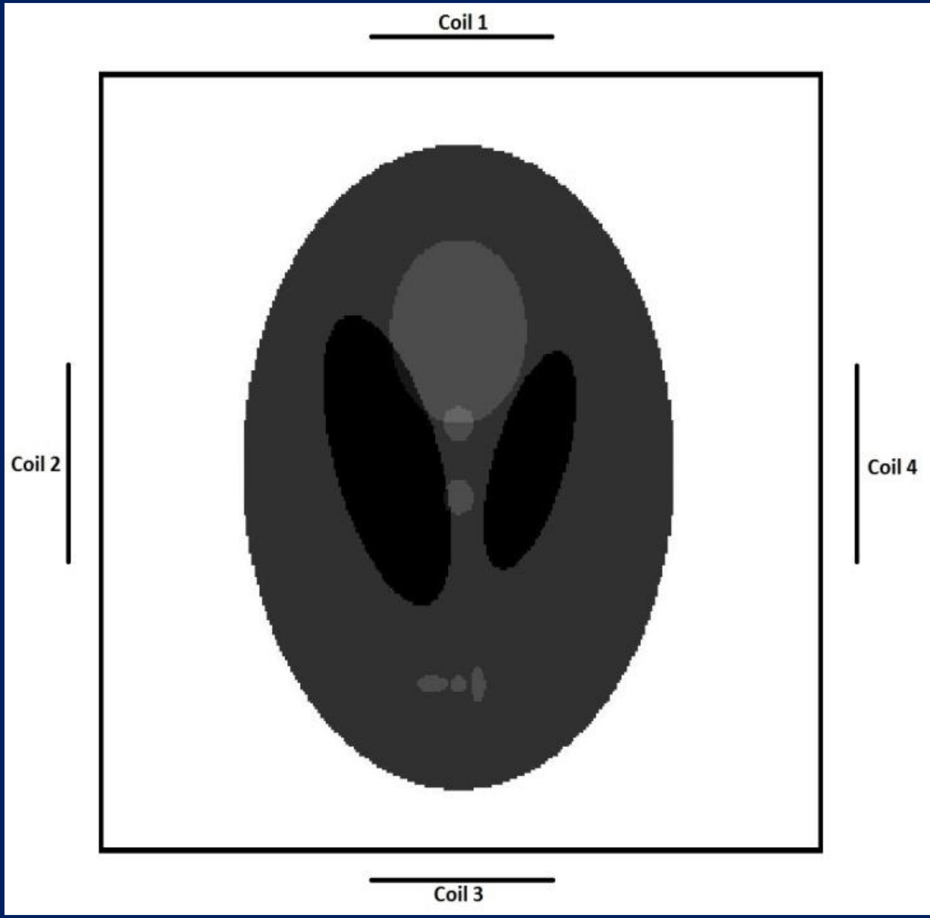
$$(\Omega_{yR} + i\Omega_{yI}) \ast (F_R + iF_I) \ast (\Omega_{xR} + i\Omega_{xI})^T = (V_R + iV_I)$$



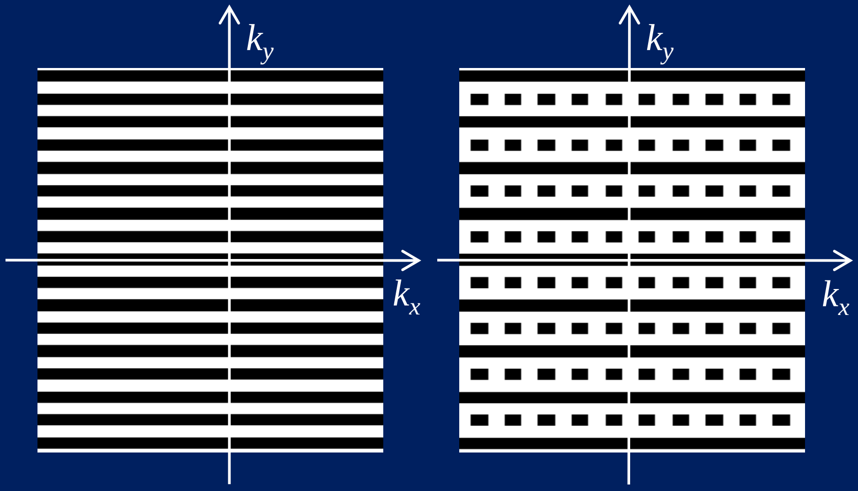
Rowe, Nencka, Hoffmann, Signal and noise of Fourier reconstructed fMRI data. JNSM 159:361-369, 2007.

Background

In parallel imaging there is more than one receive coil.



Each coil measures a k-space array where every A^{th} line is skipped.

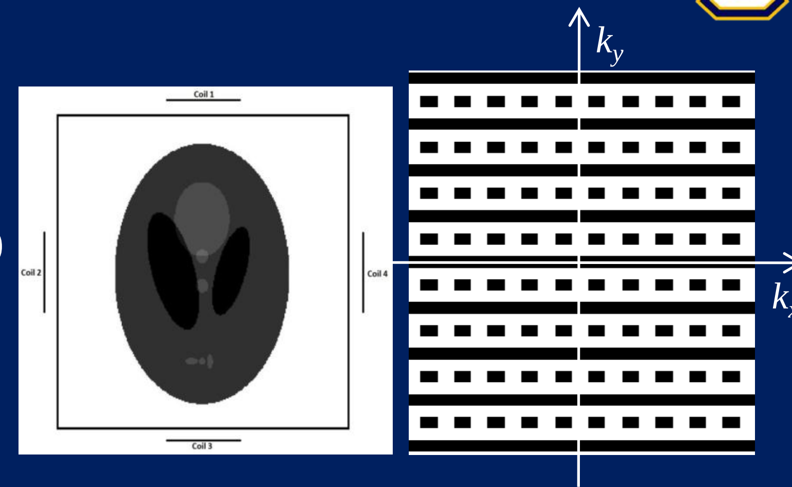


Full k-space.   Skipped k-space.

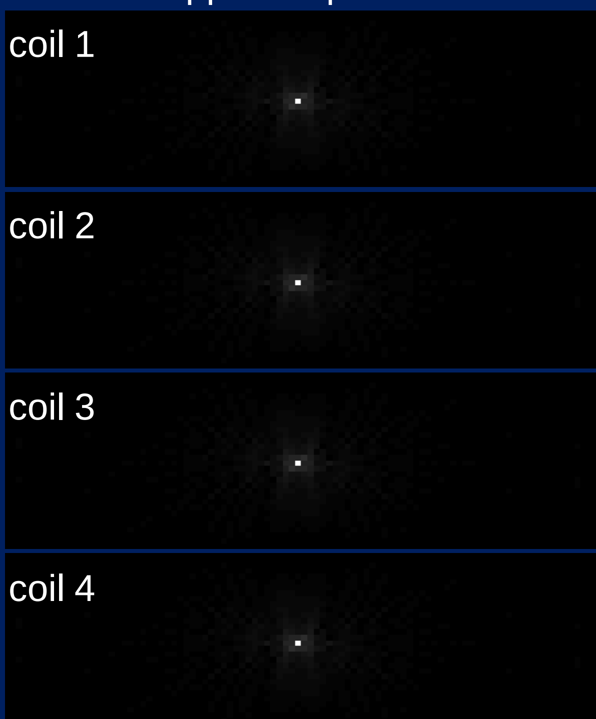
Bruce, Karaman, and Rowe: In Submission, 2011.

Background

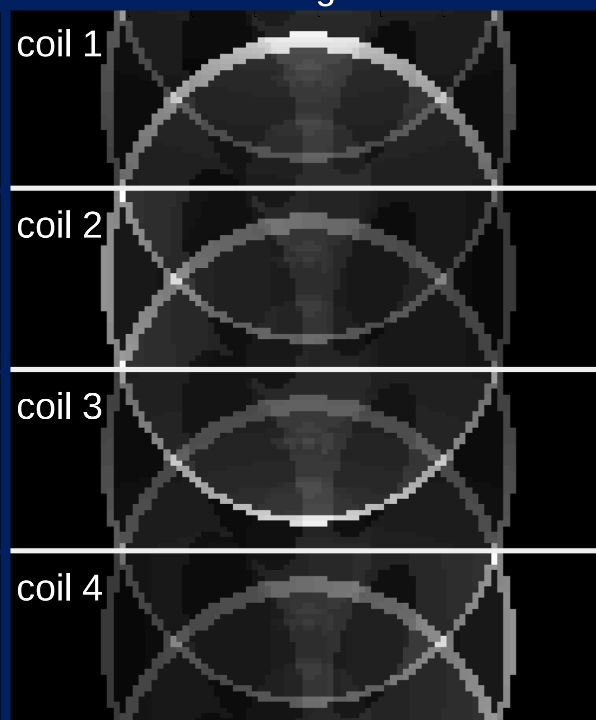
The k-space arrays where every A^{th} line is skipped are reconstructed into an aliased image to be combined to form a single image.



