The Distribution of Magnitude and Complex Voxel Values in MRI

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Waisman Laboratory
for Brain Imaging
and Behavior

OUTLINE

1. Image Reconstruction

2. Statistics-Ricean & Normal

3. Estimation-Ricean & Normal

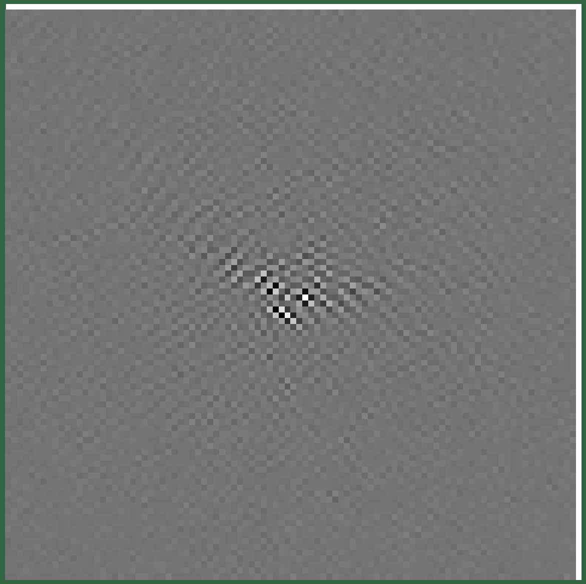
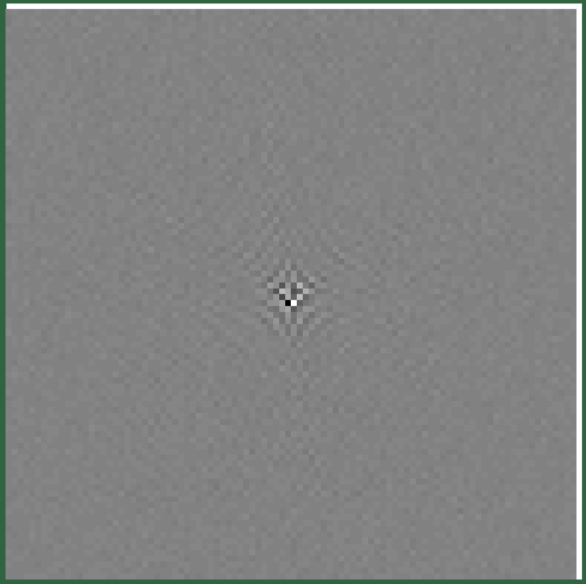
4. Estimation-Bivariate Normal

5. Discussion

Reconstruction:

Ideally measure complex-valued FT of the object.

$$S(k_x,k_y) = S_R(k_x,k_y) + i\, S_I(k_x,k_y)$$

$S(k_x,k_y) =$  $+i$ 

Complex: 96×96 Real: 96×96 Imaginary: 96×96

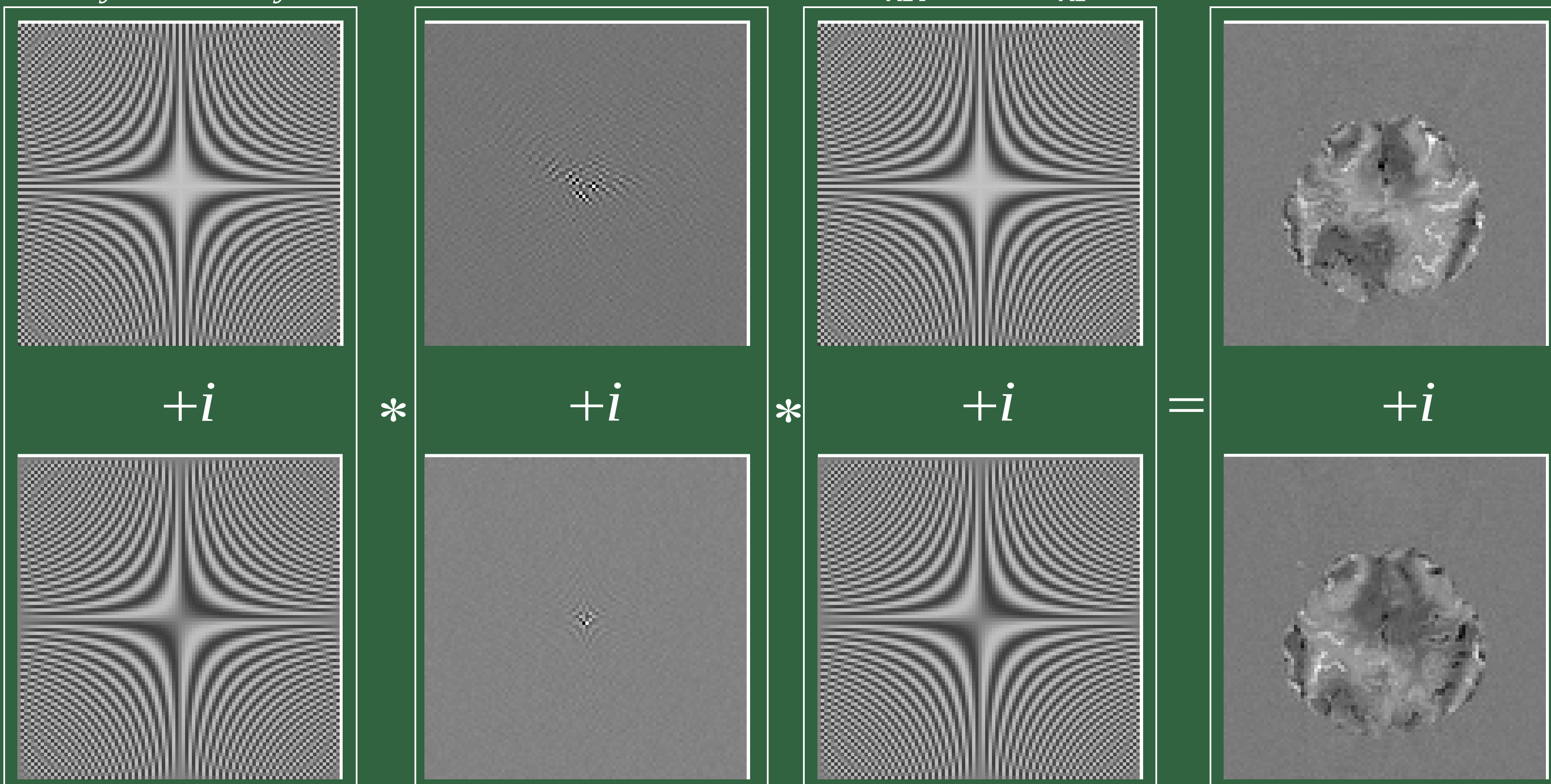
Actual data!

p = 9216
# of voxels

Reconstruction:

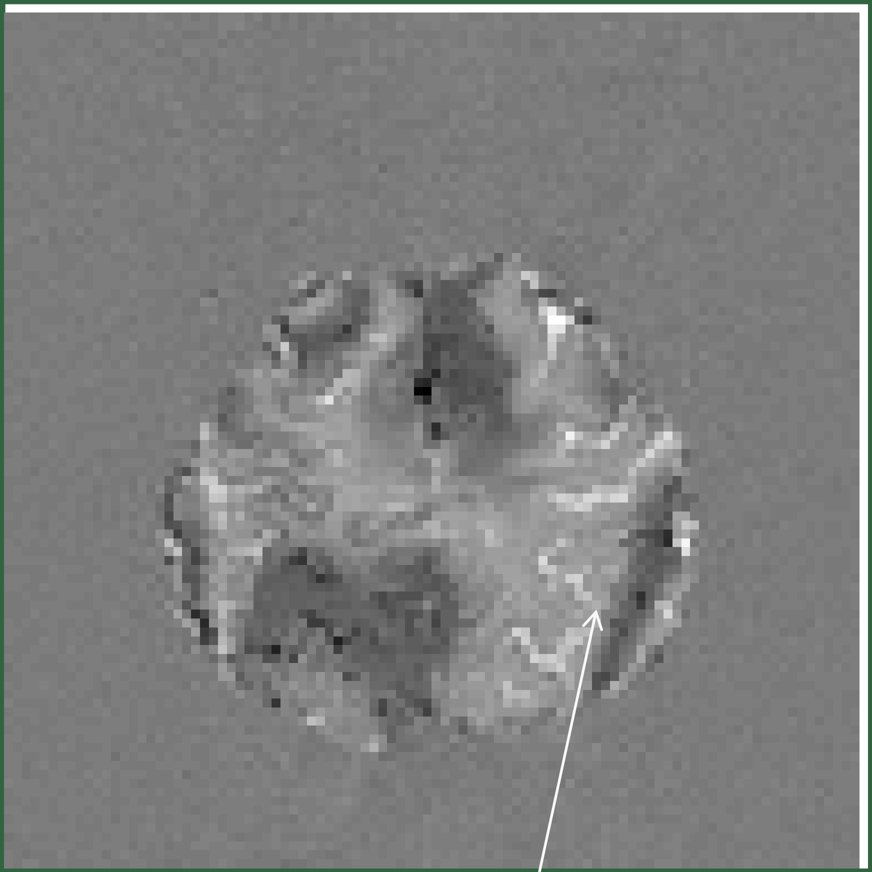
By complex-valued inverse FT of the object.

