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Significant FMRI Neurologic Synchrony Using Monte Carlo Methods

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Abstract — It is well known that regions of the brain exhibit functional synchrony. A well established method is the average cross correlation termed the COSLOF index which has proved useful as a noninvasive quantitative marker of hippocampal synchrony for the preclinical stage of Alzheimer's disease. This paper presents the COMDET, an alternative index of functional synchrony, and compares it to the COSLOF with their statistical underpinnings. Logarithmic functions of these two statistics are presented with their asymptotic chi squared distributions. These two statistics are empirically compared under five correlation structures. It is found that the COMDET performs better than the COSLOF except under very specific cases. Critical values of the COMDET and COSLOF as well as their logarithmic functions are presented which are determined via Monte Carlo simulation.

1 Introduction

The well-known phenomenon of functional synchrony [3, 6] in functional magnetic resonance imaging (FMRI) studies has been characterized by cross-correlations of resting state voxels. The average of the cross-correlations termed the coefficients of spontaneous low frequency fluctuations (COSLOF) index [7] when applied to the hippocampus has been used as a preclinical marker for Alzheimer's disease. The COSLOF index in a set of voxels yields a measure of functional synchrony. It is known that the hippocampus is responsible for the storage of declarative (factual) memory and it has been shown that functional synchrony is reduced in Alzheimer's disease patients compared to normals [7]. In this paper, the correlation matrix determinant (COMDET) statistic is proposed as an alternative to the COSLOF index and both of their statistical underpinnings are presented. These two measures of functional synchrony are compared in simulations to determine their power of detection. For several correlation structures, the number of times that the each measure rejects the null hypothesis of no synchrony is determined. Critical values of the COMDET and COSLOF are determined via Monte Carlo simulation.

2 Regression Model

The standard multivariate linear regression model [10] is used to describe the voxel time courses. The model, for a set of p voxels measured at n time points, is written as

$$\underset{n\times p}{Y} = \underset{n\times(q+1)}{X} \quad \underset{(q+1)\times p}{B} + \underset{n\times p}{E} \tag{2.1}$$

where Y is the matrix of observed time courses, X is the design matrix, B is the matrix of regression coefficients, and E is the matrix of errors. The errors of observation ϵ_i (which are rows of E) are assumed to be independent and normally distributed with p dimensional zero mean vector and $p \times p$ positive definite covariance matrix Σ. With the above model specifications, the likelihood of the observations is

$$p(Y|B,\Sigma,X) = (2\pi)^{-\frac{np}{2}}|\Sigma|^{-\frac{n}{2}}e^{-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)}. \tag{2.2}$$

It can be shown [10] that the unrestricted maximum likelihood estimate of the matrix of regression coefficients $\hat{B}$ is

$$\hat{B} = (X'X)^{-1}X'Y,$$

and that the unrestricted maximum likelihood estimate of the covariance matrix is

$$\hat{\Sigma} = \frac{1}{n}(Y - X\hat{B})'(Y - X\hat{B}).$$

Also, denote the unrestricted maximum likelihood estimate of the cross-correlation matrix as

$$\hat{R} = \hat{D}^{-1/2} \; \hat{\Sigma} \; \hat{D}^{-1/2}$$

with $jj'th$ element given by $\hat{\rho}_{jj'}$ where $\hat{D} = \text{diag}(\hat{\Sigma})$ is a diagonal matrix formed from the main diagonal of $\hat{\Sigma}$.

The spatial relationship between voxels can be anywhere from statistically independent with a diagonal covariance matrix to a full positive definite covariance matrix. Hypotheses regarding spatial structure are to be evaluated with the use of the generalized likelihood ratio test. The generalized likelihood ratio statistic is computed by maximizing the likelihood with respect to the unknown parameters under the null and alternative hypotheses. These estimated parameters are inserted into their respective likelihood functions and the ratio taken.

3 COSLOF

Often structure in a covariance matrix can aid in simplifying the spatial relationship between the voxels. One such spatial relationship is the intraclass correlation structure. The intraclass correlation structure is when the correlation matrix R has ones along the main diagonal and all other elements have the correlation parameter ρ.

The null hypothesis that the voxels are unequal variance intraclass with known correlation parameter ρ_0 (independent when ρ_0=0) versus the alternative hypothesis that the voxels are unequal variance intraclass with unknown correlation parameter $\rho \neq \rho_0$ is

$$\begin{array}{rrcl} H_0: & B & \in & \mathbb{R}^{(q+1)\times p} \\ & \text{diag}(D) & = & \mathbb{R}^{p+} \\ & R & = & R_0 \end{array} \quad vs \quad \begin{array}{rrcl} H_1: & B & \in & \mathbb{R}^{(q+1)\times p} \\ & \text{diag}(D) & = & \mathbb{R}^{p+} \\ & R & \neq & R_0 \end{array}$$

where R_0 is the intraclass correlation matrix with ρ_0 for the off diagonal elements.

The generalized likelihood ratio statistic is computed by maximizing the likelihood $p(Y|B,D,\rho,X)$ with respect to (B,Σ) under the two hypotheses to obtain null and alternative estimates $(\tilde{B},\tilde{D},\rho_0)$ and $(\hat{B},\hat{D},\hat{\rho})$.

The likelihood in Equation 2.2 can be written as

$$p(Y|B,\Sigma,X)=(2\pi)^{-\frac{np}{2}}|D|^{-\frac{n}{2}}|R|^{-\frac{n}{2}}e^{-\frac{1}{2}trD^{-\frac{1}{2}}R^{-1}D^{-\frac{1}{2}}(Y-XB)'(Y-XB)}.$$

Using the two intraclass matrix identities

$$|R|=(1-\rho)^{p-1}[1+(p-1)\rho] \qquad R^{-1}=\frac{I_p}{(1-\rho)}-\frac{\rho e_p e_p'}{(1-\rho)[1+(p-1)\rho]}$$

where e_p is a p-dimensional vector of ones and taking the natural logarithm, the log likelihood can be written as

$$\begin{aligned} LL = & -\tfrac{np}{2}\ln(2\pi)-\tfrac{n}{2}\textstyle\sum_{j=1}^{p}\ln\sigma_j^2-\tfrac{n(p-1)}{2}\ln(1-\rho)-\tfrac{n}{2}\ln[1+(p-1)\rho] \\ & -\tfrac{1}{2}\textstyle\sum_{j=1}^{p}(\sigma_j^2)^{-1}G_{jj}\left[\tfrac{1}{(1-\rho)}\right]+\tfrac{1}{2}\tfrac{\rho}{(1-\rho)[1+(p-1)\rho]}\left[\textstyle\sum_{j'=1}^{p}(\sigma_j^2)^{-\frac{1}{2}}\sum_{j'=1}^{p}G_{jj'}(\sigma_{j'}^2)^{-\frac{1}{2}}\right]. \end{aligned}$$

In the above, $G=(Y-XB)'(Y-XB)$ was defined and the property $tr(ABC)=tr(B'A'C')$ was used.

It can be shown that estimates of B, D, and ρ under the two hypotheses are

$$\begin{aligned} \tilde{B} &= (X'X)^{-1}X'Y, \quad & \tilde{D} &= \tfrac{1}{1-\rho_0}\left[1-\tfrac{\rho_0}{1+(p-1)\rho_0}\right]\ \mathrm{diag}\tfrac{\tilde{G}}{n}, \quad & \tilde{\rho} &= \rho_0 \\ \hat{B} &= (X'X)^{-1}X'Y, \quad & \hat{D} &= \tfrac{1}{1-\hat{\rho}}\left[1-\tfrac{\hat{\rho}}{1+(p-1)\hat{\rho}}\right]\ \mathrm{diag}\tfrac{\hat{G}}{n}, \quad & \hat{\rho} &= \bar{\rho} \end{aligned}$$

where $\tilde{G}$ and $\hat{G}$ which are equivalent are G above with $\tilde{B}$ or $\hat{B}$ substituted in for B and

$$\bar{\rho}=\frac{1}{p(p-1)}\left[\sum_{j=1}^{p}\sum_{j'=1}^{p}\frac{G_{jj'}}{\sqrt{G_{jj}G_{j'j'}}}-p\right].$$

The maximum likelihood estimate of the variances $\hat{D}$ is an approximate value. The maximum likelihood estimate of the correlation coefficient under the alternative hypothesis has been well approximated by its method of moments estimate because an explicit expression can not be found.