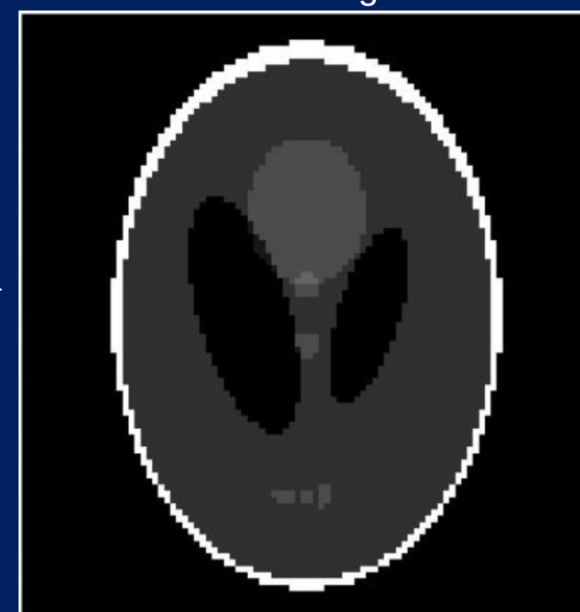
Skipped k-space.



Aliased images.



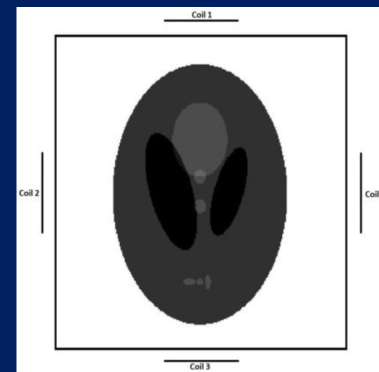
Combined image.

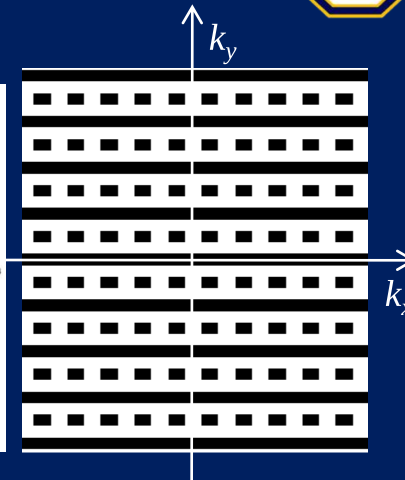


Bruce, Karaman, and Rowe: In Submission, 2011.

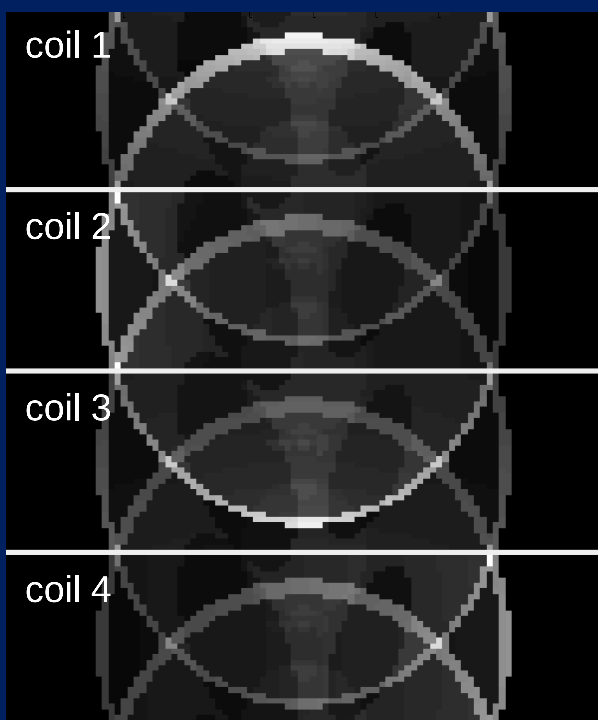
Background

The combination of aliased images to form a single image utilizes coil sensitivities.

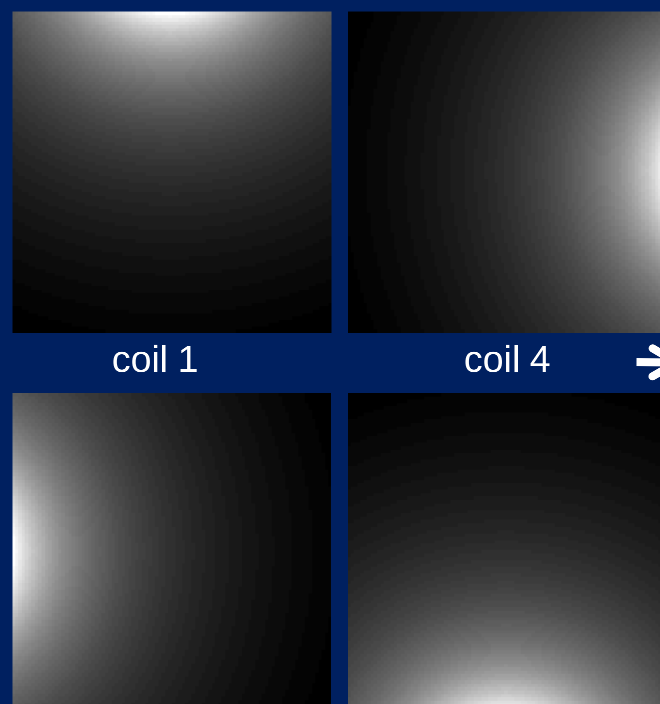




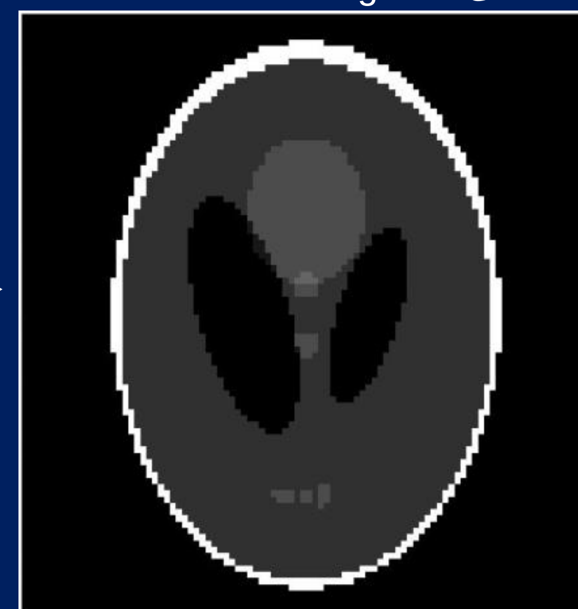
Aliased images. a_C



Coil sensitivities. S_C



Combined image. v_C



Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

The SENSE model for aliased voxel values from n coils is

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \; \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C} \quad , \quad \varepsilon_C \sim CN(0,\Psi_C)$$

$$\Psi_C = \Psi_R + i\Psi_I$$

where for each voxel

a_C is a vector of the n complex-valued aliased voxel values

$$a_C = a_R + i a_I$$

v_C is a vector of the A unaliased voxel value

$$v_C = v_R + i v_I$$

S_C is an nxA matrix of complex-valued coil sensitivities

$$S_C = S_R + i S_I$$

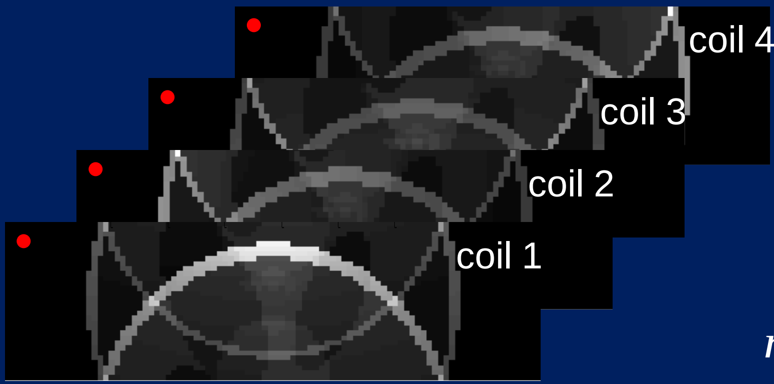
ε_C is a vector of the n complex-valued error values

$$\varepsilon_C = \varepsilon_R + i\varepsilon_I$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