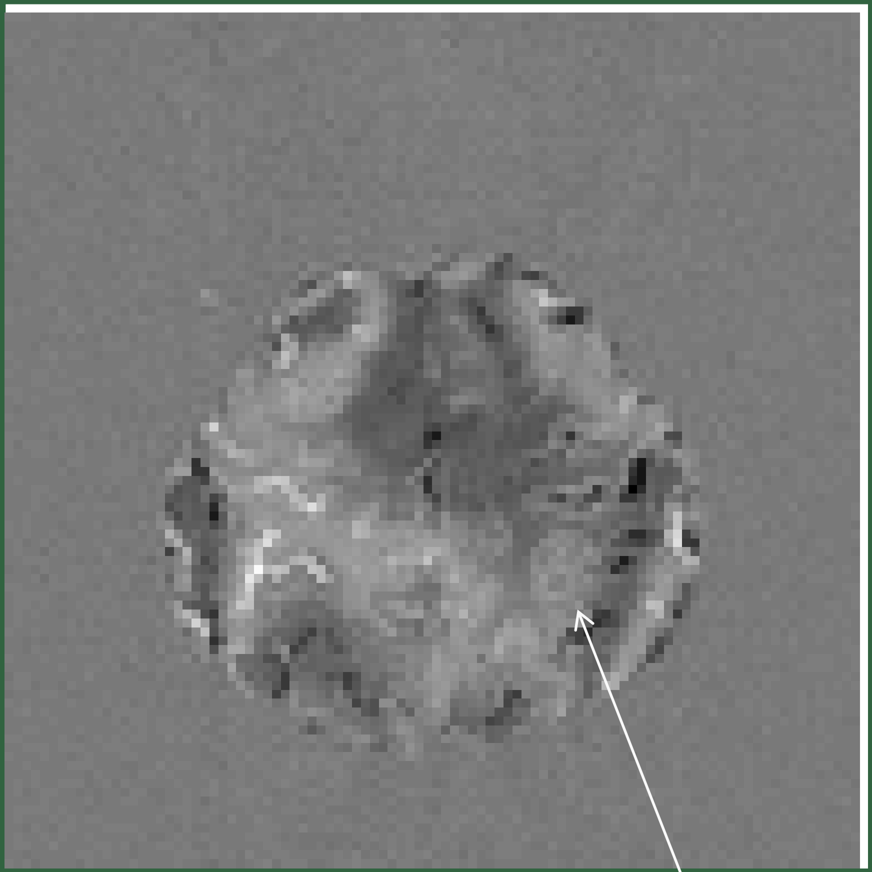
$$(\Omega_{yR} + i\Omega_{yI}) * (S_R + iS_I) * (\Omega_{xR} + i\Omega_{xI})^T = (Y_R + iY_I)$$



Reconstruction:

Due to imperfect reconstruction (noise, T_2*, ΔB, …), image is complex-valued, $Y_C(x,y) = Y_R(x,y) + iY_I(x,y)$.


Real Image

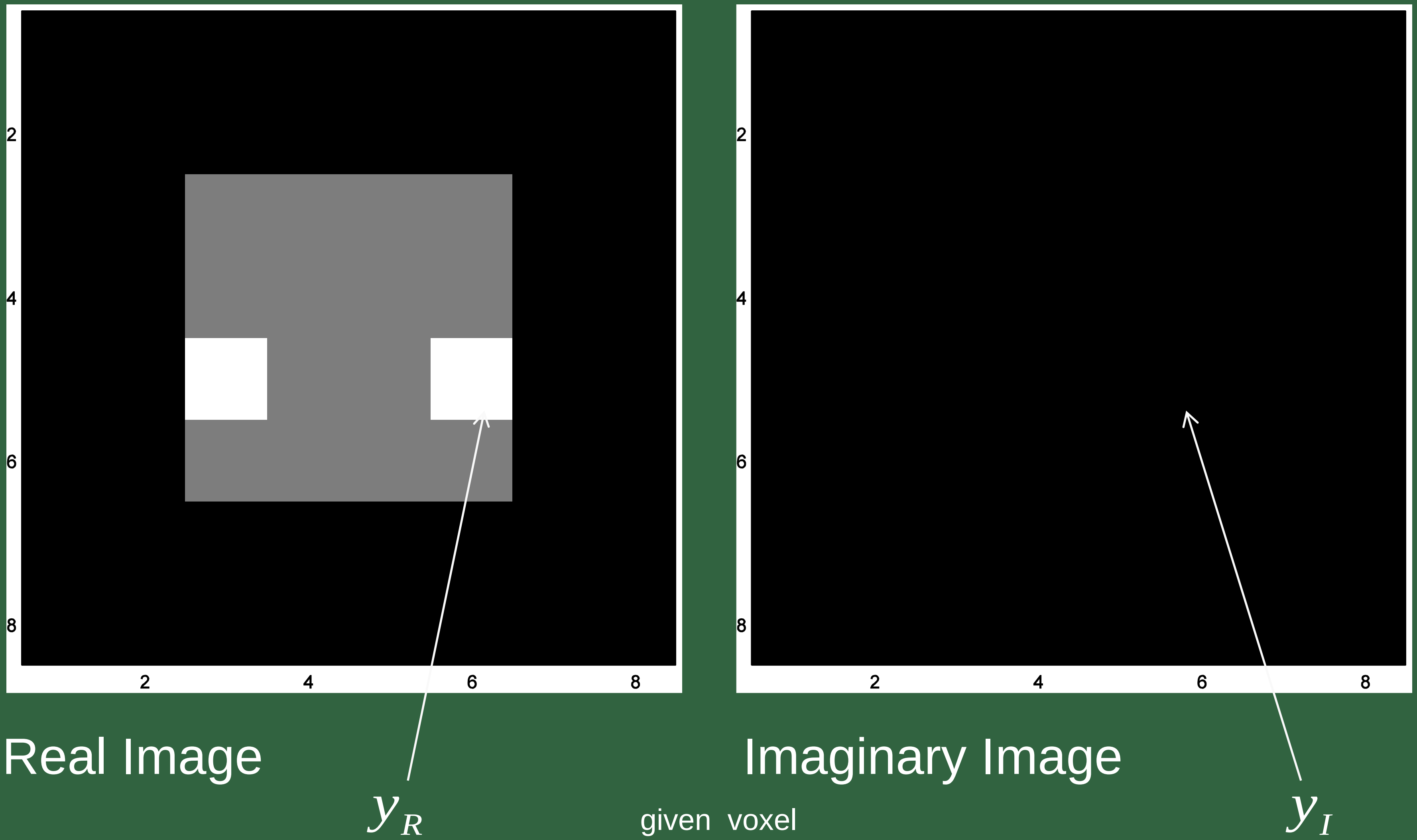

Imaginary Image

y_R given voxel y_I

Reconstruction:

Toy Example 8×8, image is complex-valued,

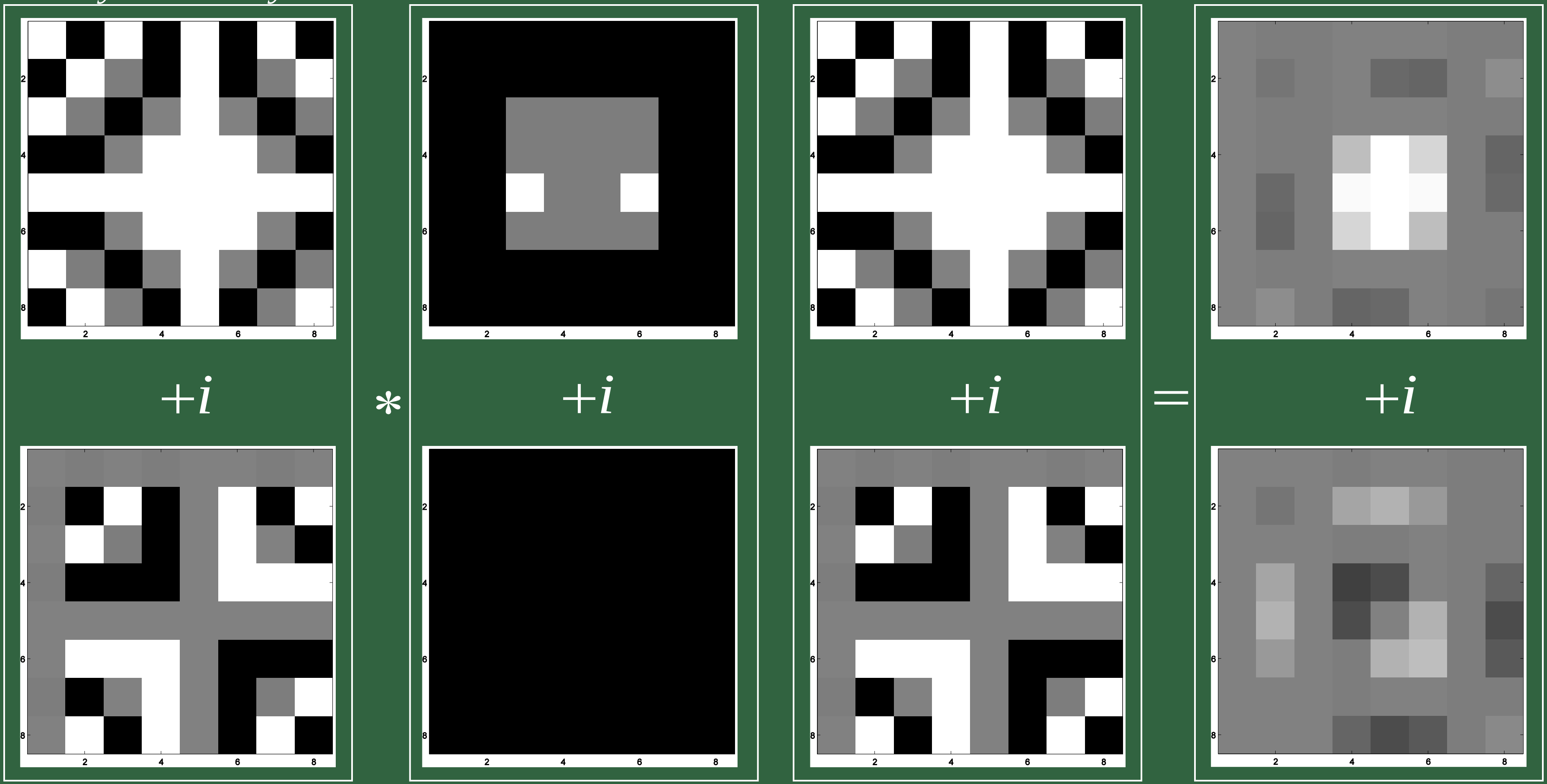
$$Y_C(x,y) = Y_R(x,y) + iY_I(x,y).$$



Reconstruction:

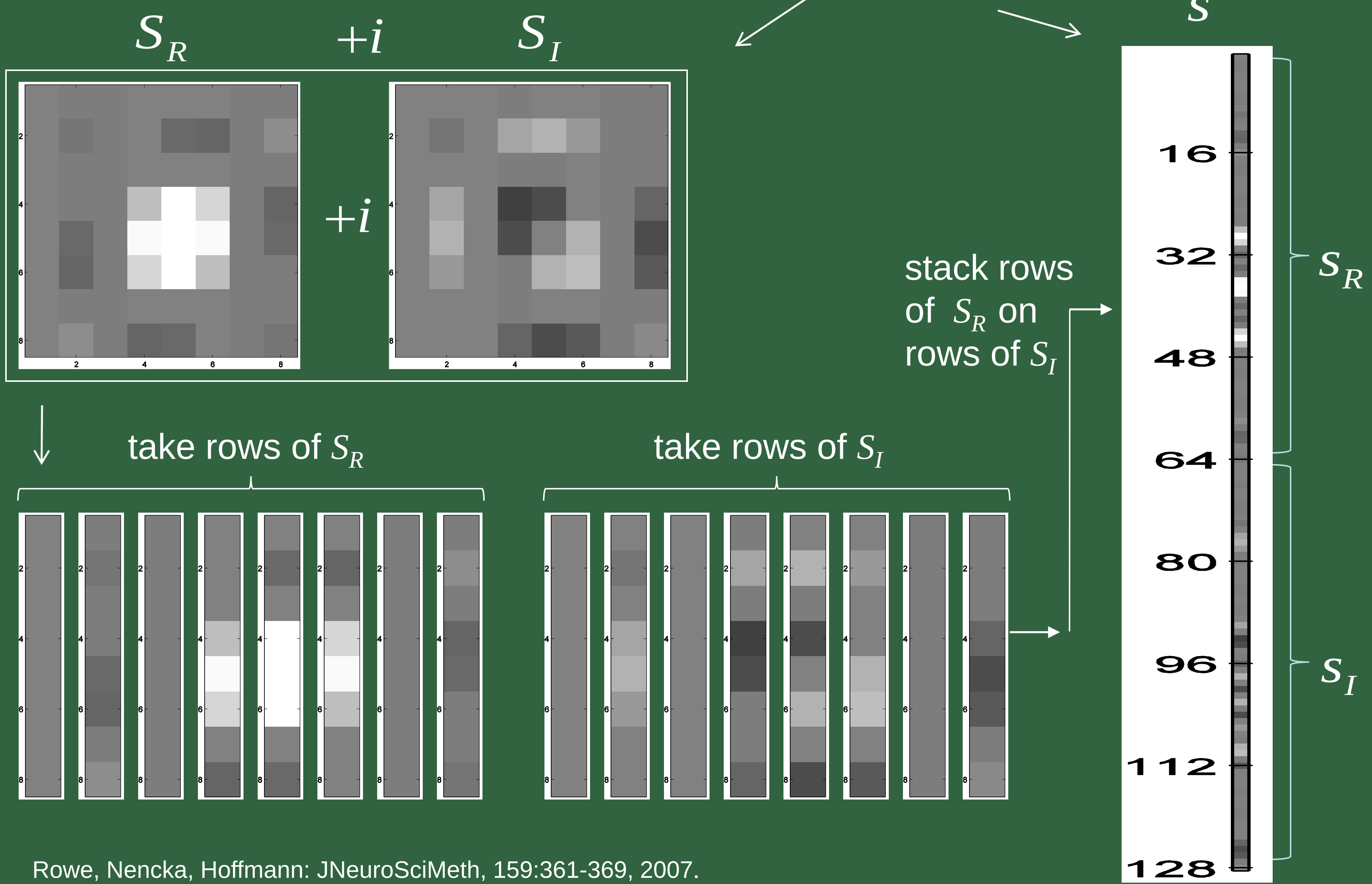
By complex-valued forward FT of the object. $\Omega\bar{\Omega} = I$

$$(\bar{\Omega}_{yR} + i\bar{\Omega}_{yI}) * (Y_R + iY_I) * (\bar{\Omega}_{xR} + i\bar{\Omega}_{xI})^T = (S_R + iS_I)$$



Reconstruction:

On Cartesian grid



Rowe, Nencka, Hoffmann: JNeuroSciMeth, 159:361-369, 2007.

Reconstruction:

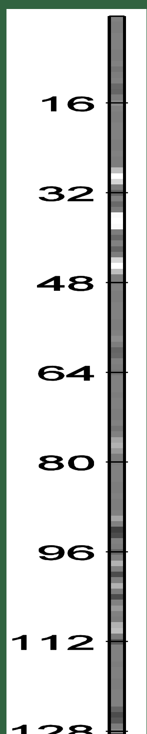
reconstruction matrix

$$\underset{2p\times 1}{s} = \begin{pmatrix} s_R \\ s_I \end{pmatrix}$$

$$\Omega = \begin{bmatrix} \Omega_R & -\Omega_I \\ \Omega_I & \Omega_R \end{bmatrix}$$

k-space vector

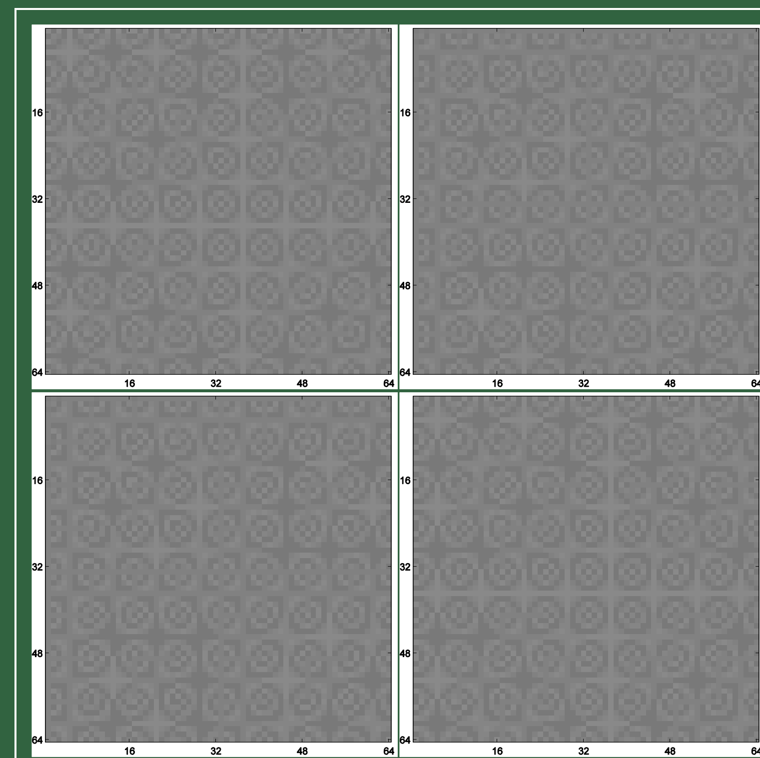
$s =$



s_R

s_I

$\Omega =$



$$\Omega_R = [(\Omega_{yR} \otimes \Omega_{xR}) - (\Omega_{yI} \otimes \Omega_{xI})]$$