The generalized likelihood ratio statistic for the above hypothesis test is

$$\begin{aligned} \lambda &= \frac{p(Y|\tilde{B},\tilde{D},\tilde{\rho},X)}{p(Y|\hat{B},\hat{D},\hat{\rho},X)} \\ &= \frac{(2\pi)^{-\frac{np}{2}}|\tilde{\Sigma}|^{-\frac{n}{2}}e^{-\frac{1}{2}tr\tilde{\Sigma}^{-1}(Y-X\tilde{B})(Y-X\tilde{B})'}}{(2\pi)^{-\frac{np}{2}}|\hat{\Sigma}|^{-\frac{n}{2}}e^{-\frac{1}{2}tr\hat{\Sigma}^{-1}(Y-X\hat{B})(Y-X\hat{B})'}} \\ &= \frac{\left|\frac{1}{1-\rho_0}\left(1-\frac{\rho_0}{1+(p-1)\rho_0}\right)diag\frac{\tilde{G}}{n}\right|^{-n/2}|R_0|^{-n/2}}{\left|\frac{1}{1-\hat{\rho}}\left(1-\frac{\hat{\rho}}{1+(p-1)\hat{\rho}}\right)diag\frac{\hat{G}}{n}\right|^{-n/2}|\hat{R}|^{-n/2}} \\ &= \\ \lambda &= \frac{(1-\rho_0)^{n/2}[1+(p-1)\hat{\rho}]^{n/2}[1-\frac{\hat{\rho}}{1+(p-1)\hat{\rho}}]^{np/2}}{[1+(p-1)\rho_0]^{n/2}[1-\frac{\rho_0}{1+(p-1)\rho_0}]^{np/2}(1-\hat{\rho})^{n/2}} \end{aligned}$$

It is readily seen that a test statistic for the hypothesis test should be a function of the COSLOF or average cross corelation coefficient. A natural choice for the test statistic is $\hat{\rho}$. It can be seen that $\hat{\rho}/[1+(p-1)\hat{\rho}]$ is approximately zero then $\hat{\rho}$ can be explicitly solved for from the generalized likelihood ratio. The test statistic $-2\ln\lambda$ can alternatively be used which has an asymptotic χ^2 distribution with one degree of freedom.

4 COMDET

Often it is believed that there is no structure in a covariance matrix or that the spatial relationship between the voxels is general and not independent. The general covariance (correlation) structure is when each element voxel has its own distinct variance and covariance with every other voxel.

The null hypothesis that the voxels are statistically independent versus the alternative hypothesis that the voxels have a general covariance is

$$\begin{array}{lrcl} H_0: & B & \in & \mathbb{R}^{(q+1)\times p} \\ & \Sigma & = & diag(\Sigma) \end{array} \quad vs \quad \begin{array}{lrcl} H_1: & B & \in & \mathbb{R}^{(q+1)\times p} \\ & \Sigma & > & 0 \end{array}$$

that the voxels are independent.

The generalized likelihood ratio statistic is computed by maximizing the likelihood $p(Y|B,\Sigma,X)$ with respect to (B,Σ) under the two hypotheses to obtain null and alternative estimates $(\tilde{B},\tilde{\Sigma})$ and $(\hat{B},\hat{\Sigma})$.

The maximum likelihood estimates of B and Σ under the two hypotheses are

$$\begin{array}{rcl} \tilde{B}' & = & (X'X)^{-1}X'Y \\ \hat{B}' & = & (X'X)^{-1}X'Y \end{array} \qquad \begin{array}{rcl} \tilde{\Sigma} & = & \frac{1}{n}\text{diag}(Y-X\tilde{B}')'(Y-X\tilde{B}') \\ \hat{\Sigma} & = & \frac{1}{n}(Y-X\hat{B}')'(Y-X\hat{B}'). \end{array}$$

The generalized likelihood independence ratio statistic

$$\begin{aligned} \lambda &= \frac{p(Y|\tilde{B},\tilde{\Sigma},X)}{p(Y|\hat{B},\hat{\Sigma},X)} \\ &= \frac{(2\pi)^{-\frac{np}{2}}|\tilde{\Sigma}|^{-\frac{n}{2}}e^{-\frac{1}{2}tr\tilde{\Sigma}^{-1}(Y-X\tilde{B})(Y-X\tilde{B})'}}{(2\pi)^{-\frac{np}{2}}|\hat{\Sigma}|^{-\frac{n}{2}}e^{-\frac{1}{2}tr\hat{\Sigma}^{-1}(Y-X\hat{B})(Y-X\hat{B})'}} \\ \lambda^{-\frac{2}{n}} &= \frac{|\frac{1}{n}\text{diag}(Y-X\tilde{B})(Y-X\tilde{B})'|}{|\frac{1}{n}(Y-X\hat{B})(Y-X\hat{B})'|} \\ \lambda^{-\frac{2}{n}} &= \frac{|\frac{1}{n}\text{diag}(Y-X\hat{B})(Y-X\hat{B})'|}{|\frac{1}{n}(Y-X\hat{B})(Y-X\hat{B})'|} \\ \hat{\varrho} &= |\hat{R}| \end{aligned}$$

It is readily seen that $\hat{\varrho}$ which is the coefficient matrix determinant (COMDET) is a test statistic for the general covariance (correlation) structure. The exact distribution $p(|R|)$ under the null hypothesis that the voxels are independent has been derived [8] which is of an extremely complicated form. Along with this exact distribution $p(|R|)$ is the asymptotic distribution of

$$v = -[\nu - (2p+5)/6]\ln|R|$$

which is χ^2 with $\gamma = p(p-1)/2$ degrees of freedom. Exact critical 5% and 1% values for v (which have been replicated in Tables 9 and 11) were presented [8] for $\nu = 3, \ldots, 20$, $p = 3, \ldots, 10$, $p \leq \nu$ where $\nu = n - q - 1$ along with the asymptotic critical values. Note that the statistic $-2n \ln |R|$ was not used. Monte Carlo investigations have found that the distribution and hence critical values of $-[\nu - (2p+5)/6] \ln |R|$ was closer to the true sampling distribution than $-2 \ln |R|$.

5 Significance Critical Values

The COSLOF, $\bar{\rho}$ and COMDET $|\hat{R}|$, which are derived from generalized likelihood ratio tests do not follow well known and easily integrable distribution functions. In order for the COMDET statistic to be computable $\hat{R}$ must be positive definite and is only so when $p \leq \nu$. The probability distribution function of ϱ was derived under the null hypothesis with exact 5% and 1% critical values along with the asymptotic χ^2 distribution of v [8]. Both statistics $\hat{\rho}$ and $\hat{\varrho}$ are measures of functional synchrony. The distribution of the COSLOF ρ is not known or easily computable and critical values have not previously been presented. Monte Carlo generated critical values are presented in the appendix.

Given a set of data Y, the statistics $\hat{\rho}$ and $\hat{\varrho}$ (or $\hat{u}$ and $\hat{v}$) are computed. These test statistics are compared to critical values in the appendix. The null hypothesis is rejected at a level α if the test statistic $\hat{\rho}$ is larger than the tabulated α level critical value or $\hat{\varrho}$ is smaller than the tabulated value. Asymptotic hypothesis for population differences can also be made as outlined in the appendix.

If functional synchrony were being measured in the presence of a presented stimulus or task, then a third column would be added to the design matrix X corresponding to a hemodynamic reference function.

The exact distributions of $\hat{\rho}$, u, $\hat{\varrho}$, and v under the null hypothesis that the voxels are independent are extremely complicated as is the determination of exact critical values. Exact 5% and 1% critical values for v (and hence $\hat{\varrho}$ have only previously been given [8] for very specific combinations of ν and p. For the evaluation of functional synchrony, combinations of ν and p which are not previously presented are needed as are critical values for $\hat{\rho}$ and u. The previously presented values of v were replicated by a Monte Carlo sampling based technique. A large number $L = 10^6$ was selected. For each $l = 1, \ldots, L$, a set of n random vectors $x_{1l}, \ldots, x_{nl}$ of dimension p from a standard multivariate normal distribution were generated. The sample correlation matrix was computed from each set of vectors which resulted in $\hat{R}_1, \ldots, \hat{R}_L$ and $\hat{v}_1, \ldots, \hat{v}_L$. These v's were ranked then the $.90 * L^{th}$, $.95 * L^{th}$, $.975 * L^{th}$, $.99 * L^{th}$ and $.999 * L^{th}$ ones retained as the 10%, 5%, 2.5%, 1%, and .1% critical values. This sampling based procedure was repeated for each previously reported combination of n and p. The sampling based critical values which are in agreement with the previously reported values are presented in Tables 8-12 and are greatly expanded. In addition, the same critical values were computed for the COSLOF index ρ and presented in Tables 3-7. The critical values are needed to determine significant functional synchrony. Statistic p-values can be interpolated from the presented critical values.