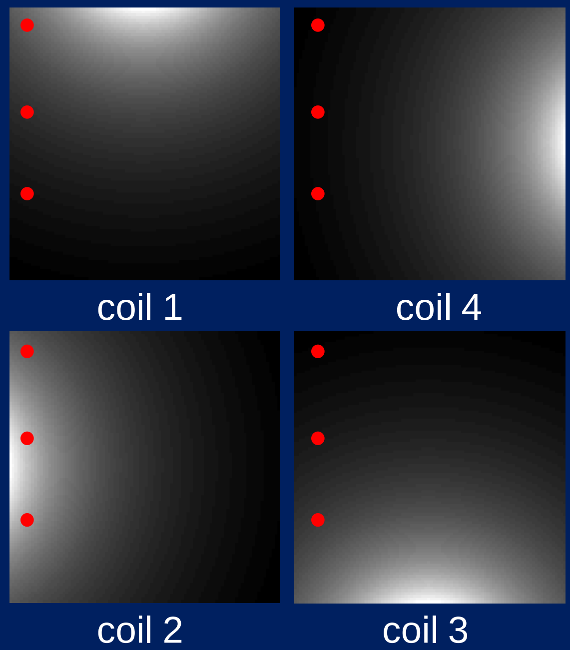
$n \times 1$



$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C}\;\underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C}$$

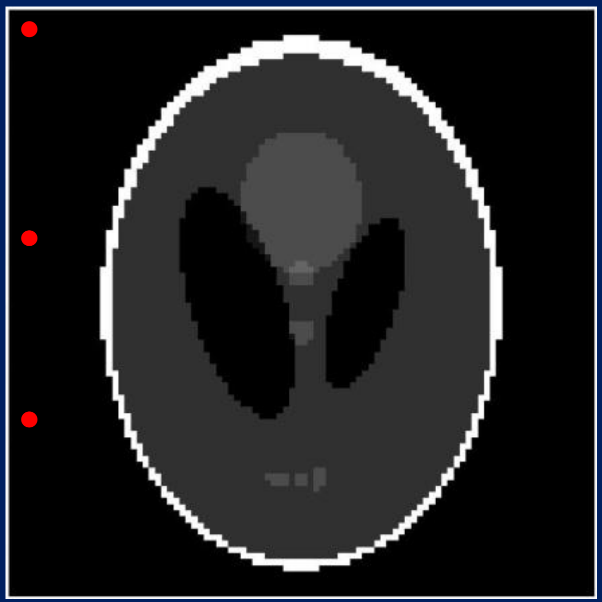
$n \times A$

$=$



$*$

$A \times 1$

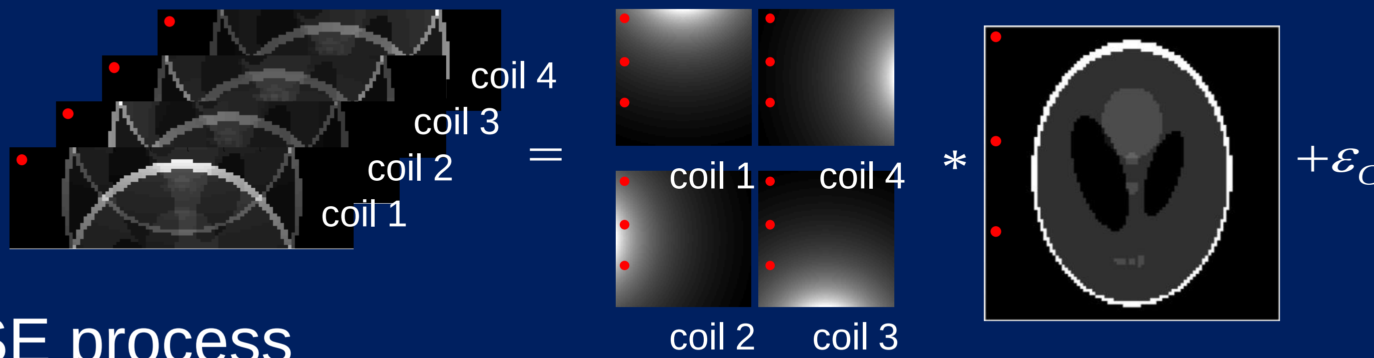


$n \times 1$

$+\varepsilon_C$

Bruce, Karaman, and Rowe: In Submission, 2011.

Methods



The SENSE process

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C} \,, \qquad \varepsilon_C \sim CN(0, \Psi_C)$$

$$\Psi_C = \Psi_R + i\Psi_I$$

uses the complex-valued normal distribution

H is the conjugate transpose (Hermetian)

$$f(\varepsilon_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2\,\varepsilon_C^H \Psi_C^{-1} \varepsilon_C} \,,$$

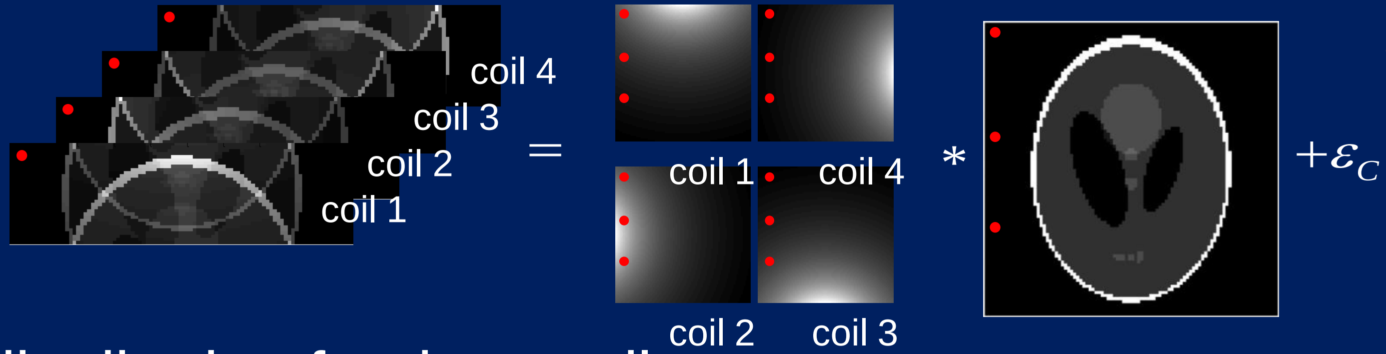
and for n coil measurements

$$f(a_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2(a_C - S_C v_C)^H \Psi_C^{-1} (a_C - S_C v_C)}$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

Rowe

Methods



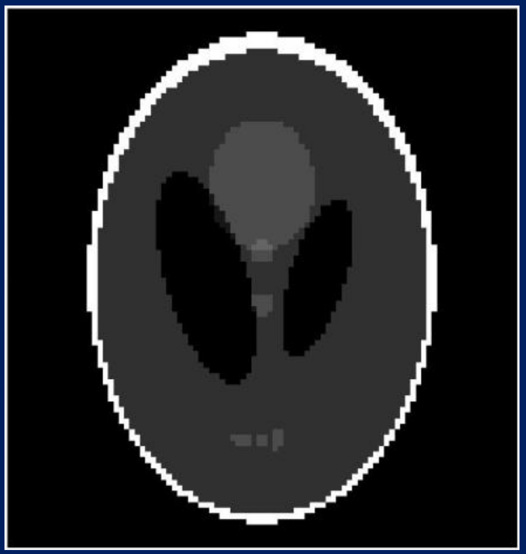
From the distribution for the n coil measurements

$$f(a_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2(a_C - S_C v_C)^H \Psi_C^{-1}(a_C - S_C v_C)}$$

the voxel values can be estimated as

$$v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C$$

with knowledge of S_C and Ψ_C.



Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

Instead of writing the model with complex numbers as

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \; \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C} \; ,$$

$$a_C = a_R + i a_I \quad S_C = S_R + i S_I \quad v_C = v_R + i v_I \quad \varepsilon_C = \varepsilon_R + i \varepsilon_I$$

we can write the model using an isomorphism as

$$\underset{2n\mathrm{x}1}{a} = \underset{2n\mathrm{x}2A}{S} \; \underset{2A\mathrm{x}1}{v} + \underset{2n\mathrm{x}1}{\varepsilon}$$

$$a = \begin{pmatrix} a_R \\ a_I \end{pmatrix} \quad S = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix} \quad v = \begin{pmatrix} v_R \\ v_I \end{pmatrix} \quad \varepsilon = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix} .$$