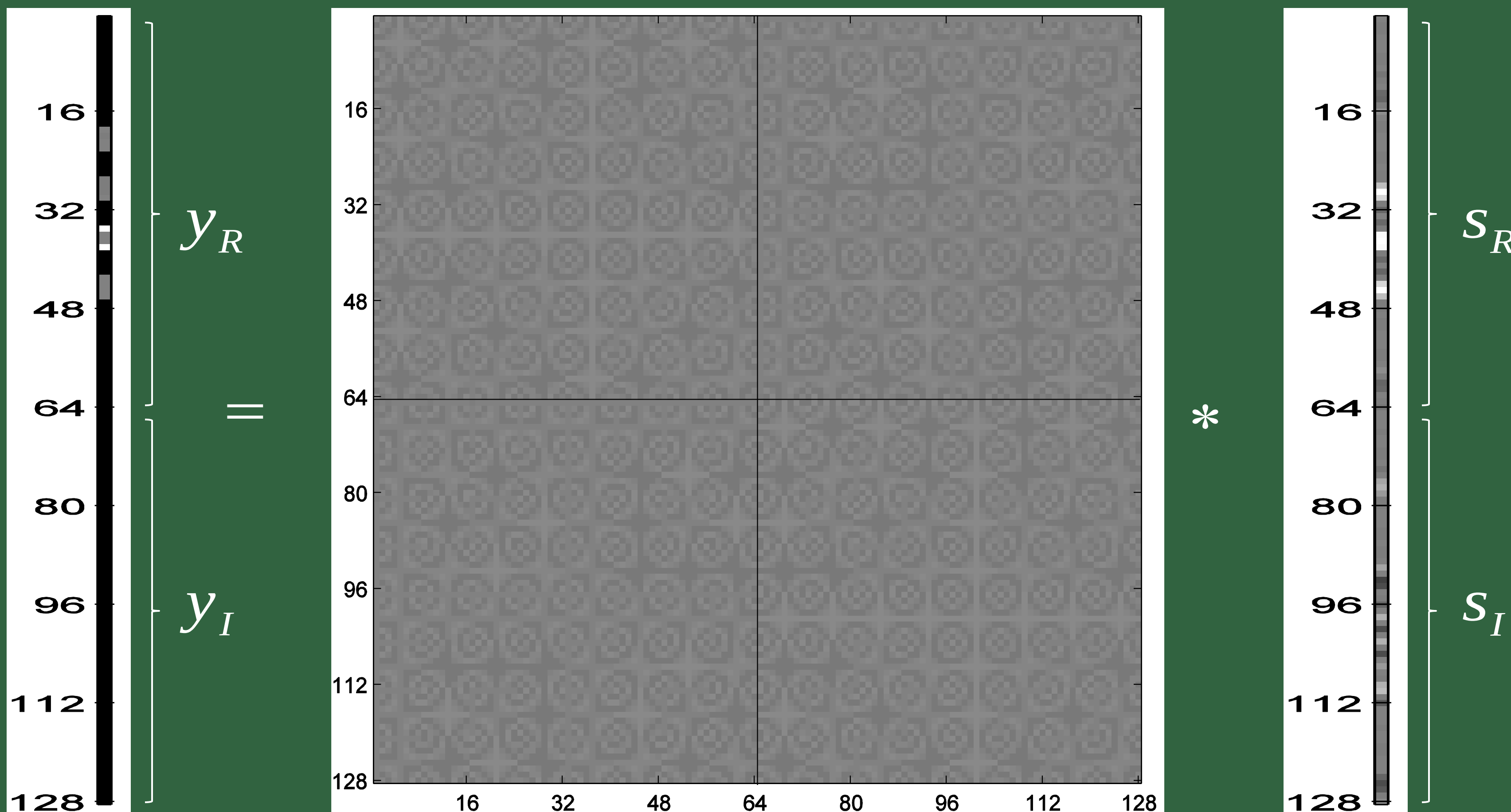
$$\Omega_I = [(\Omega_{yR} \otimes \Omega_{xI}) + (\Omega_{yI} \otimes \Omega_{xR})]$$

Rowe, Nencka, Hoffmann: JNeuroSciMeth, 159:361-369, 2007.

Reconstruction:

Inverse FT reconstruction can be equivalently described as:

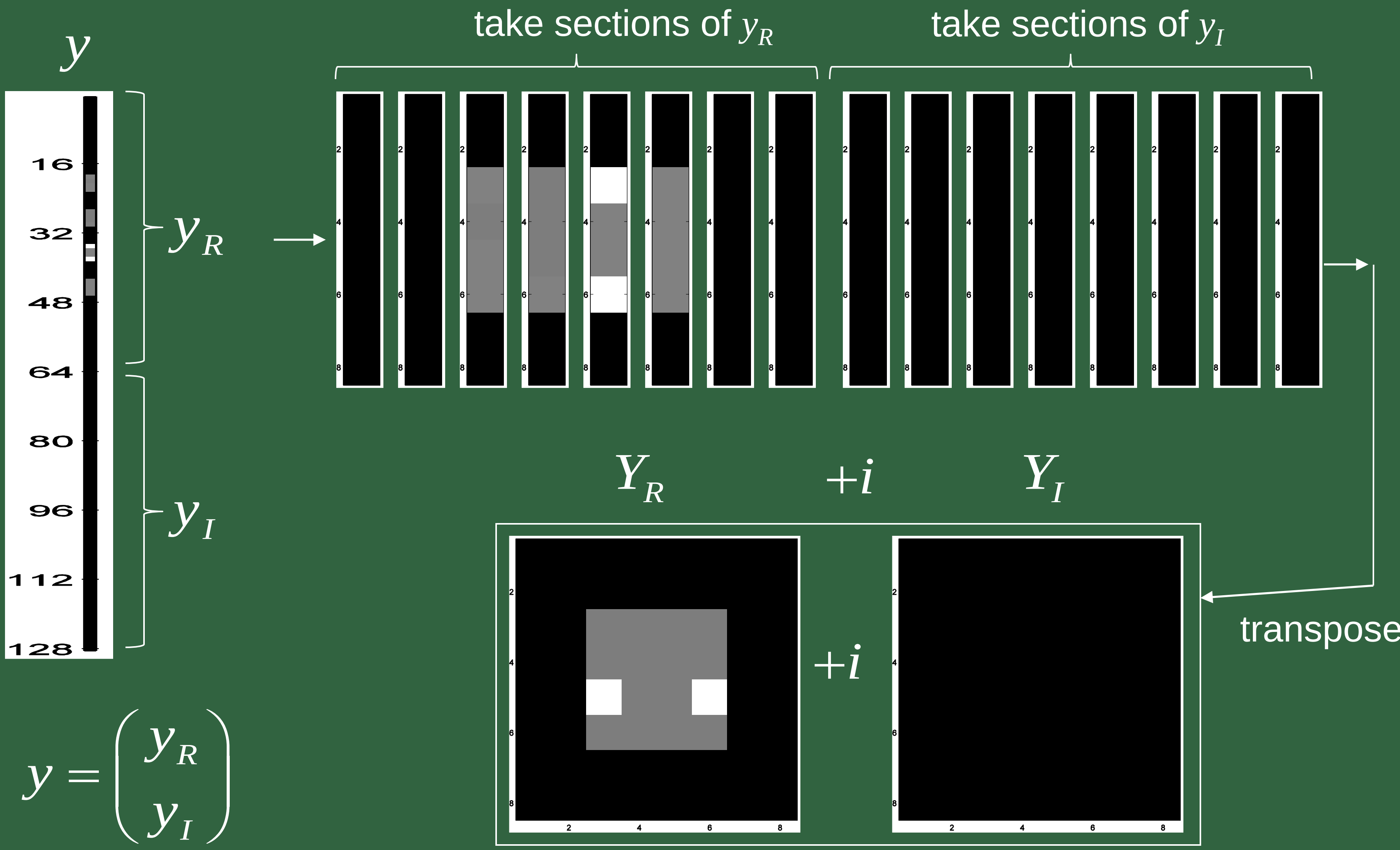
$$y = \Omega * s$$



Rowe, Nencka, Hoffmann: JNeuroSciMeth, 159:361-369, 2007.

Real-valued isomorphism

Reconstruction:



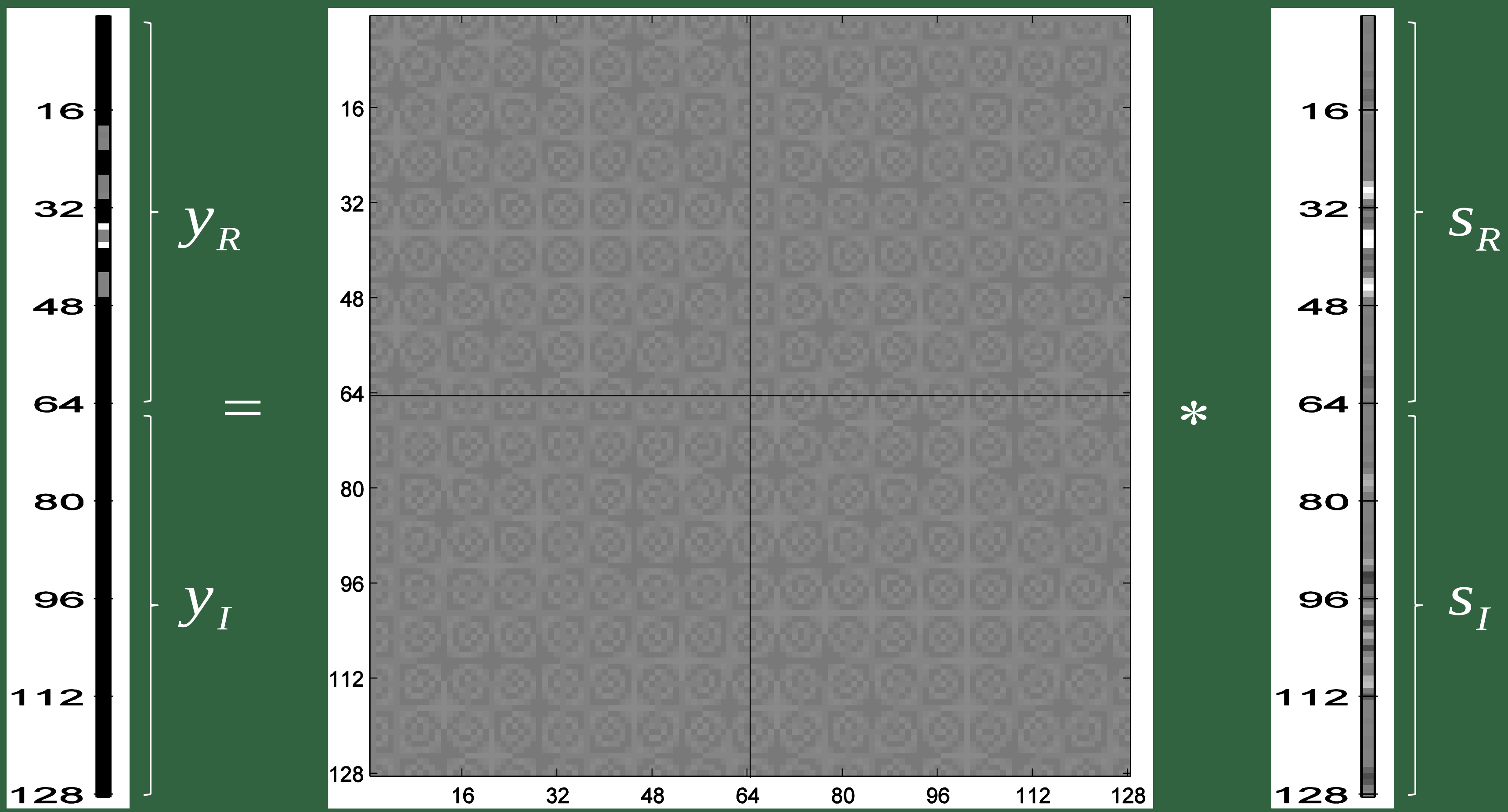
$$y = \begin{pmatrix} y_R \\ y_I \end{pmatrix}$$