6 Example

For an example, data from a simulated FMRI experiment of a 5×5 ROI is generated. The voxels in the ROI are numbered sequentially from top left to bottom right and stacked in increasing numerical order. The simulated data consists of $n = 177$ data points ($\nu = 175$) for $p = 25$ voxels. The simulated data is generated according to Equation 2.1 where the design matrix X is a $n \times 2$ matrix whose first column is an n dimensional vector of ones and the second column is at an n dimensional vector of the first counting numbers. The true regression coefficient matrix B is given in Table 1. The voxels were specified to have spatial distributed lag (SDL) one correlation structure with a voxel having the same cross-correlation with its four-neighbors of $c = 0.2$.

Table 1: True regression intercept and slope coefficients.

β_0, β_1	1	2	3	4	5
1	.2, .5	.7, .1	.4, .9	.3, .2	.9, .6
2	.4, .8	.5, .3	.2, .7	.9, .1	.1, .3
3	.5, .5	.1, .6	.6, .4	.4, .2	.4, .5
4	.8, .9	.2, .5	.7, .1	.4, .9	.3, .2
5	.3, .2	.9, .6	.4, .8	.5, .3	.2, .7

For this simulated data set, the COSLOF was computed to be $\hat{\rho} = 0.0177$ ($u = 129.64$) and COMDET to be $\hat{\varrho} = .0328$ ($v = 566.76$). The COSLOF statistic is significant at the $\alpha = 10^{-2}$ level while the COMDET statistic is clearly significant at the $\alpha = 10^{-3}$ level. Because of the large degrees of freedom of v, the asymptotic normal distribution can be used to see that the above sample value is significant at $\alpha = 5 \times 10^{-6}$ [1]. These are two distinct measures of functional synchrony that are computed from the cross-correlations. The COMDET is an index of volume from the cross-correlation matrix while the COSLOF is simply the average cross-correlation. In the next section, the COSLOF and COMDET are compared for sensitivity and power to detect varying correlation structures.

7 Power Analysis

For a power analysis of sensitivity to different correlation structures, data for $p = 25$, $q = 2$, and $n = 77, 102$, or 177 is generated with five different correlation structures. The correlation structures are intraclass (INT) with equal cross correlations, spatial distributed lag (SDL) one with a voxel having the same cross-correlation with its four-neighbors, what will be called spatial distributed lag one-minus (SDM) where the first thirteen voxels are SDL(1) while the last twelve are spatial SDL(1) with c replaced by $-c$, Markov (MKV) or temporal autoregressive one where the correlation between the numbered voxels is the correlation raised to the power of the difference in their numbering, temporal distributed lag (TDL) one or tridiagonal where the correlation matrix has the correlation on the first super and sub diagonal. The last two may not be appropriate for the current application but are commonly used for others and are included for completeness. For each of the correlation structures, 10^4 data sets with the same model as above were generated with correlation parameter $c = 0.15$ or $c = 0.25$.

Table 2 contains the number of rejected null hypotheses for the two functional synchrony measures with each correlation structure at the $\alpha = .001$ level. The sensitivity

Table 2: Correlation structure power analysis. $p = 25$

		$c = .15$					$c = .25$				
		INT	SDL	SDM	MKV	TDL	INT	SDL	SDM	MKV	TDL
$n = 75$	COSLOF	10000	3478	61	9784	918	10000	7889	167	9999	3415
	COMDET	9522	2570	2543	10000	495	10000	10000	10000	10000	7992
$n = 100$	COSLOF	10000	4725	92	9961	1307	10000	9052	241	10000	4784
	COMDET	9979	5871	5818	10000	1392	10000	10000	10000	10000	9838
$n = 175$	COSLOF	10000	7914	163	10000	3140	10000	9957	511	10000	8002
	COMDET	10000	9941	9936	10000	6279	10000	10000	10000	10000	10000

power analysis indicates that for moderate to large sample sizes or moderate to large spatial correlations, always use the COMDET; but for low sample sizes and low spatial correlations, use the COSLOF. Further, when there are both positive and negative correlations use the COMDET and not the COSLOF. The COSLOF and COMDET performed similarly with no correlation with about ten rejected hypotheses. In neuroscience, voxels are most often positively correlated. In other applications, there may be both positive and negative correlations which could sum to zero and result in a COSLOF which is not significant but a COMDET which is very significant.

8 Conclusion

The COMDET, a new measure of functional synchrony was presented and compared with the previous measure, the COSLOF. For a simulated data set, both measures of functional synchrony were computed and found the correlation between the voxels to be significant, but the COMDET declared it to be significant at a much higher level. A power analysis of sensitivity to different correlation structures was performed with the aid of Monte Carlo generated critical values for significance. It was found that for moderate to large sample sizes or moderate to large correlations, the COMDET should be used, but for low sample sizes and low correlations the COSLOF may be prefered. It was also found that when there are both positive and negative correlations, the COMDET should be used.

9 Appendix: Population Hypotheses

Suppose we now wish to test hypotheses regarding the COSLOF and COMDET for a population or the difference in populations. Let $s = 1, \ldots, S$ denote the subjects in a population. Denote the estimated COSLOF and COMDET for subject s by $\hat{\rho}_s$ and $\hat{\varrho}_s$. For a single population, there are two functional synchrony hypotheses which can be evaluated. Associated with the COSLOF, is the hypothesis

$$\begin{array}{rrcl}H_0: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & \text{diag}(D_s) & = & \mathbb{R}^{p_s+} \\ & \rho_s & = & \rho_{s'}\end{array} \quad vs \quad \begin{array}{rrcl}H_1: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & \text{diag}(D_s) & = & \mathbb{R}^{p_s+} \\ & \rho_s & \neq & \rho_{s'}\end{array}$$

that the average cross-correlation for the subjects in the population is the same with test statistic $\hat{u} = \sum_{s=1}^{S} \hat{u}_s$. Asymptotically, $\hat{u} \sim \chi^2(S)$ where S is the number of subjects. The large degrees of freedom approximation $\hat{u} \sim N(S, 2S)$ can also be used and the test statistic $Z = \frac{\hat{u}-S}{\sqrt{2S}}$. Associated with the COMDET is the hypothesis

$$\begin{array}{rrcl}H_0: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & \Sigma_s & = & diag(\Sigma_s)\end{array} \quad vs \quad \begin{array}{rrcl}H_1: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & \Sigma_s & > & 0\end{array}$$

that the voxels for the subjects in the population are independent with test statistic $\hat{v} = \sum_{s=1}^{S} \hat{v}_s$. Asymptotically, $\hat{v} \sim \chi^2(\gamma)$ where $\gamma = \sum_{s=1}^{S} \gamma_s$ and $\gamma_s = p_s(p_s-1)/2$. The large degrees of freedom result that $\hat{v} \sim N(\gamma, 2\gamma)$ and the test statistic $z = \frac{\hat{v}-\gamma}{\sqrt{2\gamma}}$ can also be used.

Hypotheses can be evaluated for two populations, say a and b. If $\hat{u}_a$ and $\hat{v}_a$ denote the above population a statistics while $\hat{u}_b$ and $\hat{v}_b$ denote the above population b statistics then there are two functional synchrony hypotheses which can be evaluated. Associated with the COSLOF, is the hypothesis

$$\begin{array}{rrcl}H_0: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & B_t & \in & \mathbb{R}^{(q+1)\times p_t} \\ & \text{diag}(D_s) & = & \mathbb{R}^{p_s+} \\ & \text{diag}(D_t) & = & \mathbb{R}^{p_t+} \\ & \rho_a & = & \rho_b\end{array} \quad vs \quad \begin{array}{rrcl}H_1: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & B_t & \in & \mathbb{R}^{(q+1)\times p_t} \\ & \text{diag}(D_s) & = & \mathbb{R}^{p_s+} \\ & \text{diag}(D_t) & = & \mathbb{R}^{p_t+} \\ & \rho_a & \neq & \rho_b\end{array}$$

that the voxels in the populations have the same correlation, with test statistic $-2\ln\lambda$ which has an approximate χ^2 distribution with a single degree of freedom, where

$$\lambda = \frac{\prod_{s=1}^{S} p(Y_s|\tilde{B}_s, \tilde{D}_s, \tilde{\rho}, X) \prod_{t=1}^{T} p(Y_t|\tilde{B}_t, \tilde{D}_t, \tilde{\rho}, X)}{\prod_{s=1}^{S} p(Y_s|\hat{B}_s, \hat{D}_s, \hat{\rho}_a, X) \prod_{t=1}^{T} p(Y_t|\hat{B}_t, \hat{D}_t, \hat{\rho}_b, X)},$$

where

$$\hat{\rho}_a = \tfrac{1}{S}\sum_{s=1}^{S} \bar{\rho}_s, \quad \hat{\rho}_b = \tfrac{1}{T}\sum_{t=1}^{T} \bar{\rho}_t, \quad \text{and} \quad \tilde{\rho} = \tfrac{S\hat{\rho}_a + T\hat{\rho}_b}{S+T}.$$