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

Then the distribution for n coil measurements is

$$f(a) = (2\pi)^{-n} \left|\Psi\right|^{-1/2} \mathrm{e}^{-1/2(a-Sv)'\Psi^{-1}(a-Sv)} \quad ,$$

with

$$\underset{2n\text{x}1}{a} = \begin{pmatrix} a_R \\ a_I \end{pmatrix} \quad \underset{2n\text{x}2A}{S} = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix} \quad \underset{2A\text{x}1}{v} = \begin{pmatrix} v_R \\ v_I \end{pmatrix} \quad \underset{2n\text{x}1}{\varepsilon} = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix} \quad ,$$

and the imposed skew-symmetric covariance structure

$$\Psi = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix} \quad .$$

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

The SENSE voxel values can be estimated by

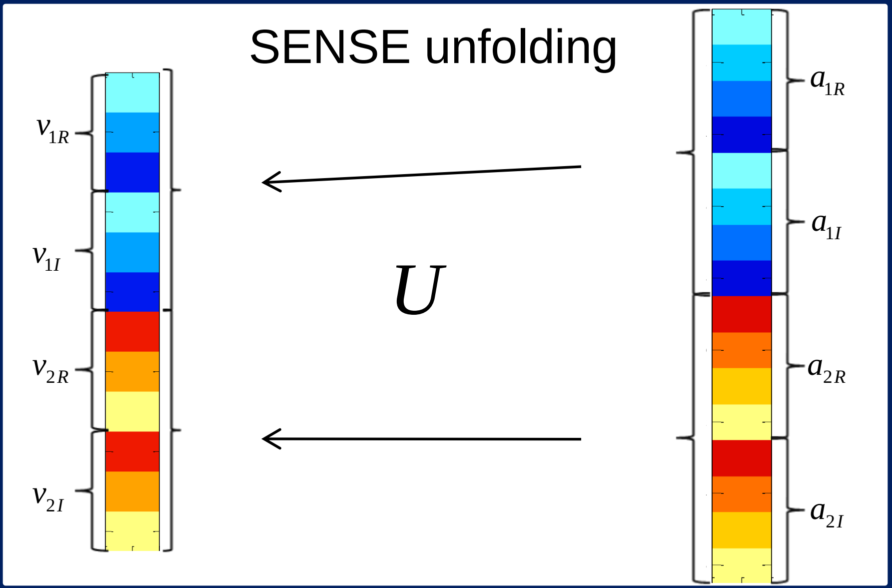
$$v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C$$

or in terms of an isomorphism

$$\underset{2A\times 1}{\begin{pmatrix} v_R \\ v_I \end{pmatrix}} = \left[\underset{2A\times 2n}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}^T} \underset{2n\times 2n}{\begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}^{-1}} \underset{2n\times 2A}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}} \right]^{-1} \underset{2A\times 2n}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}^T} \underset{2n\times 2n}{\begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}^{-1}} \underset{2n\times 1}{\begin{pmatrix} a_R \\ a_I \end{pmatrix}}$$

$$\underset{2A\times 1}{\begin{pmatrix} v_R \\ v_I \end{pmatrix}} = \underset{2A\times 2n}{U} * \underset{2n\times 1}{\begin{pmatrix} a_R \\ a_I \end{pmatrix}}$$

n=4 real / n=4 imaginary true aliased voxel values

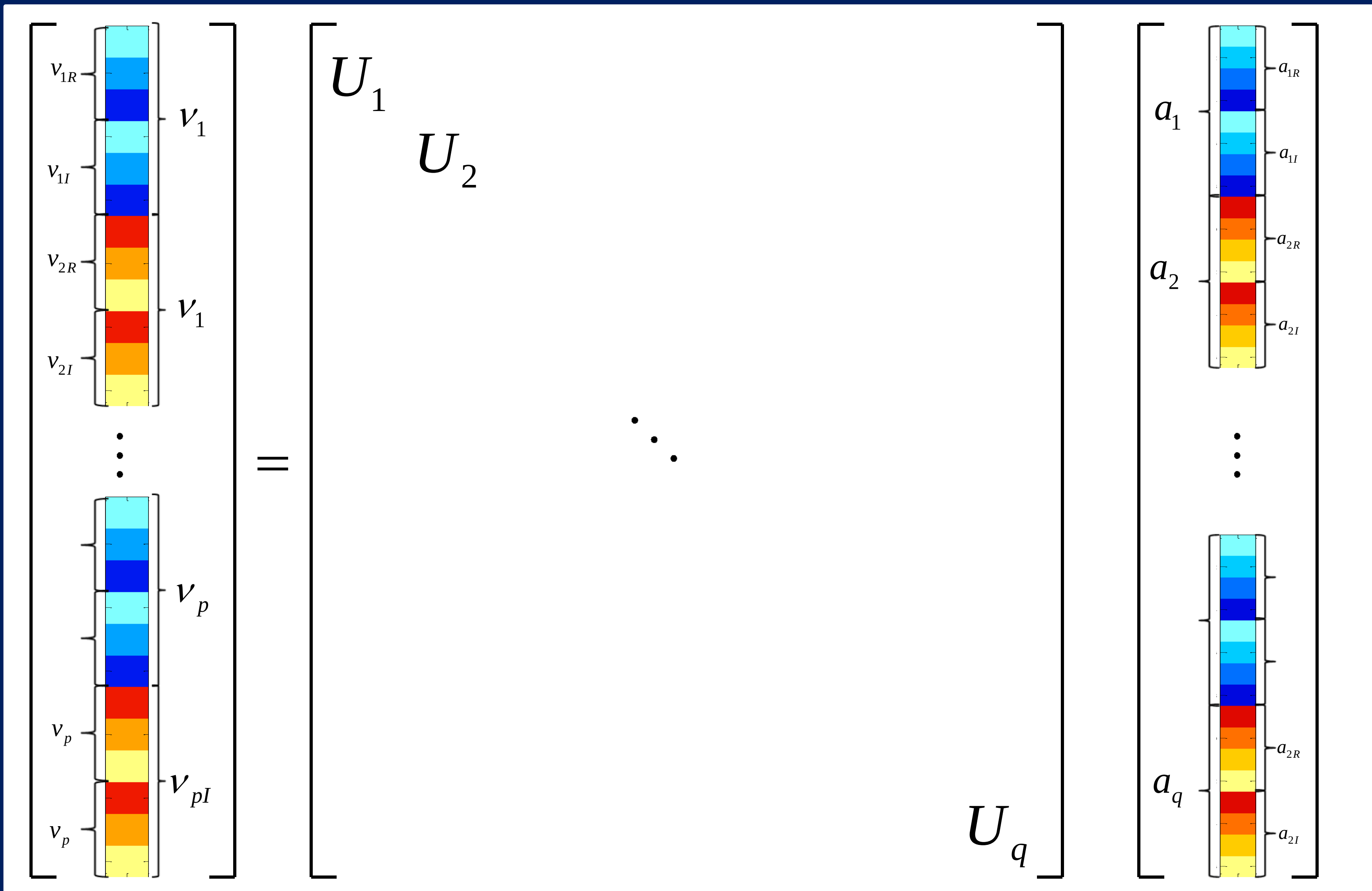


Acceleration A
A=3 real / A=3 imaginary un-aliased fold values

Bruce, Karaman, and Rowe: In Submission, 2011.

SENSE unfolding $\begin{pmatrix} v_R \\ v_I \end{pmatrix} = U * \begin{pmatrix} a_R \\ a_I \end{pmatrix}$