Rowe, Nencka, Hoffmann: JNeuroSciMeth, 159:361-369, 2007.

Reconstruction:

Inverse FT reconstruction can be performed as:

$$y = \Omega * s$$



Rowe, Nencka, Hoffmann: JNeuroSciMeth, 159:361-369, 2007.

Real-valued isomorphism

Statistics: Expectation and Covariance.

If E(s) = s_0, then for $y = \Omega s$, E(y) = E(Ωs) = Ωs_0.

If cov(s) = Γ, then for $y = \Omega s$, cov(y) = cov(Ωs) = $\Omega\Gamma\Omega'$.

This means that with $\Gamma = \sigma_k^2 I$,

and because $\Omega\Omega' = \sigma^2 I$ where $\sigma^2 = (\sigma_k^2 / p^2)$

$$\mathrm{cov}(y) = \sigma^2 \underset{2p\times 2p}{I} .$$

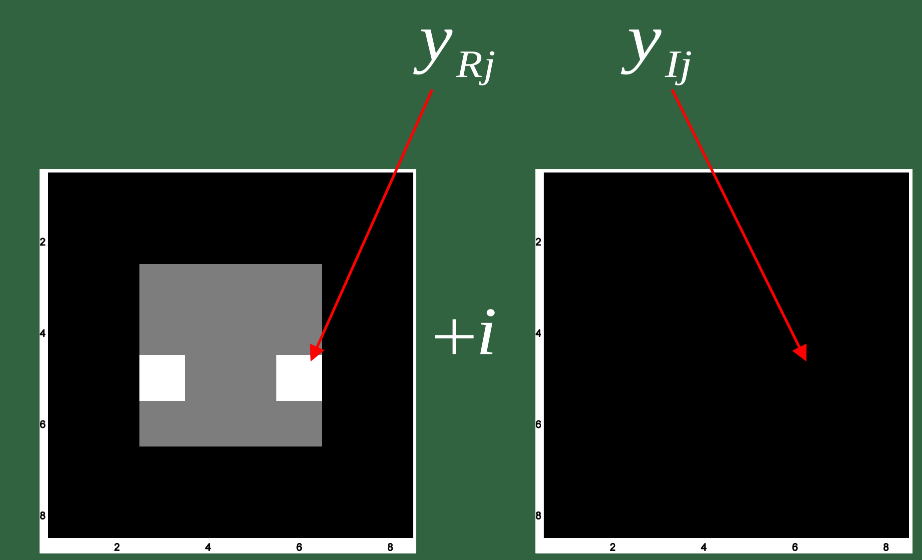
Statistics: Expectation and Covariance.

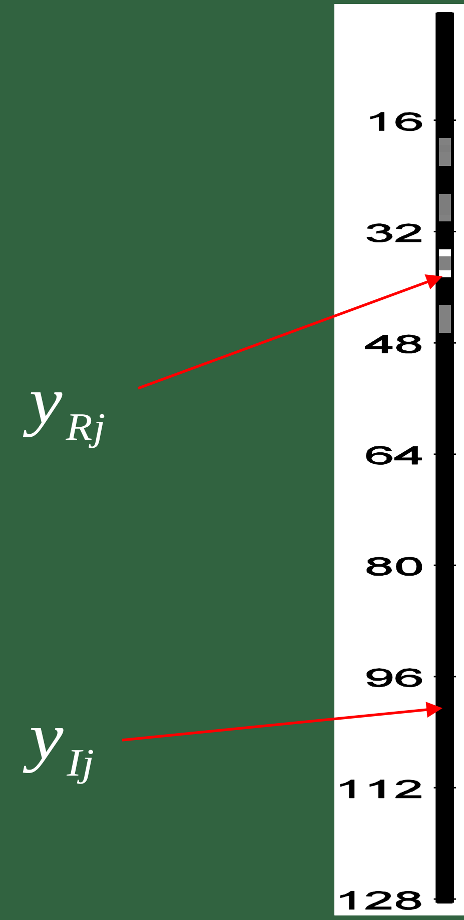
When we use normal distribution from thermal noise

$$s = s_0 + \varepsilon, \quad \varepsilon \sim N(0, \sigma_k^2 I)$$

$\underset{2p\times 1}{s} \sim \mathrm{N}(\underset{2p\times 1}{s_0}, \underset{2p\times 2p}{\sigma_k^2 I})$, then $\underset{2p\times 1}{y} \sim \mathrm{N}(\underset{2p\times 1}{\Omega s_0}, \underset{2p\times 2p}{\sigma^2 I})$.

This means that if we choose a voxel, say j





Statistics: Expectation and Covariance.

from $y \sim \mathrm{N}(\Omega s_0, \sigma^2 I)$, the distribution of y_{Rj} and y_{Ij} is

$$\begin{pmatrix} y_{Rj} \\ y_{Ij} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{Rj} \\ \mu_{Ij} \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right)$$

where

$$\mu_{Rj} = \omega_j s_0$$

$$\mu_{Ij} = \omega_{p+j} s_0$$

j^{th} row of Ω

$(p+j)^{th}$ row of Ω

$$y_{Cj} = y_{Rj} + i y_{Ij}$$

the pdf is

$$p(y_{Rj}, y_{Ij}) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2} \left[(y_{Rj} - \mu_{Rj})^2 + (y_{Ij} - \mu_{Ij})^2 \right] \right\}$$

product of two normal pdfs

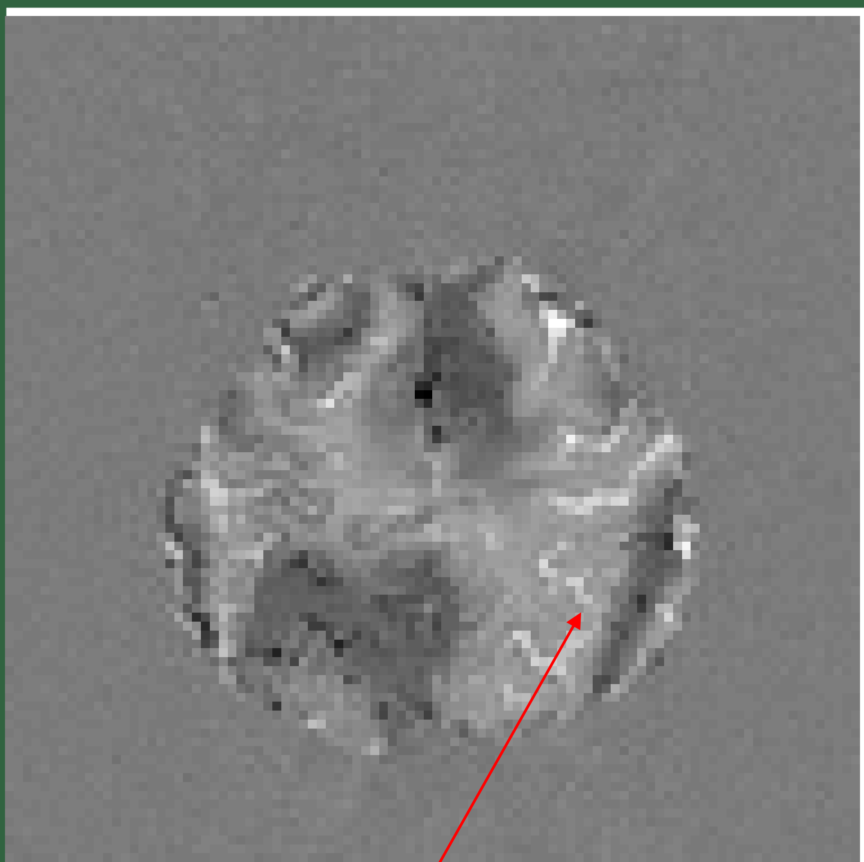
with phase coupled means

$$\mu_{Rj} = \rho_j \cos\theta_j$$

$$\mu_{Ij} = \rho_j \sin\theta_j$$

Statistics:

Real Image



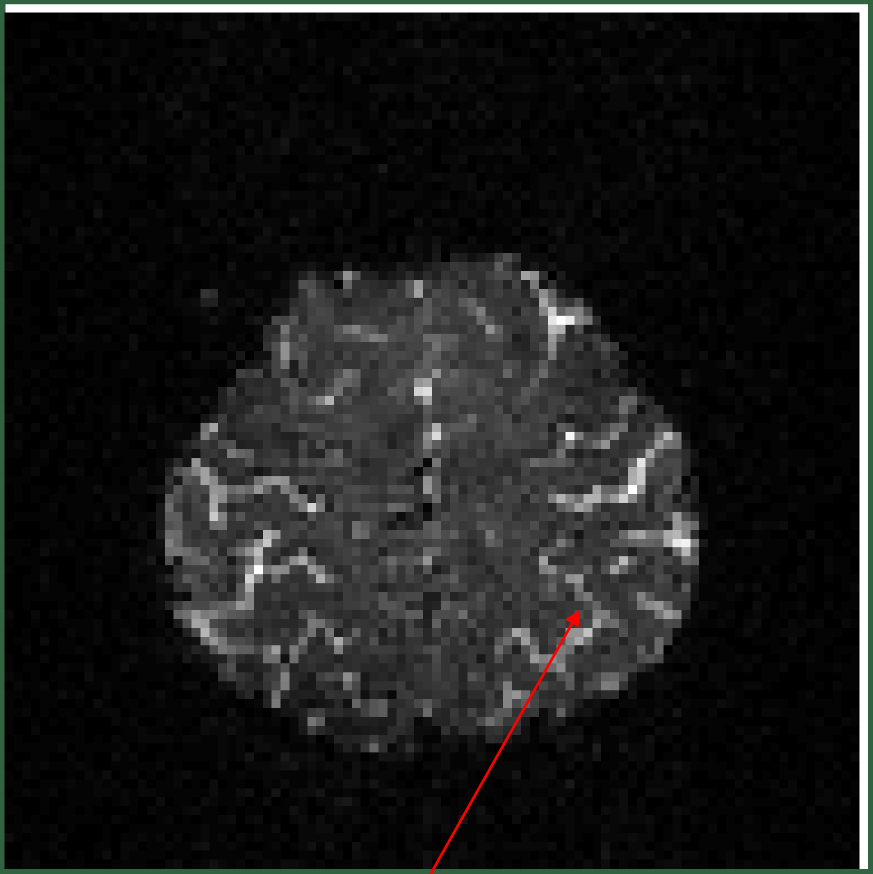
Imaginary Image

voxel j

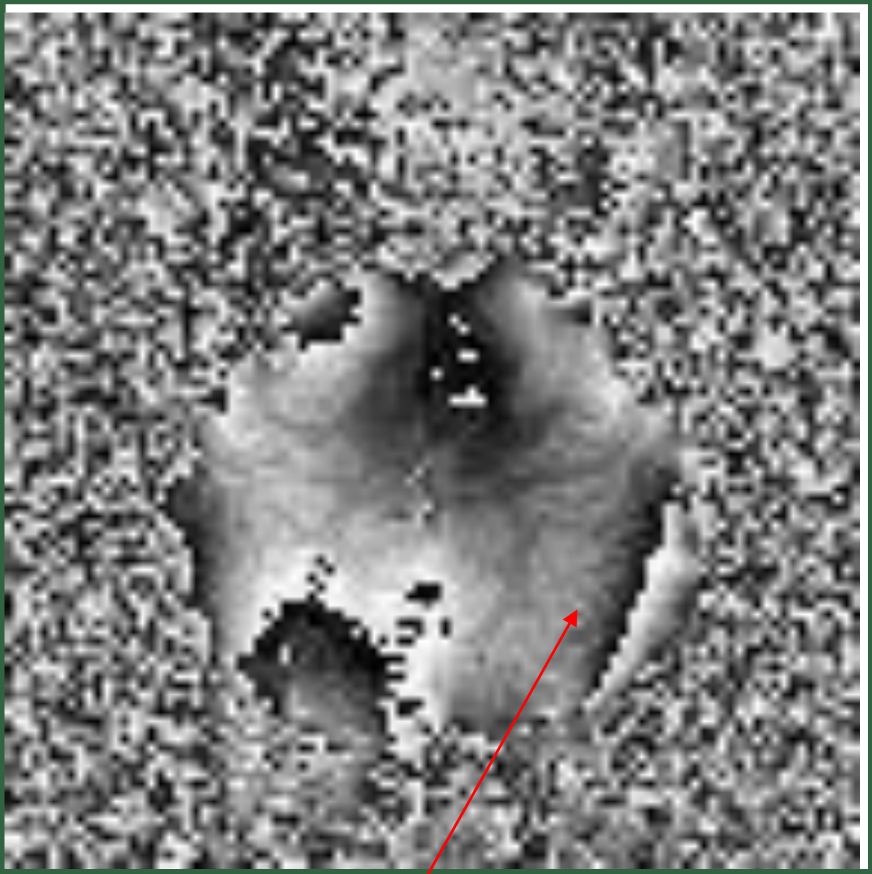
y_{Rj} y_{Ij}

Statistics:

Magnitude Image



Phase Image



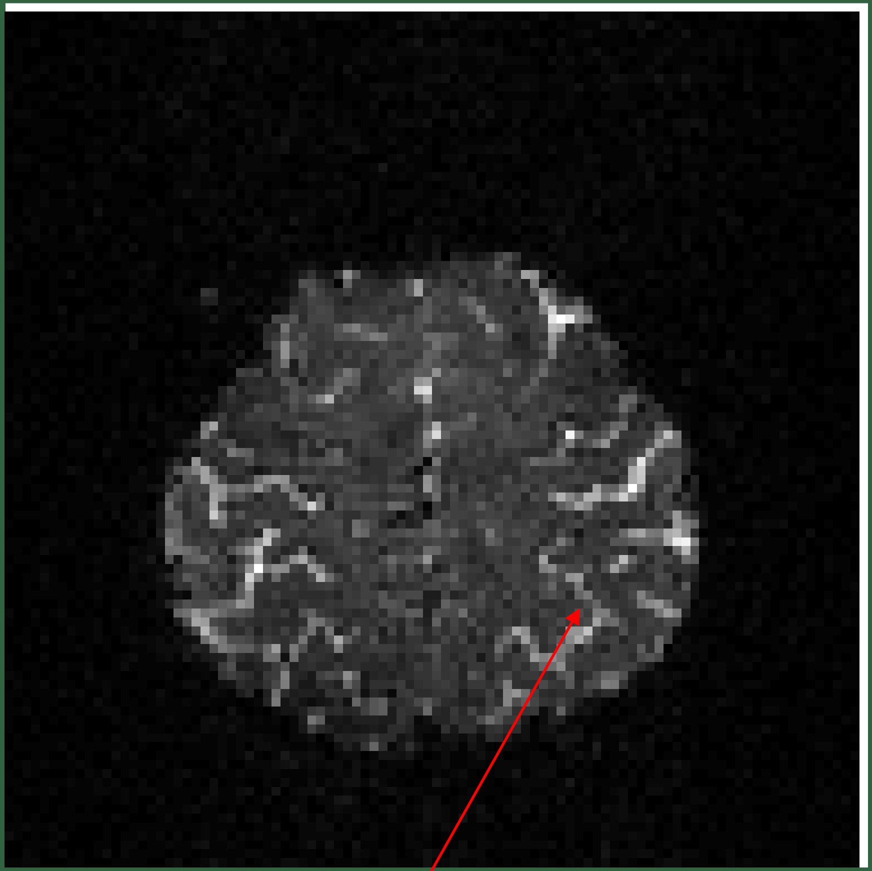
voxel j

$$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$$

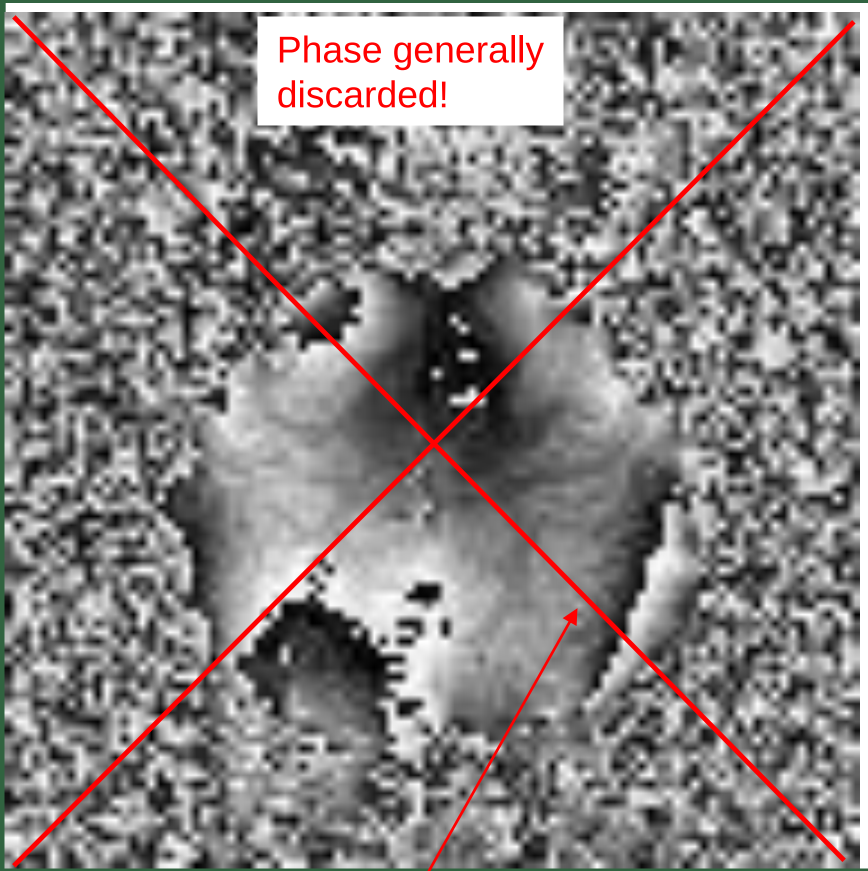
$$\varphi_j = \tan^{-1}(y_{Ij} / y_{Rj})$$