Associated with the COMDET is the hypothesis

$$\begin{array}{rrcl}H_0: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & B_t & \in & \mathbb{R}^{(q+1)\times p_t} \\ & \Sigma_b - \Sigma_a & > & 0\end{array} \quad vs \quad \begin{array}{rrcl}H_1: & B_s & \in & \mathbb{R}^{(q+1)\times p_s} \\ & B_t & \in & \mathbb{R}^{(q+1)\times p_t} \\ & \Sigma_b - \Sigma_a & \not> & 0\end{array}$$

where $\Sigma_b - \Sigma_a$ denotes that the voxels for the subjects in the population a are more independent than those in population b. The test statistic

$$F = \frac{\hat{v}_a/\gamma_a}{\hat{v}_b/\gamma_b}$$

which is the ratio of two approximate χ^2 variables divided by their degrees of freedom has an approximate F distribution with γ_a and γ_a numerator and denominator degrees of freedom.

Tables of Critical Values

Monte Carlo based critical values for ρ, u, and v for specific combinations of ν and p are presented. The tabulated critical values are percentages points for upper tail probabilities. Critical values for ϱ can be uniquely found from those for v.

Table 3: Critical 10% COSLOF values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	.9512																							
3	.8006	.4998																						
4	.6878	.4167	.2892																					
5	.6079	.3629	.2549	.1951																				
6	.5511	.3272	.2304	.1771	.1440																			
7	.5068	.3000	.2119	.1637	.1326	.1121																		
8	.4709	.2783	.1966	.1519	.1242	.1047	.0906																	
9	.4437	.2616	.1848	.1431	.1165	.0984	.0850	.0748																
10	.4185	.2466	.1738	.1350	.1101	.0930	.0805	.0710	.0636															
11	.3980	.2347	.1658	.1282	.1048	.0884	.0767	.0676	.0603	.0545														
12	.3801	.2236	.1586	.1227	.1002	.0845	.0732	.0645	.0578	.0522	.0475													
13	.3647	.2143	.1522	.1175	.0961	.0810	.0702	.0620	.0554	.0500	.0456	.0420												
14	.3502	.2058	.1459	.1133	.0924	.0781	.0677	.0597	.0534	.0484	.0440	.0404	.0375											
15	.3377	.1986	.1405	.1092	.0892	.0754	.0653	.0575	.0515	.0465	.0424	.0391	.0362	.0337										
16	.3275	.1921	.1361	.1056	.0862	.0727	.0630	.0556	.0498	.0449	.0410	.0379	.0350	.0326	.0305									
17	.3167	.1858	.1321	.1026	.0839	.0708	.0611	.0539	.0481	.0436	.0397	.0367	.0339	.0315	.0295	.0278								
18	.3080	.1803	.1282	.0993	.0810	.0685	.0593	.0523	.0468	.0423	.0387	.0355	.0328	.0307	.0287	.0270	.0254							
19	.2994	.1756	.1244	.0965	.0787	.0666	.0577	.0509	.0455	.0412	.0376	.0345	.0320	.0298	.0279	.0262	.0246	.0234						
20	.2909	.1712	.1213	.0938	.0769	.0649	.0561	.0495	.0444	.0401	.0366	.0336	.0312	.0291	.0271	.0255	.0240	.0227	.0215					
25	.2604	.1521	.1077	.0836	.0683	.0578	.0501	.0443	.0395	.0358	.0326	.0300	.0277	.0258	.0242	.0228	.0214	.0203	.0192	.0183	.0175	.0167	.0160	.0153
50	.1826	.1065	.0754	.0586	.0480	.0405	.0352	.0310	.0277	.0250	.0229	.0211	.0195	.0182	.0170	.0160	.0150	.0142	.0135	.0128	.0122	.0117	.0112	.0107
75	.1488	.0866	.0613	.0476	.0389	.0330	.0286	.0252	.0225	.0204	.0186	.0171	.0158	.0147	.0138	.0130	.0123	.0116	.0110	.0104	.0122	.0117	.0091	.0087
100	.1285	.0748	.0531	.0412	.0336	.0285	.0247	.0217	.0194	.0176	.0161	.0148	.0137	.0127	.0119	.0112	.0106	.0100	.0095	.0090	.0086	.0082	.0079	.0075
125	.1151	.0667	.0474	.0367	.0301	.0254	.0220	.0194	.0174	.0158	.0143	.0132	.0122	.0114	.0106	.0100	.0094	.0089	.0085	.0080	.0077	.0074	.0070	.0067
150	.1048	.0609	.0433	.0335	.0274	.0232	.0200	.0177	.0158	.0144	.0131	.0121	.0111	.0104	.0097	.0091	.0086	.0081	.0077	.0074	.0070	.0067	.0064	.0061
175	.0971	.0564	.0400	.0310	.0254	.0214	.0186	.0164	.0147	.0132	.0121	.0111	.0103	.0096	.0090	.0084	.0079	.0075	.0071	.0068	.0065	.0062	.0059	.0057
200	.0908	.0527	.0374	.0290	.0236	.0201	.0173	.0153	.0137	.0124	.0113	.0104	.0096	.0090	.0084	.0079	.0074	.007	.0067	.0064	.0061	.0058	.0055	.0053

Table 4: Critical 5% COSLOF values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	.9877																							
3	.9001	.6430																						
4	.8064	.5402	.3921																					
5	.7296	.4730	.3438	.2682																				
6	.6697	.4272	.3098	.2414	.1977																			
7	.6211	.3913	.2836	.2225	.1813	.1537																		
8	.5816	.3633	.2629	.2053	.1686	.1429	.1241																	
9	.5499	.3417	.2465	.1926	.1579	.1340	.1162	.1025																
10	.5205	.3218	.2319	.1814	.1487	.1263	.1095	.0969	.0869															
11	.4965	.3067	.2202	.1720	.1418	.1195	.1039	.0919	.0821	.0745														
12	.4761	.2919	.2105	.1642	.1349	.1144	.0993	.0875	.0787	.0711	.0648													
13	.4576	.2797	.2018	.1572	.1293	.1092	.0950	.0841	.0751	.0680	.0623	.0571												
14	.4401	.2683	.1936	.1513	.1238	.1049	.0915	.0807	.0723	.0654	.0597	.0550	.0509											
15	.4253	.2588	.1858	.1457	.1196	.1014	.0881	.0777	.0696	.0630	.0575	.0529	.0491	.0457										
16	.4127	.2509	.1801	.1409	.1152	.0979	.0848	.0751	.0673	.0607	.0554	.0513	.0473	.0442	.0413									
17	.3995	.2422	.1745	.1361	.1119	.0948	.0822	.0726	.0648	.0588	.0537	.0495	.0457	.0426	.0399	.0376								
18	.3891	.2350	.1691	.1320	.1083	.0920	.0795	.0704	.0630	.0571	.0520	.0481	.0444	.0414	.0388	.0365	.0343							
19	.3780	.2287	.1641	.1284	.1050	.0891	.0773	.0685	.0613	.0555	.0507	.0466	.0432	.0402	.0377	.0354	.0333	.0315						
20	.3681	.2230	.1599	.1247	.1024	.0866	.0750	.0664	.0595	.0538	.0493	.0452	.0420	.0392	.0365	.0344	.0325	.0307	.0291					
25	.3305	.1983	.1418	.1106	.0905	.0770	.0667	.0591	.0529	.0479	.0437	.0401	.0371	.0347	.0324	.0306	.0288	.0272	.0259	.0247	.0234	.0224	.0215	.0206
50	.2333	.1383	.0986	.0768	.0631	.0534	.0464	.0410	.0366	.0331	.0303	.0278	.0258	.0241	.0225	.0212	.0199	.0188	.0178	.0170	.0162	.0155	.0148	.0142
75	.1903	.1123	.0800	.0622	.0510	.0432	.0375	.0331	.0366	.0268	.0245	.0225	.0208	.0194	.0182	.0171	.0161	.0153	.0145	.0138	.0162	.0126	.0121	.0115
100	.1644	.0969	.0691	.0536	.0440	.0372	.0323	.0285	.0254	.0231	.0211	.0194	.0179	.0167	.0156	.0147	.0138	.0131	.0124	.0118	.0113	.0108	.0103	.0099
125	.1472	.0865	.0615	.0478	.0392	.0331	.0287	.0254	.0227	.0206	.0188	.0172	.0160	.0149	.0139	.0131	.0123	.0117	.0111	.0105	.0100	.0096	.0092	.0088
150	.1345	.0789	.0562	.0436	.0357	.0302	.0261	.0231	.0207	.0187	.0171	.0158	.0145	.0136	.0127	.0119	.0113	.0106	.0101	.0096	.0092	.0088	.0084	.0080
175	.1244	.0728	.0519	.0402	.0330	.0279	.0242	.0213	.0191	.0173	.0158	.0145	.0134	.0125	.0117	.0110	.0104	.0098	.0093	.0089	.0085	.0081	.0077	.0074
200	.1164	.0680	.0483	.0376	.0308	.0260	.0226	.0199	.0178	.0162	.0148	.0136	.0126	.0117	.0109	.0103	.0097	.0092	.0087	.0083	.0079	.0075	.0072	.0069