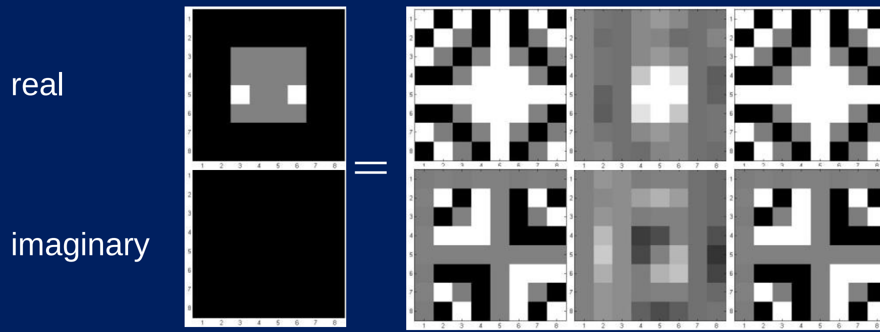
Methods

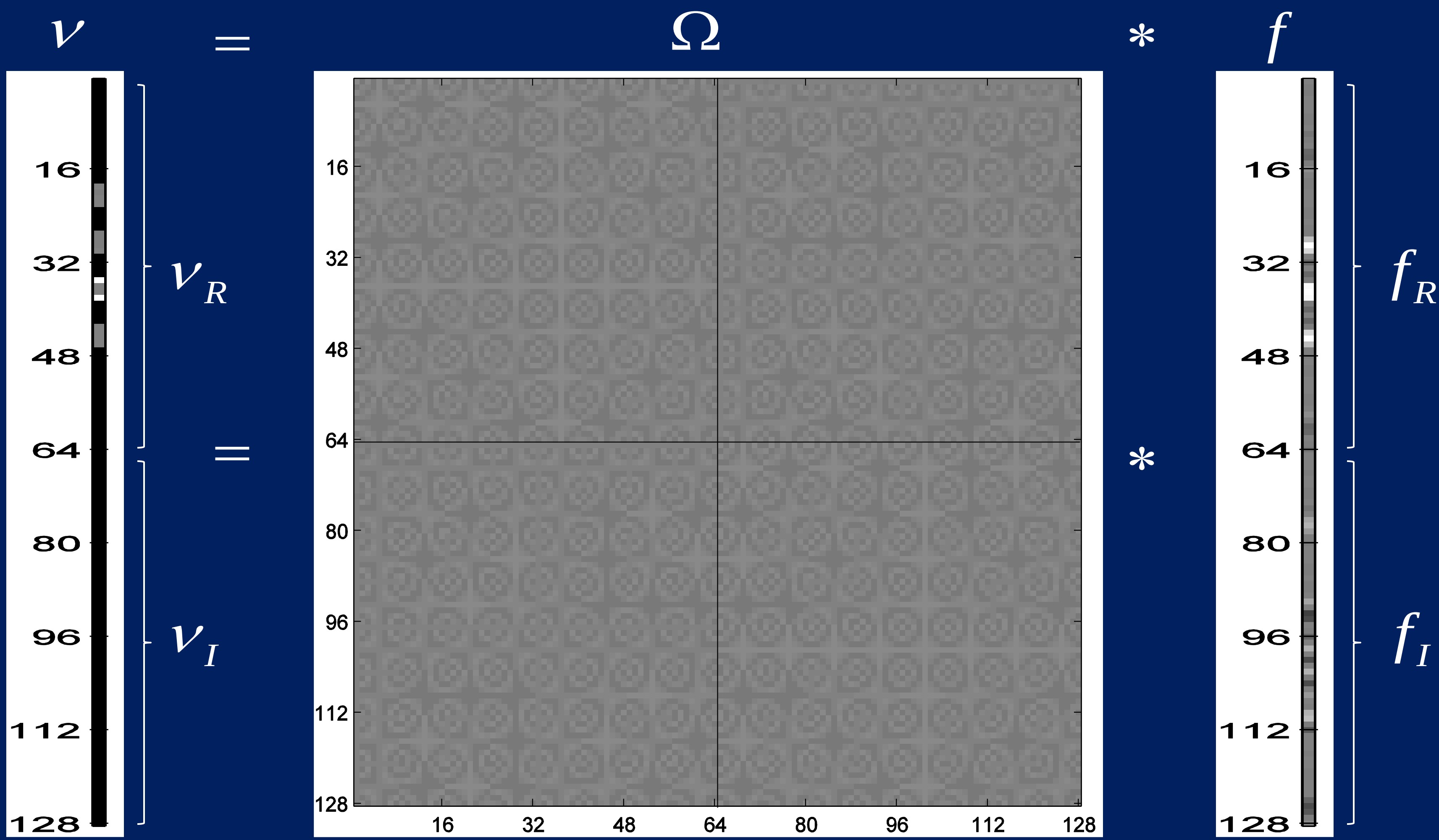


Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

Real-valued isomorphism

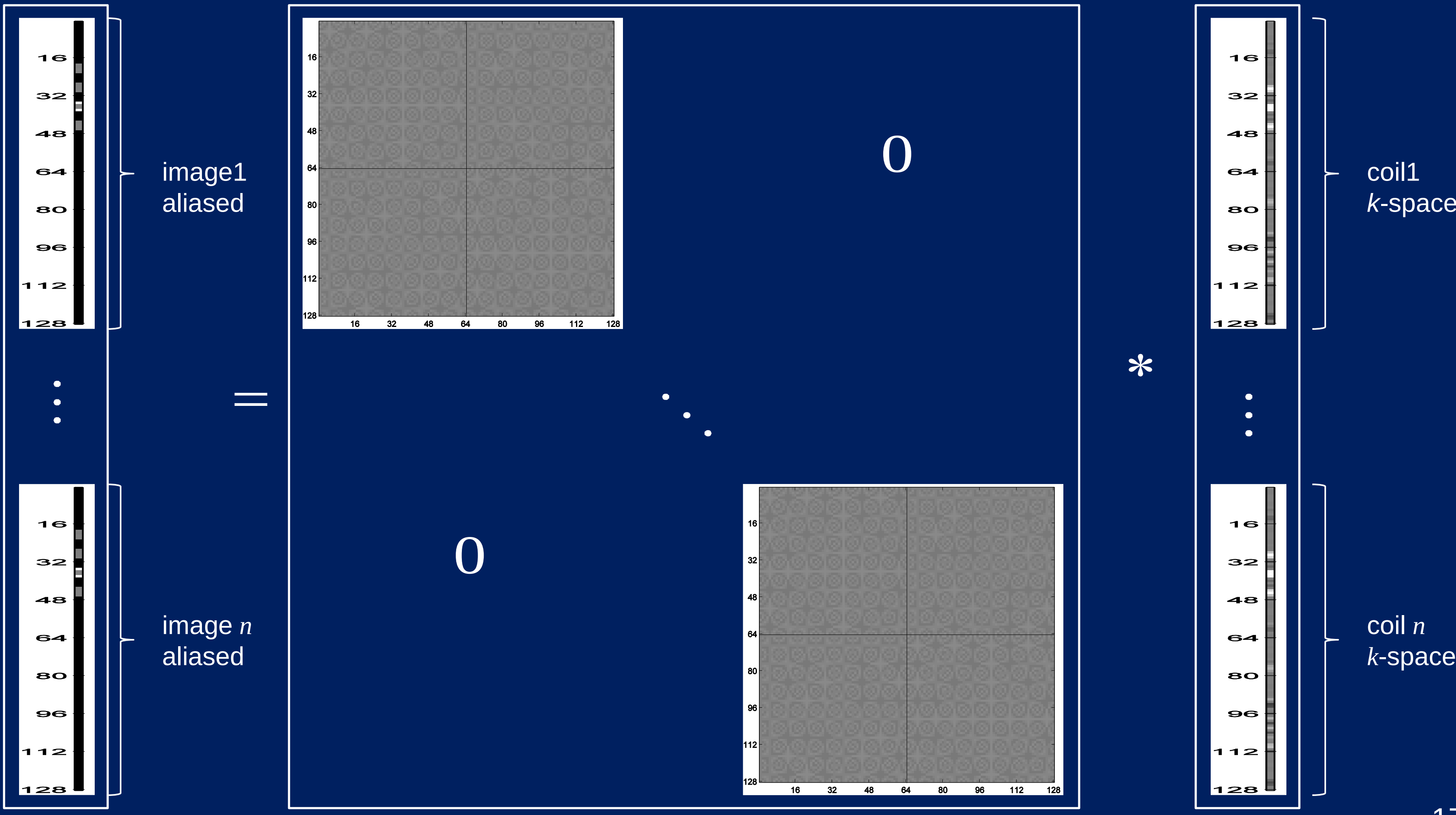




Rowe, Nencka, Hoffmann,Signal and noise of Fourier reconstructed fMRI data. JNSM 159:361-369, 2007.

Methods

$$v = I_n \otimes \Omega * f$$

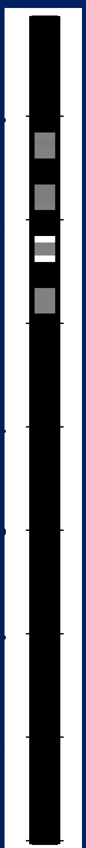


Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

here is a for each voxel

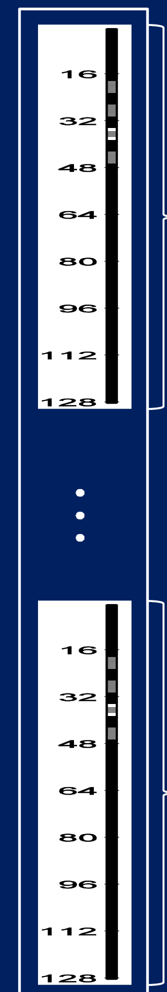
$$y = P_u \; U \; a \; P_S P_C \; v$$



$= P_U$

$$U = \begin{bmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_p \end{bmatrix}$$