Statistics:

Magnitude Image



Phase Image

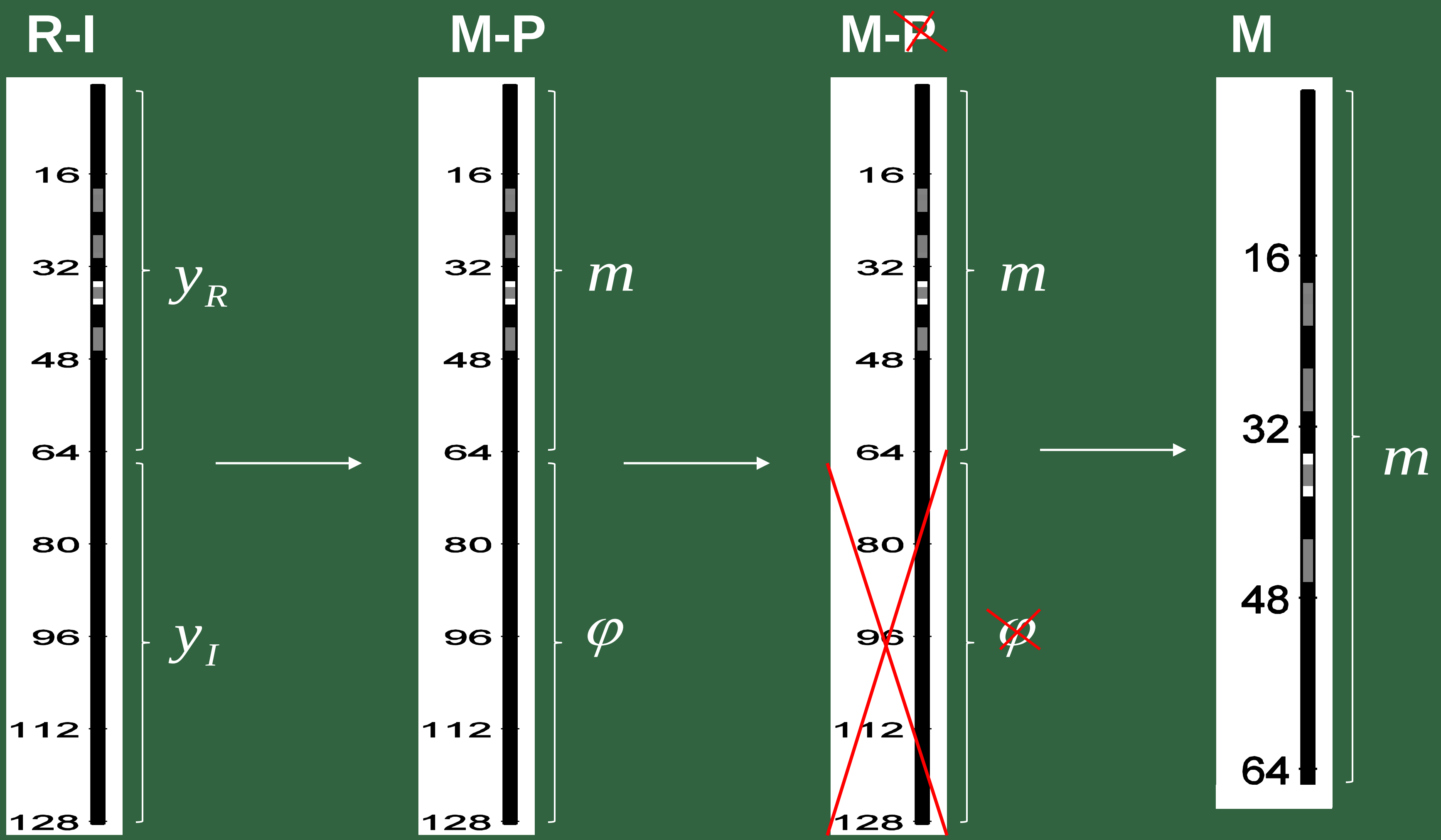


voxel j

$$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$$

$$\varphi_j = \tan^{-1}(y_{Ij} / y_{Rj})$$

Statistics:



$$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$$

Statistics:

Get $p(m_j)$ from $p(y_{Rj}, y_{Ij})$.

$$\mu_{Rj} = \rho_j \sin\theta_j$$

$$\mu_{Ij} = \rho_j \cos\theta_j$$

Convert from y_{Rj}, y_{Ij} to m_j, φ_j .

$$p(y_{Rj}, y_{Ij}) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(y_{Rj} - \rho_j \cos\theta_j)^2 + (y_{Ij} - \rho_j \sin\theta_j)^2\right]\right\}$$

$$p(m_j, \varphi_j) = \frac{m_j}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[m_j^2 + \rho_j^2 - 2m_j\rho_j \cos(\varphi_j - \theta_j)\right]\right\}$$

$$p(m_j) = \frac{m_j}{\sigma^2} \exp\left\{-\frac{m_j^2 + \rho_j^2}{2\sigma^2}\right\} I_0\left(\frac{\rho_j m_j}{\sigma^2}\right)$$

zeroth order modified Bessel function of first kind

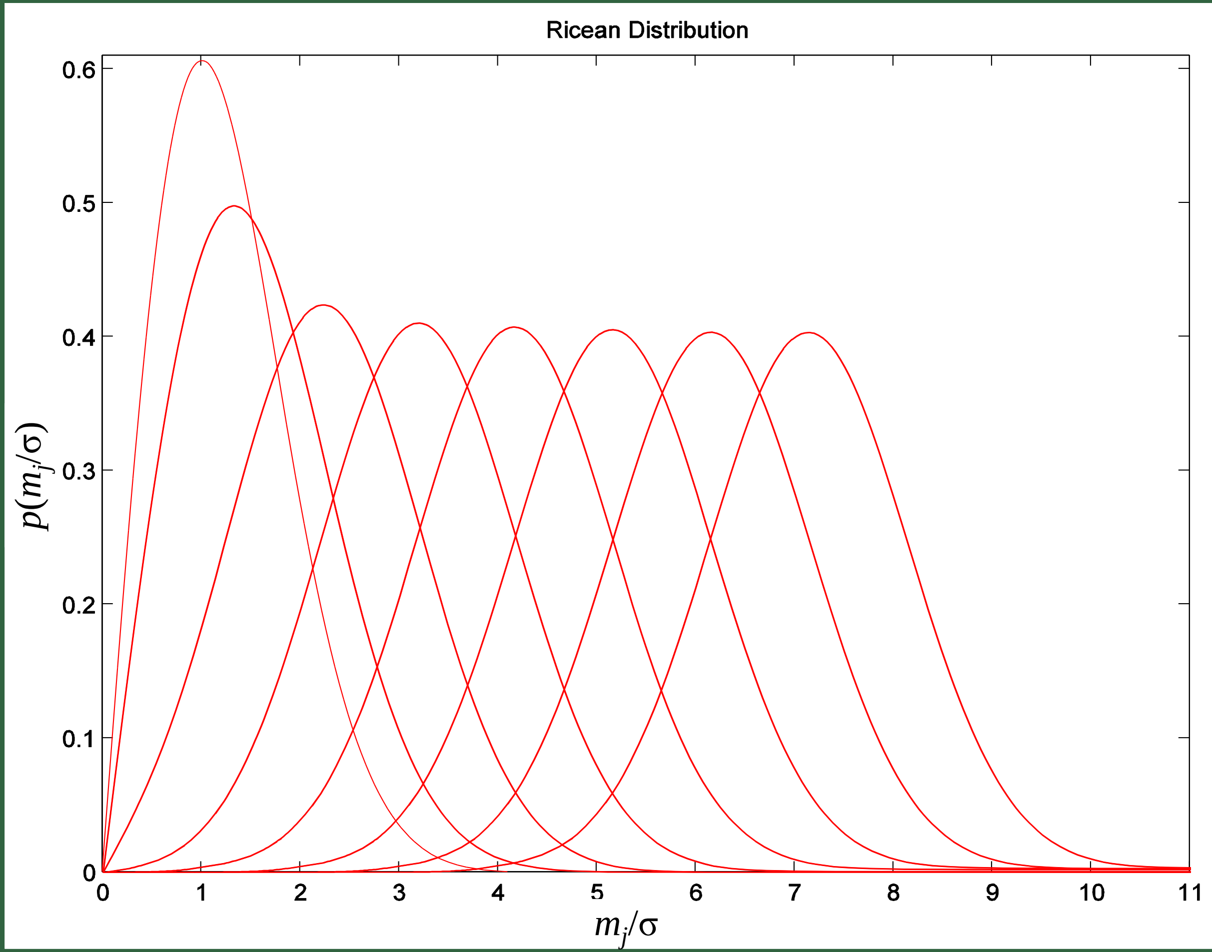
$$\frac{1}{2\pi}\int_{\varphi_j=-\pi}^{\pi} e^{\frac{\rho_j m_j}{\sigma^2}\cos(\varphi_j - \theta_j)} d\varphi_j$$

Rice,S.O., Bell Syst. Tech. 23:282, 1944.
Gudbjartsson, Patz. MRM 34:910–914, 1995.
Rowe and Logan: NIMG, 23:1078-1092, 2004.