Table 5: Critical 2.5% COSLOF values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	.9969																							
3	.9500	.7461																						
4	.8792	.6375	.4796																					
5	.8113	.5617	.4209	.3330																				
6	.7559	.5097	.3781	.2989	.2469																			
7	.7059	.4679	.3464	.2741	.2256	.1920																		
8	.6652	.4357	.3208	.2531	.2090	.1782	.1546																	
9	.6317	.4093	.3004	.2372	.1954	.1663	.1446	.1281																
10	.6022	.3858	.2824	.2230	.1840	.1565	.1358	.1205	.1085															
11	.5757	.3681	.2685	.2113	.1747	.1479	.1291	.1142	.1022	.0930														
12	.5525	.3506	.2559	.2017	.1663	.1414	.1231	.1088	.0976	.0884	.0807													
13	.5324	.3360	.2451	.1921	.1590	.1348	.1176	.1041	.0933	.0842	.0773	.0710												
14	.5139	.3227	.2356	.1850	.1524	.1294	.1131	.0996	.0896	.0812	.0740	.0683	.0634											
15	.4961	.3109	.2263	.1785	.1468	.1247	.1085	.0960	.0861	.0781	.0714	.0656	.0609	.0567										
16	.4815	.3010	.2187	.1723	.1416	.1203	.1046	.0926	.0830	.0753	.0687	.0635	.0586	.0549	.0513									
17	.4676	.2916	.2117	.1661	.1370	.1164	.1011	.0895	.0800	.0728	.0665	.0612	.0566	.0528	.0495	.0468								
18	.4558	.2828	.2055	.1615	.1330	.1128	.0979	.0868	.0777	.0704	.0642	.0595	.0549	.0513	.0479	.0452	.0425							
19	.4430	.2748	.1994	.1563	.1285	.1092	.0950	.0842	.0755	.0684	.0626	.0575	.0535	.0498	.0465	.0438	.0411	.0390						
20	.4324	.2681	.1940	.1519	.1254	.1064	.0921	.0816	.0733	.0665	.0607	.0559	.0519	.0484	.0452	.0425	.0402	.0381	.0360					
25	.3891	.2383	.1720	.1347	.1107	.0944	.0817	.0724	.0650	.0590	.0538	.0493	.0458	.0426	.0400	.0375	.0354	.0336	.0319	.0304	.0289	.0276	.0265	.0254
50	.2759	.1659	.1191	.0929	.0764	.0649	.0564	.0498	.0444	.0404	.0368	.0339	.0315	.0293	.0274	.0258	.0243	.0229	.0218	.0207	.0198	.0189	.0181	.0174
75	.2263	.1346	.0966	.0752	.0617	.0524	.0455	.0401	.0359	.0326	.0298	.0274	.0253	.0236	.0221	.0208	.0196	.0185	.0176	.0167	.0160	.0153	.0146	.0140
100	.1951	.1160	.0832	.0647	.0530	.0449	.0391	.0344	.0308	.0279	.0255	.0235	.0217	.0203	.0189	.0178	.0168	.0159	.0151	.0143	.0137	.0131	.0125	.0120
125	.1752	.1035	.0740	.0576	.0473	.0399	.0347	.0307	.0275	.0248	.0227	.0209	.0194	.0181	.0168	.0158	.0149	.0142	.0134	.0128	.0122	.0116	.0111	.0107
150	.1601	.0946	.0674	.0524	.0430	.0364	.0315	.0279	.0250	.0226	.0207	.0190	.0176	.0164	.0153	.0144	.0136	.0129	.0122	.0116	.0111	.0106	.0101	.0097
175	.1482	.0872	.0623	.0483	.0397	.0336	.0292	.0257	.0231	.0209	.0190	.0175	.0162	.0151	.0142	.0133	.0125	.0119	.0113	.0107	.0102	.0098	.0093	.0090
200	.1387	.0815	.0580	.0452	.0370	.0314	.0272	.0240	.0215	.0195	.0178	.0164	.0152	.0141	.0132	.0124	.0117	.0111	.0105	.0100	.0095	.0091	.0087	.0083

Table 6: Critical 1% COSLOF values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	.9995																							
3	.9797	.8386																						
4	.9349	.7348	.5769																					
5	.8820	.6551	.5071	.4088																				
6	.8333	.5993	.4578	.3663	.3051																			
7	.7892	.5515	.4177	.3347	.2787	.2378																		
8	.7490	.5154	.3884	.3093	.2572	.2198	.1922																	
9	.7153	.4853	.3626	.2902	.2406	.2049	.1793	.1598																
10	.6854	.4594	.3421	.2725	.2261	.1929	.1680	.1494	.1345															
11	.6590	.4372	.3242	.2582	.2144	.1828	.1592	.1414	.1270	.1155														
12	.6334	.4163	.3102	.2458	.2039	.1740	.1518	.1350	.1207	.1100	.1003													
13	.6119	.4002	.2962	.2339	.1944	.1659	.1450	.1284	.1154	.1044	.0957	.0881												
14	.5925	.3846	.2850	.2250	.1867	.1589	.1393	.1228	.1102	.1003	.0918	.0845	.0784											
15	.5731	.3702	.2736	.2168	.1799	.1530	.1335	.1182	.1061	.0963	.0882	.0811	.0754	.0702										
16	.5573	.3586	.2645	.2096	.1729	.1474	.1285	.1139	.1022	.0928	.0848	.0783	.0725	.0677	.0635									
17	.5411	.3476	.2555	.2019	.1673	.1425	.1242	.1100	.0985	.0896	.0817	.0754	.0699	.0653	.0612	.0578								
18	.5289	.3375	.2482	.1959	.1625	.1379	.1203	.1066	.0955	.0865	.0791	.0734	.0678	.0632	.0591	.0558	.0524							
19	.5143	.3281	.2407	.1895	.1569	.1339	.1164	.1032	.0927	.0842	.0770	.0709	.0660	.0613	.0575	.0542	.0507	.0483						
20	.5037	.3202	.2344	.1847	.1527	.1297	.1129	.0999	.0899	.0814	.0746	.0690	.0638	.0596	.0556	.0524	.0496	.0469	.0445					
25	.4549	.2846	.2076	.1634	.1350	.1149	.1000	.0885	.0796	.0722	.0661	.0605	.0561	.0524	.0491	.0462	.0436	.0413	.0392	.0373	.0357	.0339	.0326	.0312
50	.3250	.1982	.1435	.1120	.0924	.0785	.0683	.0603	.0539	.0489	.0448	.0412	.0383	.0357	.0334	.0315	.0296	.0279	.0265	.0253	.0241	.0231	.0221	.0212
75	.2670	.1608	.1160	.0904	.0744	.0633	.0549	.0485	.0435	.0394	.0361	.0331	.0306	.0286	.0268	.0252	.0237	.0225	.0214	.0203	.0193	.0185	.0178	.0170
100	.2306	.1385	.0998	.0777	.0639	.0542	.0471	.0415	.0372	.0337	.0308	.0284	.0262	.0244	.0229	.0215	.0203	.0192	.0182	.0173	.0165	.0159	.0151	.0145
125	.2075	.1238	.0887	.0690	.0569	.0479	.0417	.0370	.0331	.0299	.0274	.0252	.0233	.0218	.0203	.0191	.0180	.0171	.0162	.0154	.0147	.0140	.0134	.0129
150	.1893	.1127	.0806	.0629	.0517	.0438	.0378	.0336	.0300	.0272	.0249	.0230	.0212	.0198	.0185	.0173	.0164	.0155	.0147	.0140	.0133	.0127	.0122	.0117
175	.1757	.1039	.0747	.0578	.0477	.0402	.0350	.0309	.0278	.0251	.0228	.0210	.0195	.0182	.0170	.0160	.0151	.0143	.0136	.0129	.0123	.0118	.0113	.0108
200	.1644	.0972	.0693	.0542	.0444	.0376	.0326	.0288	.0258	.0234	.0214	.0197	.0182	.0170	.0159	.0149	.0140	.0133	.0126	.0120	.0115	.0109	.0105	.0100