$P_S P_C$



Single image vector

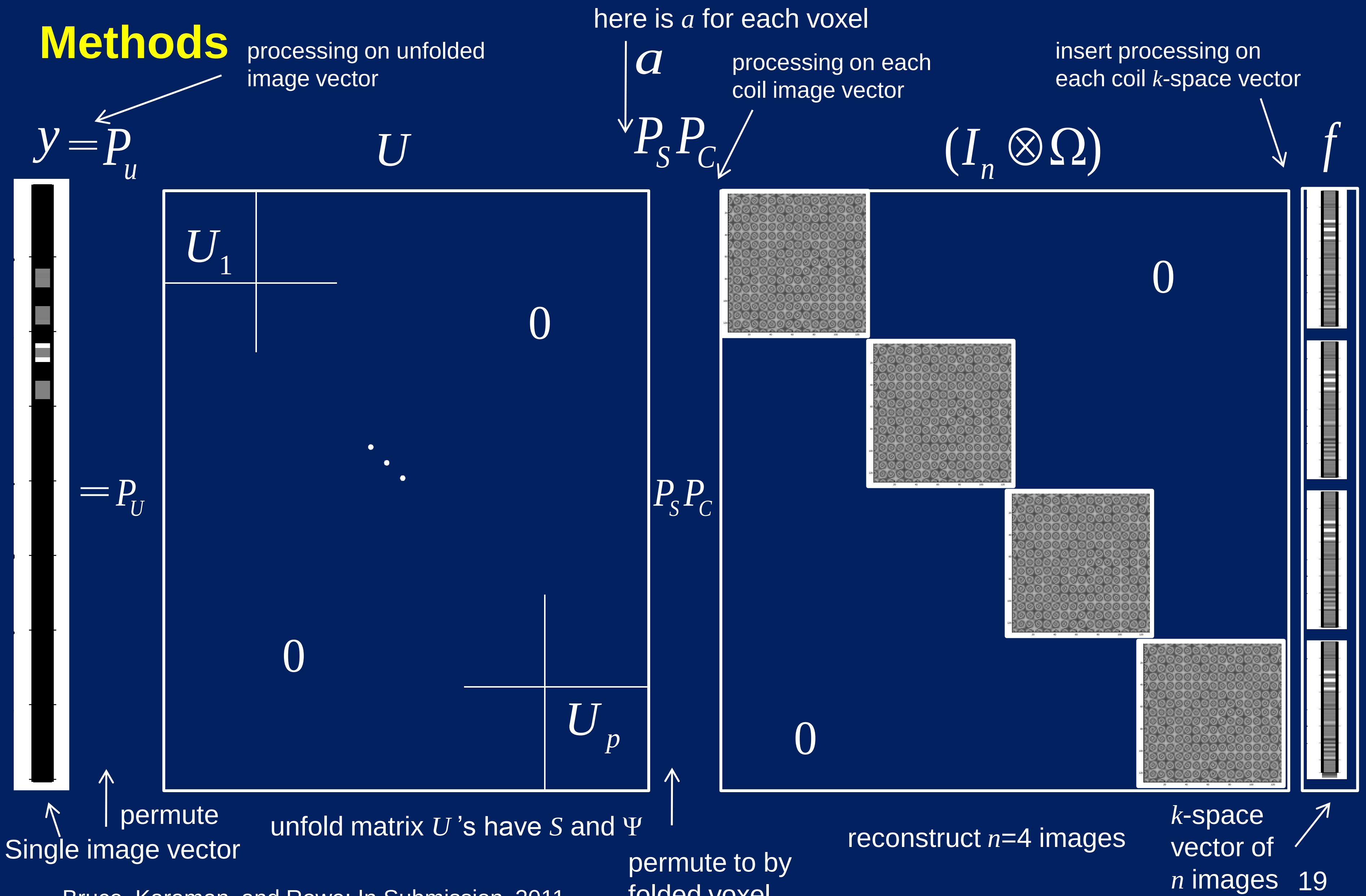
permute

unfold matrix U's have S and Ψ

permute to by folded voxel

Bruce, Karaman, and Rowe: In Submission, 2011.

Methods



Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

$$y = \underbrace{O_I \; P_u \; U \; P_S P_C \; (I_n \otimes \Omega_a O_k)}_{O} \; f$$

where

$f = (f_1, \ldots, f_n)'$ are coil *k*-space

O_k is *k*-space preprocessing

Ω_a is adj. inverse Fourier matrix

P_u , P_S , P_C , permutation matrices

U SENSE unfolding matrix

O_I is image space preprocessing

$$f = P_C \mathcal{R} C \mathcal{F}$$

k-space vector
censor blip points
row reverse
permute

$$O_k = \mathcal{A} \mathcal{Z} \mathcal{H} \underbrace{P_R^{-1} \Omega_{row}^{-1} \Phi \Omega_{row} P_R}$$

$\Omega_a = \Omega$ adjusted for ΔB and for T_2^*

$O_I = S_m$ ← Image smoothing

Bruce, Karaman, and Rowe: In Submission, 2011.

Methods

Statistical Expectation and Covariance.

If E(f)=f_0, then for Mf, E(Mf)=Mf_0.

If cov(f)=Γ, then for Mf, cov(Mf)=$M\Gamma M'$.

This means that with $y = Of$,

$E(y) = Of_0$ and $\text{cov}(y) = O\Gamma O' = \underset{2p\times 2p}{\Sigma}$

→ $\text{cor}(\nu) = D_\Sigma^{-1/2}\Sigma D_\Sigma^{-1/2}$

So even if $\Gamma = \sigma^2 I$, it is not necessarily true that $\Sigma = \sigma^2 I$!

This has H_0 fMRI noise and fcMRI connectivity implications!

Nencka, Hahn, Rowe: JNSM, 181:268-282, 2009.

Nencka, Rowe: OHBM, 2007.

Results

Since

$$y = Of,$$

we inverted and made the n coil spatial frequencies from

$$(O^T O)^{-1} O^T v = f$$

where O and v are known

v is true/noiseless Shepp-Logan phantom (scaled by 50)

$$O = S_m P_U U P_S P_C (I_n \otimes \Omega)$$

The number of coils, n, and the reduction factor, A, are specified in the dimensions of operators, O.

Bruce, Karaman, and Rowe: In Submission, 2011.

$$y = Of$$

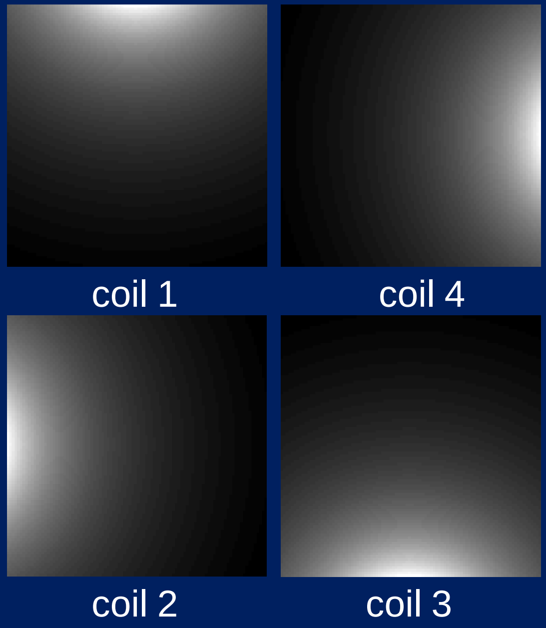
Results

Noiseless data $f = (O^T O)^{-1} O^T v$ generated for $N_X = N_Y = 96$, $n = 4$, $A = 3$

$$O = S_m P_U U P_S P_C (I_n \otimes \Omega)$$

O had diagonal blocks $U_j = (S_j^T \Psi^{-1} S_j)^{-1} S_j^T \Psi^{-1}$

Sensitivities, S



Markovian coil covariance, Ψ

$$\Psi = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix} \qquad \Psi_R = \begin{pmatrix} 1 & .33 & .11 & .33 \\ .33 & 1 & .33 & .11 \\ .11 & .33 & 1 & .33 \\ .33 & .11 & .33 & 1 \end{pmatrix}$$

Not skew-symmetric

$$\Psi_I = \Psi_R \qquad \Psi_{RI} = \begin{pmatrix} 0 & -.11 & -.07 & -.11 \\ .26 & 0 & -.11 & -.07 \\ .42 & .26 & 0 & -.11 \\ .26 & .42 & .26 & 0 \end{pmatrix}$$

Bruce, Karaman, and Rowe: In Submission, 2011.

Results

Gaussian Smoothing applied in image-space

- FWHM = 3 voxels,

-Normalized to leave variance unaffected (Scales mean by 4.516)

By definition, smoothing induces a covariance and correlation between voxels and their neighbors.

This effect is in turn transferred to the correlated voxels from each fold in SENSE.