Table 7: Critical .1% COSLOF values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	1.0000																							
3	.9980	.9486																						
4	.9860	.8765	.7471																					
5	.9627	.8097	.6677	.5601																				
6	.9350	.7511	.6088	.5078	.4274																			
7	.9053	.7033	.5603	.4592	.3928	.3388																		
8	.8758	.6613	.5237	.4272	.3609	.3120	.2752																	
9	.8459	.6292	.4884	.4042	.3373	.2907	.2561	.2293																
10	.8216	.5979	.4635	.3758	.3179	.2725	.2391	.2131	.1930															
11	.7972	.5711	.4415	.3555	.2983	.2583	.2263	.2033	.1819	.1657														
12	.7705	.5483	.4220	.3401	.2850	.2437	.2146	.1918	.1714	.1579	.1440													
13	.7504	.5277	.4025	.3230	.2705	.2345	.2047	.1824	.1645	.1498	.1375	.1265												
14	.7308	.5087	.3875	.3099	.2597	.2242	.1965	.1742	.1580	.1435	.1315	.1208	.1123											
15	.7117	.4908	.3727	.3001	.2502	.2150	.1892	.1675	.1517	.1383	.1255	.1166	.1075	.1011										
16	.6946	.4775	.3613	.2888	.2413	.2070	.1823	.1615	.1456	.1321	.1216	.1119	.1038	.0965	.0914									
17	.6768	.4605	.3476	.2783	.2329	.2000	.1755	.1559	.1400	.1278	.1165	.1075	.0994	.0938	.0874	.0823								
18	.6627	.4463	.3376	.2705	.2261	.1929	.1696	.1501	.1348	.1226	.1122	.1045	.0968	.0905	.0848	.0796	.0749							
19	.6472	.4396	.3289	.2620	.2182	.1878	.1640	.1450	.1311	.1190	.1092	.1007	.0931	.0871	.0820	.0770	.0727	.0690						
20	.6349	.4271	.3203	.2548	.2115	.1815	.1586	.1406	.1265	.1162	.1064	.0980	.0904	.0844	.0791	.0746	.0707	.0670	.0634					
25	.5795	.3810	.2819	.2252	.1871	.1604	.1400	.1242	.1116	.1011	.0933	.0854	.0792	.0741	.0695	.0655	.0619	.0583	.0560	.0528	.0505	.0485	.0464	.0445
50	.4229	.2672	.1945	.1529	.1272	.1079	.0943	.0834	.0747	.0679	.0623	.0574	.0533	.0498	.0465	.0438	.0411	.0388	.0370	.0353	.0335	.0323	.0310	.0295
75	.3493	.2162	.1568	.1237	.1019	.0865	.0754	.0665	.0598	.0542	.0497	.0455	.0425	.0394	.0369	.0346	.0329	.0311	.0296	.0280	.0266	.0255	.0246	.0235
100	.3030	.1868	.1350	.1062	.0868	.0741	.0644	.0568	.0508	.0460	.0424	.0388	.0362	.0336	.0315	.0296	.0279	.0265	.0249	.0237	.0227	.0218	.0210	.0201
125	.2719	.1665	.1191	.0939	.0773	.0654	.0571	.0507	.0454	.0408	.0374	.0345	.0321	.0298	.0279	.0263	.0247	.0236	.0222	.0212	.0203	.0193	.0185	.0177
150	.2492	.1510	.1090	.0849	.0703	.0598	.0518	.0456	.0410	.0370	.0339	.0313	.0290	.0269	.0254	.0236	.0224	.0212	.0202	.0191	.0183	.0175	.0168	.0160
175	.2316	.1395	.1001	.0782	.0643	.0548	.0476	.0419	.0380	.0342	.0310	.0288	.0266	.0248	.0233	.0219	.0206	.0195	.0185	.0176	.0168	.0161	.0153	.0146
200	.2183	.1301	.0934	.0730	.0603	.0512	.0443	.0391	.0349	.0318	.0291	.0267	.0247	.0231	.0216	.0204	.0191	.0181	.0173	.0164	.0156	.0149	.0143	.0137

Table 8: Critical 10% v values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	1.85																							
3	2.49	6.38																						
4	2.61	6.26	12.62																					
5	2.66	6.24	11.31	20.39																				
6	2.68	6.24	10.98	17.67	29.70																			
7	2.68	6.25	10.84	16.88	25.34	40.47																		
8	2.69	6.25	10.76	16.57	24.04	34.40	52.66																	
9	2.70	6.25	10.74	16.36	23.43	32.42	44.76	66.25																
10	2.69	6.24	10.72	16.28	23.09	31.46	42.09	56.47	81.28															
11	2.70	6.24	10.68	16.20	22.92	31.00	40.76	52.99	69.46	97.69														
12	2.70	6.25	10.69	16.17	22.77	30.68	40.01	51.27	65.15	83.91	115.50													
13	2.70	6.25	10.69	16.14	22.66	30.43	39.53	50.25	62.98	78.64	99.46	134.67												
14	2.70	6.24	10.68	16.09	22.62	30.27	39.18	49.59	61.65	75.96	93.42	116.50	155.36											
15	2.70	6.25	10.67	16.10	22.56	30.14	38.97	49.09	60.75	74.21	90.13	109.43	134.90	177.22										
16	2.70	6.24	10.67	16.08	22.53	30.08	38.77	48.80	60.13	73.17	88.10	105.55	126.81	154.57	200.66									
17	2.70	6.24	10.66	16.06	22.50	29.99	38.65	48.51	59.75	72.33	86.69	103.14	122.22	145.40	175.65	225.44								
18	2.71	6.25	10.66	16.05	22.47	29.96	38.54	48.32	59.36	71.77	85.66	101.42	119.37	140.29	165.40	197.83	251.45							
19	2.70	6.25	10.65	16.02	22.46	29.90	38.47	48.20	59.09	71.30	84.96	100.27	117.44	136.83	159.48	186.61	221.72	279.04						
20	2.70	6.25	10.65	16.02	22.41	29.86	38.39	48.04	58.85	70.98	84.40	99.25	115.94	134.59	155.59	179.94	208.98	246.63	307.79					
25	2.71	6.26	10.65	16.04	22.36	29.78	38.19	47.68	58.24	69.96	82.82	96.93	112.22	129.00	147.21	167.01	188.61	212.34	238.55	267.95	301.38	341.12	391.55	472.51
50	2.71	6.25	10.64	15.99	22.30	29.64	37.97	47.32	57.65	68.98	81.40	94.80	109.30	124.73	141.35	158.96	177.55	197.42	218.23	240.19	263.43	287.68	313.08	339.62
75	2.71	6.25	10.65	16.00	22.32	29.63	37.93	47.23	57.59	68.88	81.20	94.55	108.88	94.55	140.62	158.07	176.46	195.92	216.46	238.02	260.65	284.26	308.94	334.65
100	2.70	6.24	10.64	15.98	22.31	29.64	37.93	47.22	57.53	68.86	81.18	94.48	108.80	124.09	140.49	157.88	176.23	195.55	215.89	237.37	259.73	283.28	307.76	333.25
125	2.71	6.25	10.65	15.98	22.32	29.60	37.92	47.22	57.50	68.85	81.10	94.43	108.76	124.07	140.40	157.64	176.00	195.36	215.75	237.10	259.48	282.84	307.31	332.66
150	2.71	6.25	10.66	15.99	22.31	29.59	37.91	47.24	57.50	68.83	81.11	94.42	108.70	124.02	140.31	157.66	176.01	195.29	215.55	236.98	259.29	282.66	307.04	332.47
175	2.70	6.26	10.65	15.97	22.30	29.60	37.92	47.26	57.53	68.83	81.08	94.39	108.66	123.95	140.29	157.56	175.97	195.24	215.56	236.81	259.17	282.50	306.80	332.27
200	2.70	6.24	10.65	16.00	22.28	29.62	37.92	47.24	57.52	68.80	81.11	94.41	108.67	123.97	140.31	157.54	175.85	195.19	215.49	236.70	259.09	282.41	306.77	332.07
∞	2.71	6.25	10.64	15.99	22.31	29.62	37.92	47.21	57.51	68.80	81.09	94.37	108.66	123.95	140.23	157.52	175.80	195.09	215.37	236.65	258.94	282.22	306.51	331.79

Table 9: Critical 5% v values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2.54																							
3	3.49	8.02																						
4	3.69	7.83	15.25																					
5	3.77	7.80	13.47	23.96																				
6	3.80	7.81	13.03	20.45	34.33																			
7	3.80	7.80	12.85	19.42	28.73	46.05																		
8	3.80	7.80	12.74	19.04	27.11	38.41	59.25																	
9	3.82	7.81	12.71	18.79	26.36	36.02	49.45	73.83																
10	3.83	7.80	12.68	18.67	25.94	34.88	46.25	61.79	89.97															
11	3.83	7.79	12.64	18.58	25.72	34.31	44.65	57.64	75.40	107.39														
12	3.84	7.82	12.65	18.54	25.57	33.92	43.80	55.66	70.37	90.60	126.20													
13	3.83	7.81	12.65	18.49	25.43	33.68	43.21	54.46	67.86	84.44	106.80	146.45												
14	3.83	7.80	12.64	18.45	25.37	33.45	42.78	53.70	66.38	81.35	99.79	124.57	168.18											
15	3.83	7.81	12.62	18.44	25.29	33.28	42.58	53.12	65.30	79.39	96.04	116.36	143.61	191.02										
16	3.83	7.80	12.62	18.41	25.24	33.23	42.32	52.80	64.63	78.21	93.74	111.99	134.33	164.03	215.34									
17	3.83	7.81	12.62	18.40	25.22	33.12	42.19	52.47	64.19	77.23	92.18	109.28	129.16	153.51	185.79	241.28								
18	3.84	7.81	12.60	18.38	25.17	33.06	42.04	52.27	63.73	76.61	91.03	107.36	125.97	147.81	174.02	208.61	268.29							
19	3.83	7.81	12.61	18.37	25.19	33.03	41.95	52.09	63.43	76.11	90.23	106.06	123.84	143.95	167.54	195.92	233.19	296.84						
20	3.83	7.81	12.58	18.35	25.13	32.95	41.88	51.94	63.17	75.74	89.58	104.96	122.20	141.47	163.24	188.47	218.89	258.81	326.79					
25	3.85	7.82	12.60	18.36	25.06	32.87	41.63	51.52	62.45	74.60	87.87	102.40	118.17	135.38	154.07	174.41	196.59	220.95	247.88	278.22	312.71	354.13	407.309	496.95
50	3.84	7.80	12.57	18.31	25.00	32.71	41.40	51.11	61.81	73.51	86.33	100.08	114.97	130.76	147.78	165.76	184.68	204.94	226.19	248.54	272.16	296.84	322.64	349.58
75	3.85	7.81	12.59	18.31	25.00	32.70	41.35	51.00	61.73	73.41	86.09	99.77	114.51	130.30	147.02	164.74	183.57	203.44	224.28	246.19	269.29	293.20	318.31	344.44
100	3.83	7.80	12.59	18.29	24.98	32.69	41.39	51.03	61.66	73.35	86.07	99.70	114.44	130.07	146.79	164.65	183.32	203.05	223.76	245.59	268.30	292.23	317.06	342.89
125	3.84	7.81	12.59	18.29	25.03	32.64	41.37	51.00	61.67	73.36	86.01	99.66	114.36	130.02	146.75	164.32	183.09	202.76	223.55	245.31	267.99	291.79	316.55	342.29
150	3.85	7.83	12.62	18.30	24.99	32.64	41.30	51.00	61.66	73.38	86.01	99.71	114.30	130.04	146.62	164.36	183.04	202.74	223.35	245.14	267.83	291.53	316.29	342.17
175	3.84	7.83	12.61	18.30	24.99	32.64	41.34	51.01	61.69	73.36	85.95	99.62	114.32	129.95	146.64	164.25	183.04	202.68	223.36	244.99	267.67	291.41	316.05	341.95
200	3.85	7.81	12.60	18.33	24.96	32.68	41.33	51.02	61.66	73.31	85.98	99.65	114.27	129.96	146.66	164.30	182.95	202.66	223.28	244.95	267.62	291.27	315.96	341.71
∞	3.84	7.81	12.59	18.31	25.00	32.67	41.34	51.00	61.66	73.31	85.96	99.62	114.27	129.92	146.57	164.22	182.86	202.51	223.16	244.81	267.45	291.10	315.75	341.40

Table 10: Critical 2.5% v values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	3.23																							
3	4.51	9.64																						
4	4.80	9.36	17.82																					
5	4.90	9.35	15.56	27.51																				
6	4.94	9.35	15.01	23.08	38.82																			
7	4.96	9.34	14.80	21.86	31.97	51.52																		
8	4.97	9.33	14.65	21.38	30.00	42.25	65.67																	
9	5.00	9.35	14.60	21.10	29.12	39.41	53.83	81.16																
10	5.01	9.34	14.56	20.93	28.61	38.00	50.09	66.77	98.33															
11	5.02	9.34	14.53	20.83	28.35	37.40	48.23	61.98	80.97	116.74														
12	5.02	9.35	14.54	20.78	28.17	36.91	47.21	59.70	75.14	96.76	136.47													
13	5.01	9.34	14.53	20.70	28.01	36.65	46.56	58.33	72.28	89.81	113.57	157.87												
14	5.01	9.32	14.50	20.65	27.92	36.38	46.10	57.47	70.66	86.30	105.64	131.96	180.37											
15	5.00	9.34	14.48	20.64	27.86	36.18	45.87	56.83	69.44	84.11	101.37	122.75	151.60	204.35										
16	5.02	9.34	14.48	20.61	27.77	36.11	45.62	56.42	68.77	82.74	98.89	117.84	141.11	172.70	229.55									
17	5.00	9.34	14.46	20.60	27.78	36.01	45.39	56.07	68.20	81.66	97.10	114.86	135.39	160.88	195.10	256.49								
18	5.02	9.34	14.46	20.58	27.71	35.91	45.24	55.83	67.67	80.99	95.90	112.71	131.93	154.48	181.96	218.57	284.41							
19	5.02	9.38	14.47	20.56	27.74	35.89	45.17	55.64	67.35	80.37	94.98	111.32	129.67	150.41	174.78	204.31	243.80	313.83						
20	5.02	9.34	14.45	20.54	27.64	35.81	45.13	55.50	67.09	80.00	94.27	110.09	127.83	147.66	170.06	196.13	227.76	269.95	345.09					
25	5.03	9.34	14.46	20.54	27.56	35.69	44.77	55.02	65.31	78.79	92.40	107.35	123.50	141.06	160.21	181.07	203.79	209.10	230.06	287.32	322.93	365.85	421.68	520.11
50	5.01	9.35	14.44	20.51	27.49	35.52	44.55	54.56	65.59	77.61	90.75	104.80	120.04	136.08	153.54	191.16	211.69	233.31	255.98	255.98	279.87	304.88	331.12	358.45
75	5.02	9.34	14.45	20.47	27.50	35.52	44.50	54.42	65.48	77.45	90.46	104.47	119.56	135.67	152.71	170.77	189.90	210.08	231.18	253.47	276.86	301.17	326.58	353.04
100	5.01	9.32	14.45	20.49	27.46	35.50	44.52	54.50	65.42	77.47	90.51	104.44	119.44	135.45	152.47	170.62	189.59	209.65	230.68	252.84	275.85	300.13	325.27	351.38
125	5.03	9.34	14.46	20.48	27.54	35.45	44.56	54.42	65.45	77.43	90.39	104.37	119.37	135.40	152.34	170.35	189.47	209.39	230.42	252.51	275.59	299.69	324.68	350.82
150	5.05	9.37	14.48	20.48	27.49	35.48	44.43	54.43	65.41	77.47	90.37	104.46	119.34	135.36	152.33	170.34	189.29	209.34	230.21	252.42	275.35	299.39	324.46	350.65
175	5.03	9.35	14.46	20.45	27.48	35.46	44.48	54.44	65.44	77.43	90.31	104.23	119.33	135.23	152.31	170.20	189.31	209.37	230.27	252.14	275.27	299.25	324.22	350.44
200	5.04	9.36	14.47	20.50	27.43	35.52	44.45	54.50	65.40	77.37	90.32	104.35	119.26	135.38	152.36	170.21	189.23	209.23	230.15	252.17	275.14	299.12	324.12	350.12
∞	5.02	9.35	14.45	20.48	27.49	35.48	44.46	54.44	65.41	77.38	90.35	104.32	119.28	135.25	152.21	170.18	189.14	209.10	230.06	252.03	274.99	298.95	323.91	349.87