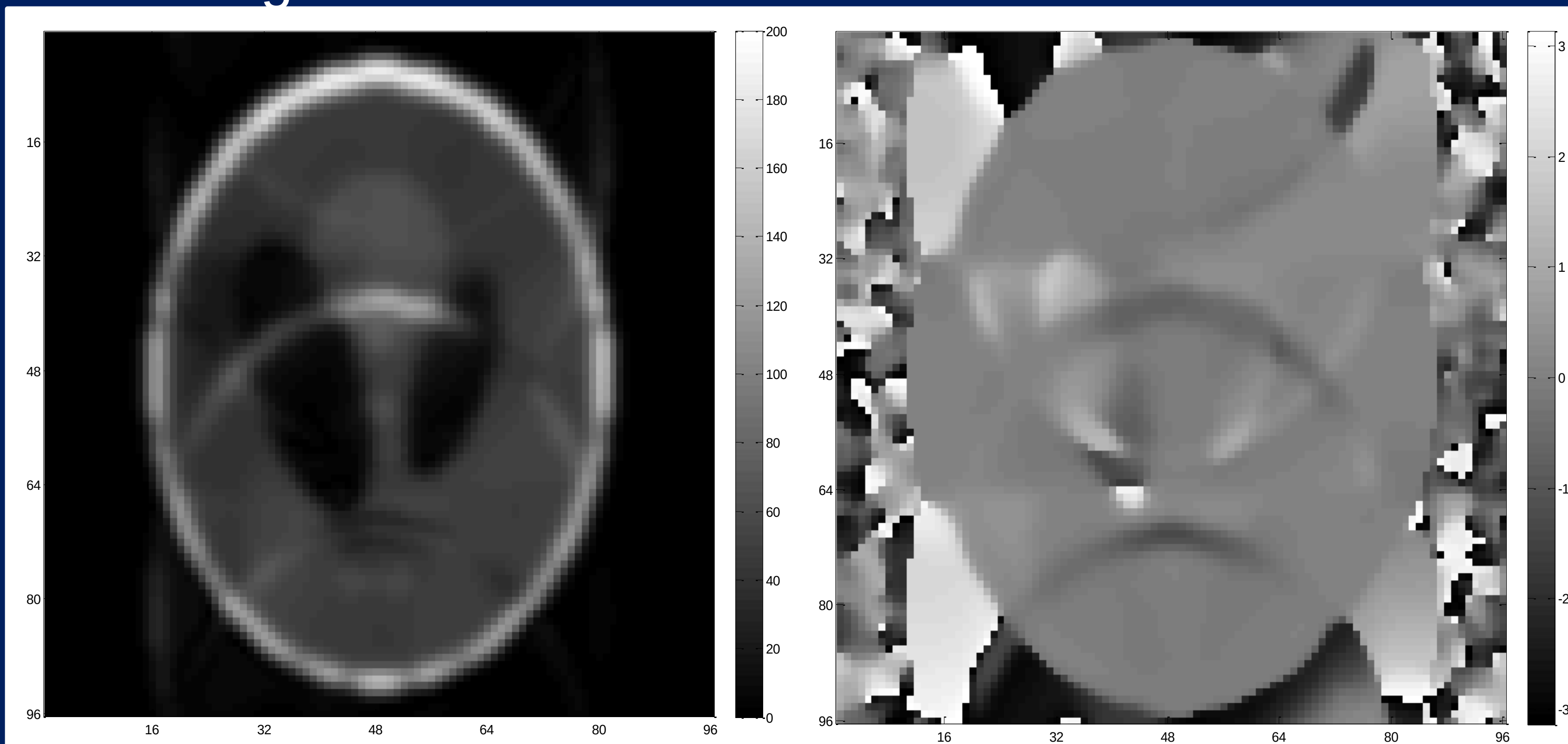
Gaussian smoothing kernel, S_m, was applied in image-space to reconstructed images.

Bruce, Karaman, and Rowe: In Submission, 2011.

Nencka et al.: A Mathematical Model for Understanding the STatistical effects of k-space (AMMUST-k) Preprocessing Operators on Observed Voxel Measurements in fcMRI and fMRI. Journal of Neuroscience Methods 181:268-282, 2009.

Results

Magnitude Phase



Ghosting because symmetric coil cov Ψ used $\rightarrow \Psi = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}$

Alternatice symmetric coil cov Ψ proposed. $\Psi = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix}$

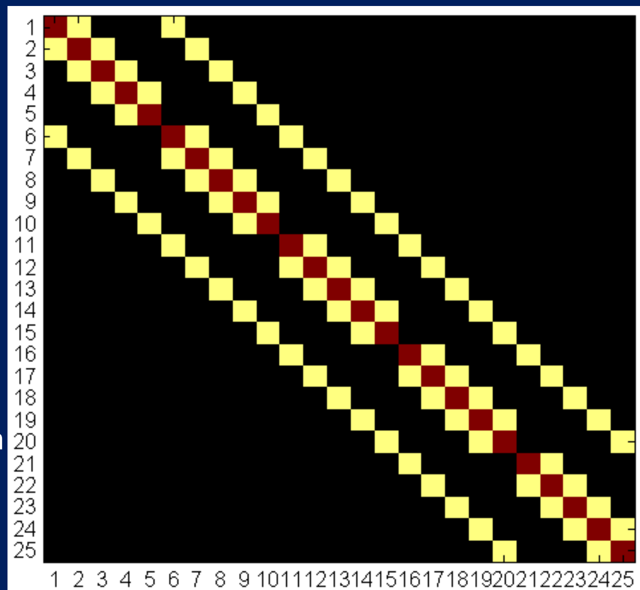
Phase is important in complex-valued fMRI!

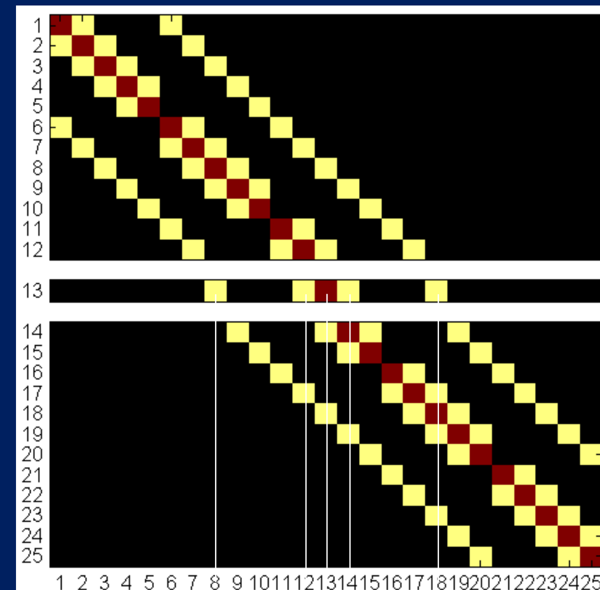
Results

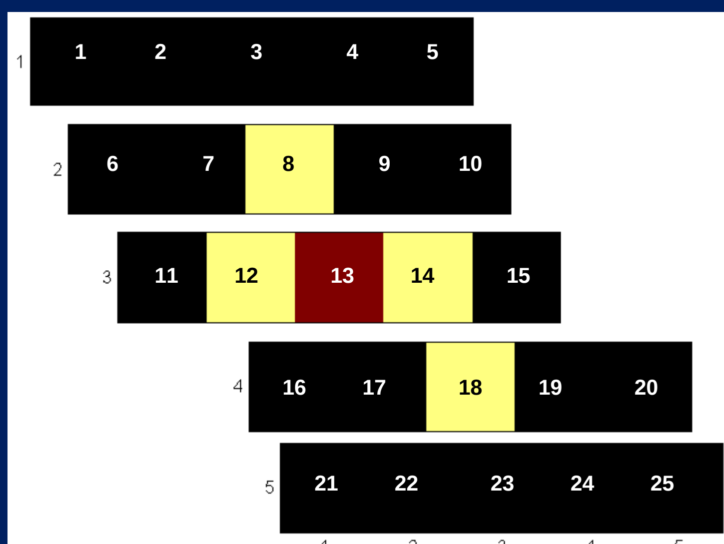
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