Table 11: Critical 1% v values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	4.15																							
3	5.86	11.78																						
4	6.27	11.39	21.23																					
5	6.43	11.36	18.26	32.22																				
6	6.51	11.34	17.50	26.49	44.68																			
7	6.54	11.33	17.26	24.95	36.08	58.59																		
8	6.55	11.32	17.08	24.29	33.64	47.09	74.11																	
9	6.58	11.34	17.05	23.94	32.55	43.55	59.44	90.79																
10	6.64	11.31	16.94	23.76	31.93	41.90	54.86	73.03	109.27															
11	6.62	11.33	16.92	23.60	31.60	41.17	52.71	67.38	87.89	128.80														
12	6.61	11.33	16.92	23.53	31.42	40.63	51.42	64.59	81.09	104.53	149.80													
13	6.62	11.32	16.92	23.48	31.17	40.28	50.76	63.07	77.75	96.30	122.14	172.35												
14	6.60	11.32	16.89	23.43	31.09	40.00	50.15	62.12	75.91	92.39	112.75	141.16	196.44											
15	6.60	11.33	16.85	23.39	31.07	39.76	49.89	61.32	74.58	89.80	107.98	130.53	161.59	221.60										
16	6.61	11.33	16.86	23.33	30.93	39.64	49.56	60.90	73.72	88.27	104.98	124.95	149.55	183.51	248.10									
17	6.61	11.36	16.82	23.30	30.92	39.51	49.39	60.43	73.10	87.10	103.14	121.59	143.06	169.95	206.63	276.19								
18	6.64	11.34	16.82	23.36	30.81	39.45	49.23	60.19	72.48	86.38	101.79	119.21	139.06	162.59	191.53	230.60	305.62							
19	6.62	11.38	16.87	23.29	30.87	39.41	49.06	59.99	72.13	85.59	100.71	117.62	136.60	158.15	183.53	214.59	256.72	336.04						
20	6.66	11.38	16.80	23.27	30.76	39.29	49.04	59.80	71.87	85.19	99.97	116.31	134.62	155.18	178.31	205.56	238.54	284.06	368.52					
25	6.65	11.35	16.83	23.27	30.67	39.17	48.59	59.24	71.01	83.85	97.90	113.35	129.88	147.88	167.48	189.00	212.17	237.75	266.16	298.38	335.10	380.05	439.50	549.37
50	6.65	11.33	16.81	23.25	30.58	38.99	48.37	58.74	70.03	82.55	96.10	110.46	126.13	142.51	160.35	179.02	198.70	219.75	241.63	264.72	289.00	314.44	341.16	368.92
75	6.65	11.31	16.80	23.15	30.58	39.01	48.31	58.59	70.05	82.42	95.80	110.12	125.50	142.06	159.43	177.86	197.36	217.98	239.51	262.13	285.80	310.55	336.38	363.36
100	6.62	11.34	16.82	23.19	30.48	38.96	48.30	58.74	70.01	82.41	95.81	110.11	125.54	141.75	159.30	177.75	196.99	217.49	238.95	261.52	284.92	309.40	335.02	361.44
125	6.65	11.34	16.78	23.20	30.60	38.93	48.41	58.59	69.99	82.32	95.67	110.02	125.37	141.87	159.06	177.45	196.95	217.75	238.58	261.05	284.63	308.89	334.43	360.92
150	6.66	11.37	16.82	23.20	30.59	38.93	48.23	58.64	69.91	82.41	95.68	110.08	125.45	141.80	159.04	177.45	196.76	217.11	238.52	260.94	284.34	308.61	334.18	360.56
175	6.64	11.33	16.83	23.17	30.57	38.92	48.35	58.60	69.93	82.37	95.62	109.88	125.32	141.58	159.06	177.35	196.72	217.19	238.52	260.76	284.14	308.58	333.90	360.49
200	6.63	11.34	16.85	23.20	30.53	38.98	48.32	58.68	69.92	82.28	95.67	109.95	125.24	141.84	159.06	177.30	196.65	217.19	238.48	260.78	284.18	308.32	333.82	360.21
∞	6.63	11.34	16.81	23.21	30.58	38.93	48.28	58.62	69.96	82.29	95.63	109.96	125.29	141.62	158.95	177.28	196.61	216.94	238.27	260.59	283.92	308.25	333.58	359.91

Table 12: Critical .1% v values from sampling.

ν, p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	6.46																							
3	9.32	17.21																						
4	10.07	16.44	29.55																					
5	10.38	16.40	24.68	43.78																				
6	10.56	16.24	23.82	34.93	59.38																			
7	10.63	16.27	23.27	32.22	46.13	76.31																		
8	10.62	16.27	22.92	31.30	42.22	58.85	94.91																	
9	10.67	16.24	22.84	30.81	40.46	53.32	72.80	114.42																
10	10.78	16.28	22.73	30.40	39.65	50.93	66.05	87.85	136.60															
11	10.72	16.18	22.57	30.29	39.10	49.98	62.84	79.33	104.75	158.80														
12	10.81	16.25	22.76	30.19	38.77	49.16	61.08	76.00	94.82	122.73	184.06													
13	10.82	16.22	22.57	30.07	38.65	48.57	60.33	73.97	90.50	111.35	142.29	209.00												
14	10.82	16.15	22.60	30.01	38.51	48.17	59.16	72.64	87.84	106.11	128.93	162.64	235.84											
15	10.72	16.31	22.60	29.95	38.49	47.92	59.12	71.43	85.83	103.04	123.03	147.96	184.89	263.91										
16	10.92	16.34	22.53	29.80	38.23	47.75	58.71	70.95	84.75	100.63	119.18	141.22	168.62	208.76	293.55									
17	10.70	16.32	22.57	29.62	38.27	47.56	58.17	70.22	83.93	99.24	116.58	136.36	160.86	190.29	233.40	324.65								
18	10.84	16.34	22.55	29.78	38.12	47.36	58.23	69.98	83.48	97.92	114.90	134.01	155.25	180.80	213.30	258.86	356.56							
19	10.80	16.26	22.45	29.70	38.12	47.61	57.89	69.94	82.92	97.22	113.71	131.85	151.94	175.43	203.62	237.33	287.13	391.26						
20	10.96	16.39	22.50	29.77	37.90	47.36	57.85	69.29	82.59	96.61	112.66	130.22	149.55	171.88	196.86	226.56	263.18	316.32	425.99					
25	10.93	16.34	22.42	29.81	37.73	47.05	57.26	68.73	81.31	94.94	110.04	126.47	143.68	162.85	183.51	206.29	230.81	257.61	287.90	322.48	362.15	410.97	480.45	623.07
50	10.77	16.26	22.46	29.55	37.80	46.84	57.10	68.24	80.37	93.49	107.76	122.89	139.37	156.47	175.47	194.67	215.04	236.81	259.79	283.63	308.64	335.75	362.94	391.64
75	10.82	16.29	22.49	29.49	37.64	46.85	56.99	67.99	79.95	93.24	107.81	122.35	138.65	155.16	174.13	193.31	213.63	234.95	257.40	280.43	305.25	330.78	357.28	385.41
100	10.81	16.30	22.59	29.51	37.59	46.80	57.05	68.15	79.99	92.97	107.12	122.64	138.78	155.89	174.01	193.07	213.37	234.23	256.65	280.42	304.46	329.52	356.54	383.24
125	10.80	16.24	22.39	29.48	37.77	46.72	56.99	67.96	80.01	93.32	107.29	122.28	138.85	155.88	173.64	192.60	213.03	234.20	256.17	279.66	303.77	329.45	354.77	382.87
150	10.90	16.32	22.47	29.82	37.62	46.77	56.73	68.08	80.11	93.46	107.31	122.55	138.61	155.59	173.92	192.99	212.93	233.77	255.92	279.84	304.01	328.99	354.72	382.01
175	10.73	16.24	22.45	29.53	37.72	46.60	56.99	68.15	79.93	93.15	107.53	122.41	138.17	155.45	173.50	192.94	213.13	234.28	256.31	279.88	303.24	328.87	354.55	381.96
200	10.87	16.31	22.50	29.44	37.62	46.74	56.95	68.10	80.07	93.10	107.17	122.54	138.05	155.71	173.60	192.86	212.61	234.26	256.16	279.26	303.80	328.24	354.74	382.02
∞	10.83	16.27	22.46	29.59	37.70	46.80	56.89	67.99	80.08	93.17	107.26	122.35	138.44	155.53	173.62	192.71	212.80	233.89	255.98	279.07	303.16	328.25	354.34	381.43

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