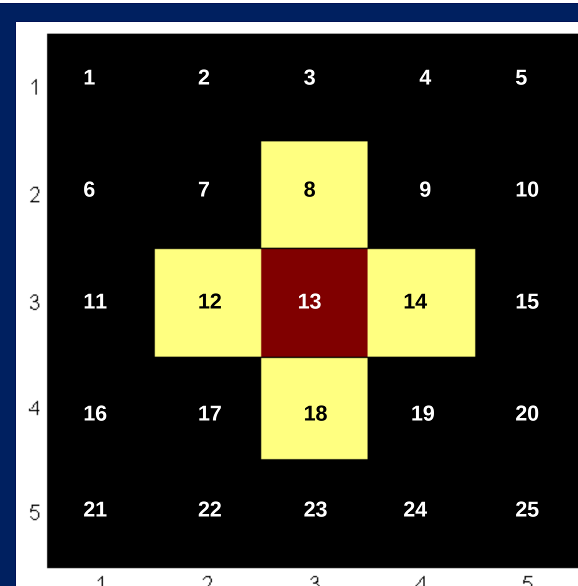
5x5 image

cor = 25x25 correlation matrix

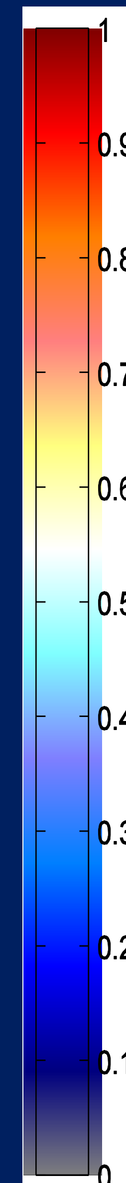








5x5 correlation image

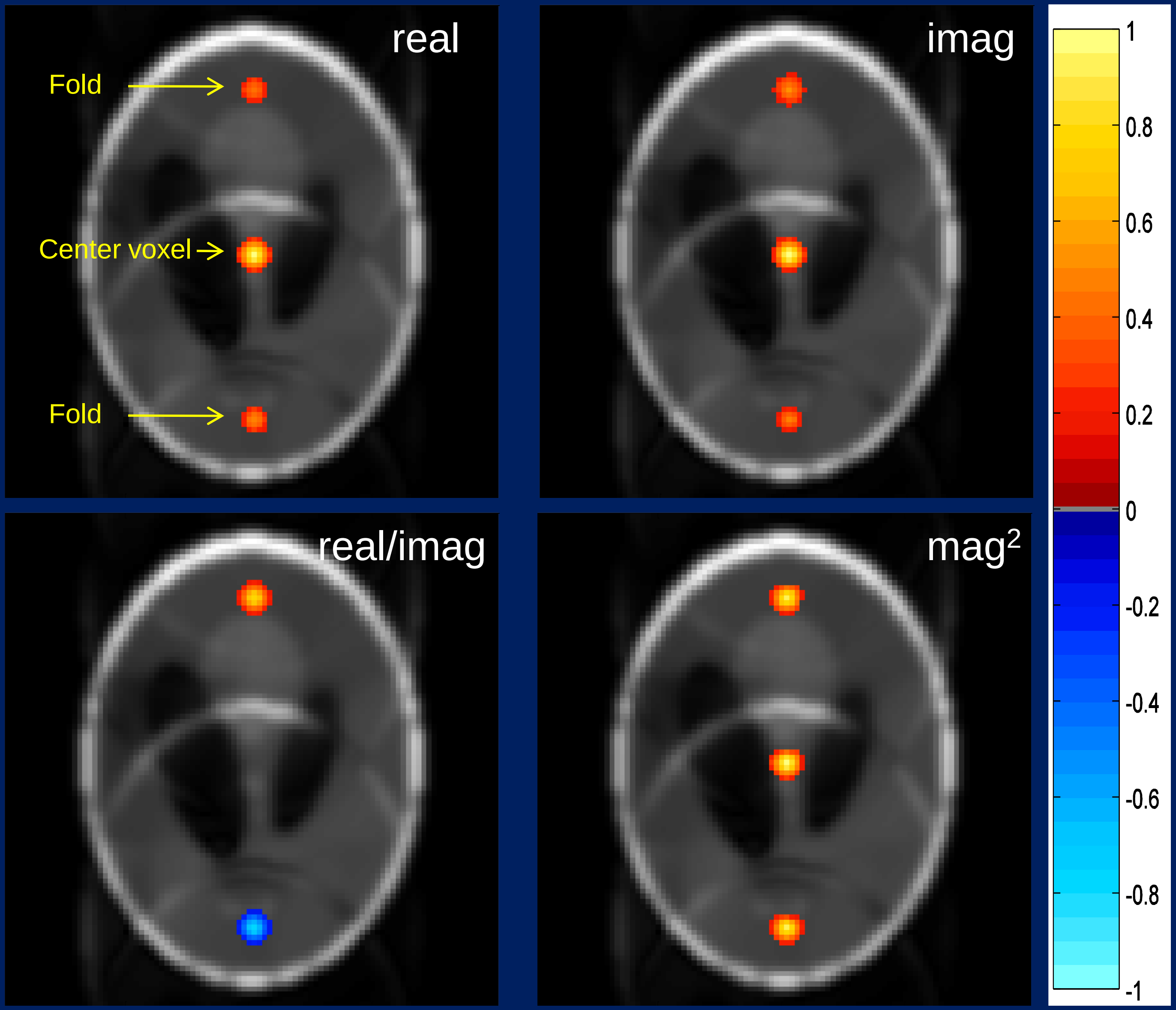


Correlations induced about the center voxel.

Results

N_X=96
N_Y=96
$n = 4$
$A = 3$

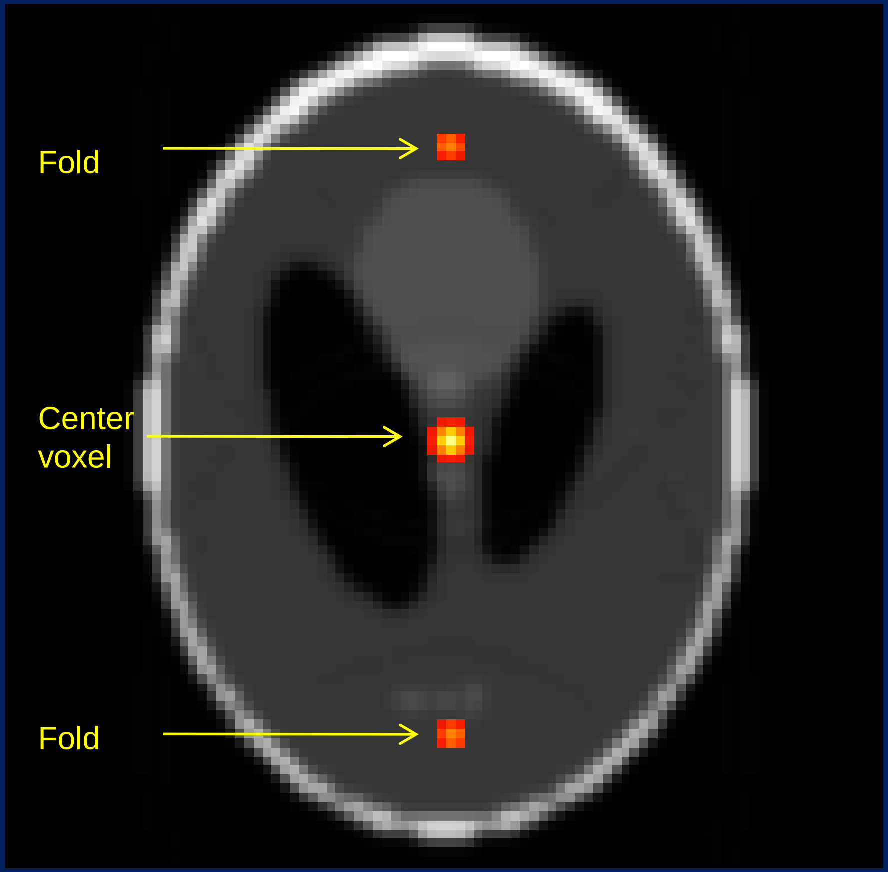
Functional connectivity implications

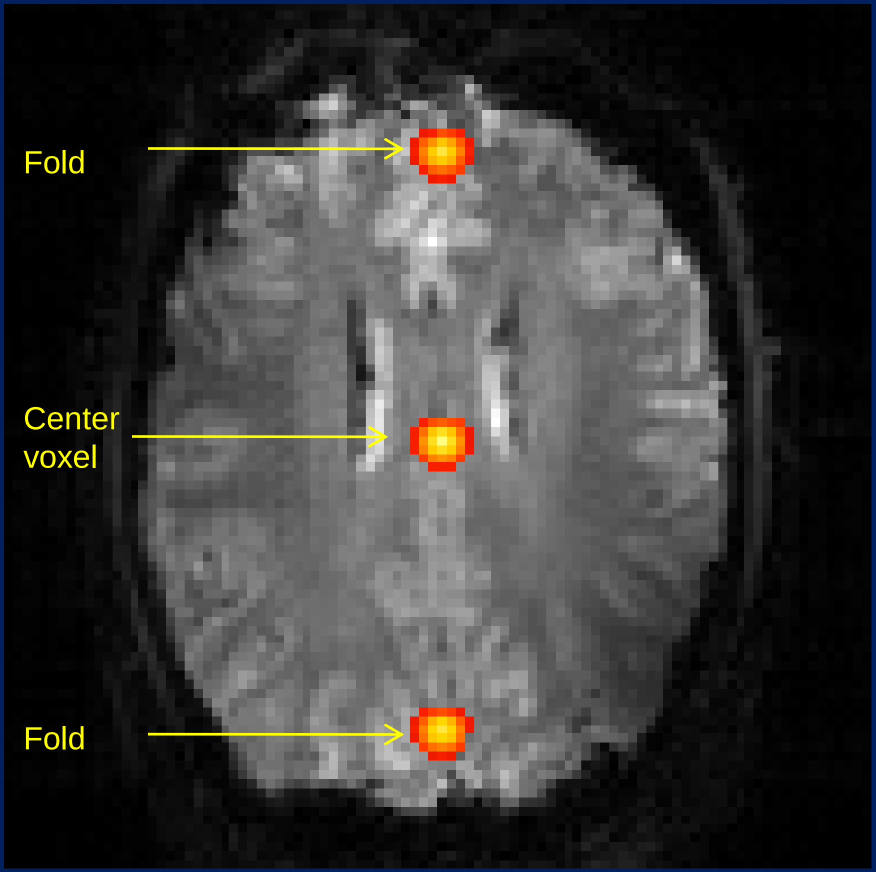


TH=0.01

Bruce, Karaman, and Rowe: In Submission, 2011.

Results





underlay artificially expanded

Extrapolate to human, mistakenly conclude regions correlated!

Discussion

The SENSE image reconstruction method was described.

Wrote SENSE reconstruction with an isomorphism

$$y = O_I P_U U P_S P_C (I_n \otimes \Omega O_k) f = Of \ .$$

The new mean $E(y) = Of_0$ and covariance $\Sigma = O\Gamma O'$ of complex-valued SENSE described.

Theoretical results of SENSE reconstruction presented.

Ghosting present in SENSE magnitude and phase images.

Induced correlation between folds of no biological origin.

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Sweet 16!

vs.

Go Marquette!