Statistics:

$$p(m_j) = \frac{m_j}{\sigma^2} \exp\left\{ -\frac{m_j^2 + \rho_j^2}{2\sigma^2} \right\} I_0\left(\frac{\rho_j m_j}{\sigma^2} \right)$$

$$SNR = \frac{\rho_j}{\sigma^2}$$



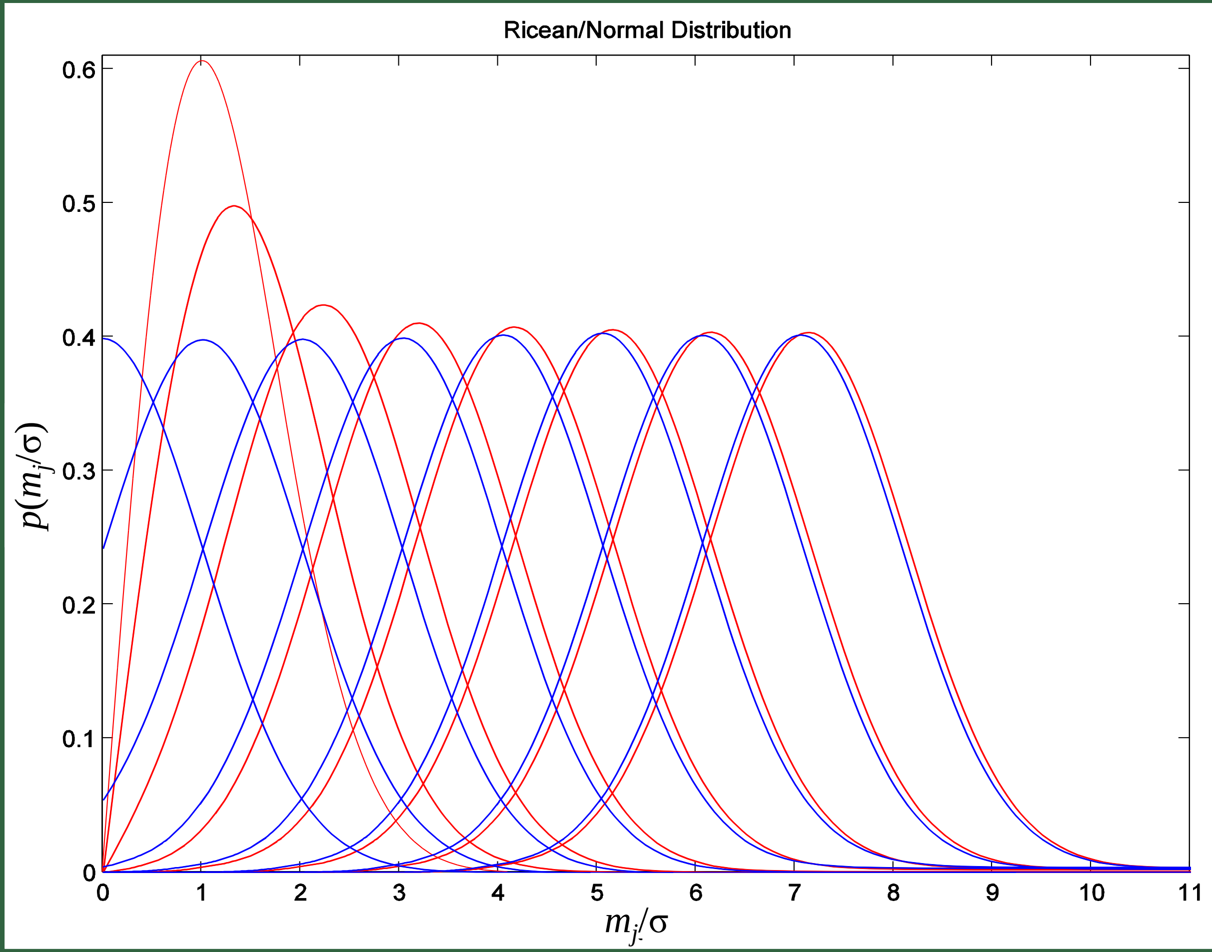
The magnitude, does not have a normal distribution!

Ricean Distribution!

Statistics:

$$p(m_j) = \frac{m_j}{\sigma^2} \exp\left\{ -\frac{m_j^2 + \rho_j^2}{2\sigma^2} \right\} I_0\left(\frac{\rho_j m_j}{\sigma^2} \right)$$

$$SNR = \frac{\rho_j}{\sigma^2}$$



The magnitude, does not have a normal distribution!

Ricean Distribution!

Ricean → Normal as the SNR ↑

Statistics:

The high SNR normality of m_j can be seen as

$$
\begin{aligned}
m_j &= \left[(y_{Rj})^2+(y_{Ij})^2\right]^{1/2} \\
&= \left[(\rho_j\cos\theta_j+\eta_{Rj})^2+(\rho_j\sin\theta_j+\eta_{Rj})^2\right]^{1/2} \\
&= \left[\rho_j^2+(\eta_{Rj}^2+\eta_{Ij}^2)+2\rho_j(\eta_{Rj}\cos\theta_j+\eta_{Rj}\sin\theta_j)\right]^{1/2} \\
&= \rho_j\left[1+2\frac{(\eta_{Rj}\cos\theta_j+\eta_{Rj}\sin\theta_j)}{\rho_j}+\cancelto{0}{\frac{(\eta_{Rj}^2+\eta_{Ij}^2)}{\rho_j^2}}\right] \\
&\approx \rho_j+\varepsilon_j
\end{aligned}
$$

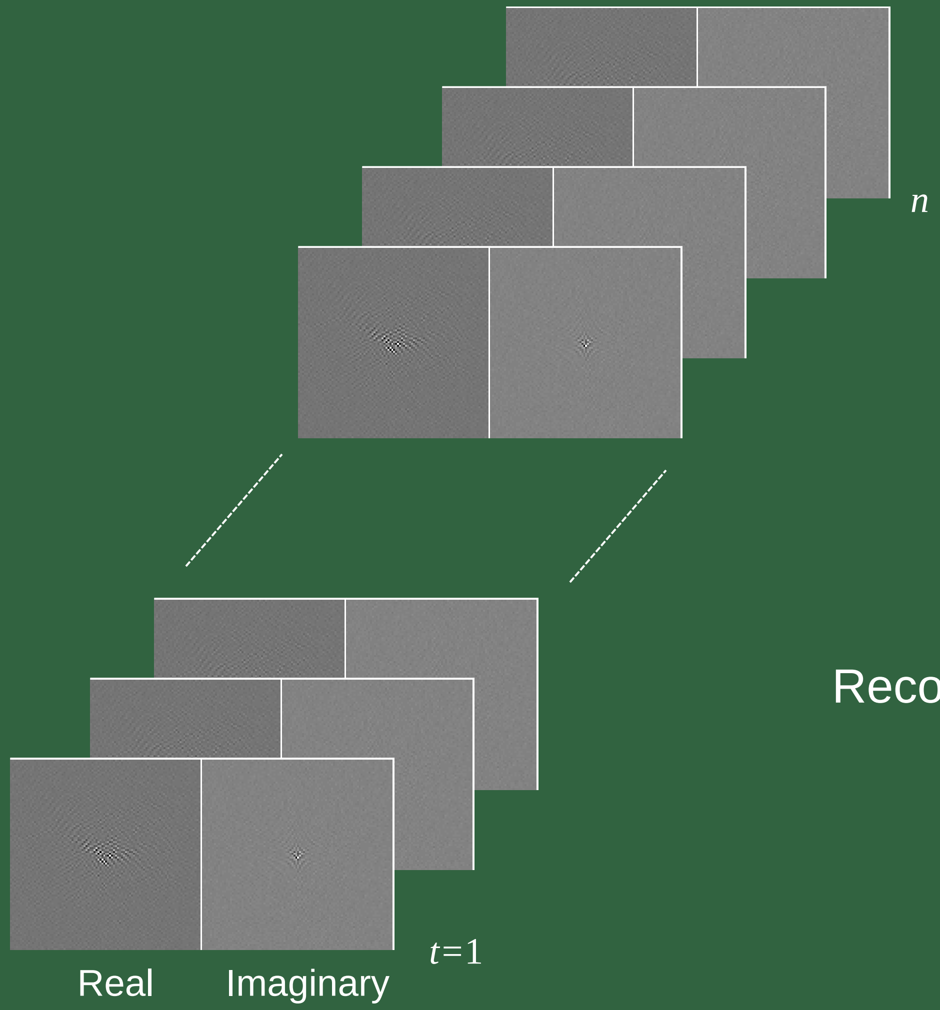
where $\varepsilon_j=\eta_{Rj}\cos\theta_j+\eta_{Rj}\sin\theta_j$

$$\varepsilon_j \sim N(0,\sigma^2)$$

$$\sqrt{1+u^2}\approx 1+u/2,\quad |u|\ll 1$$

Statistics:

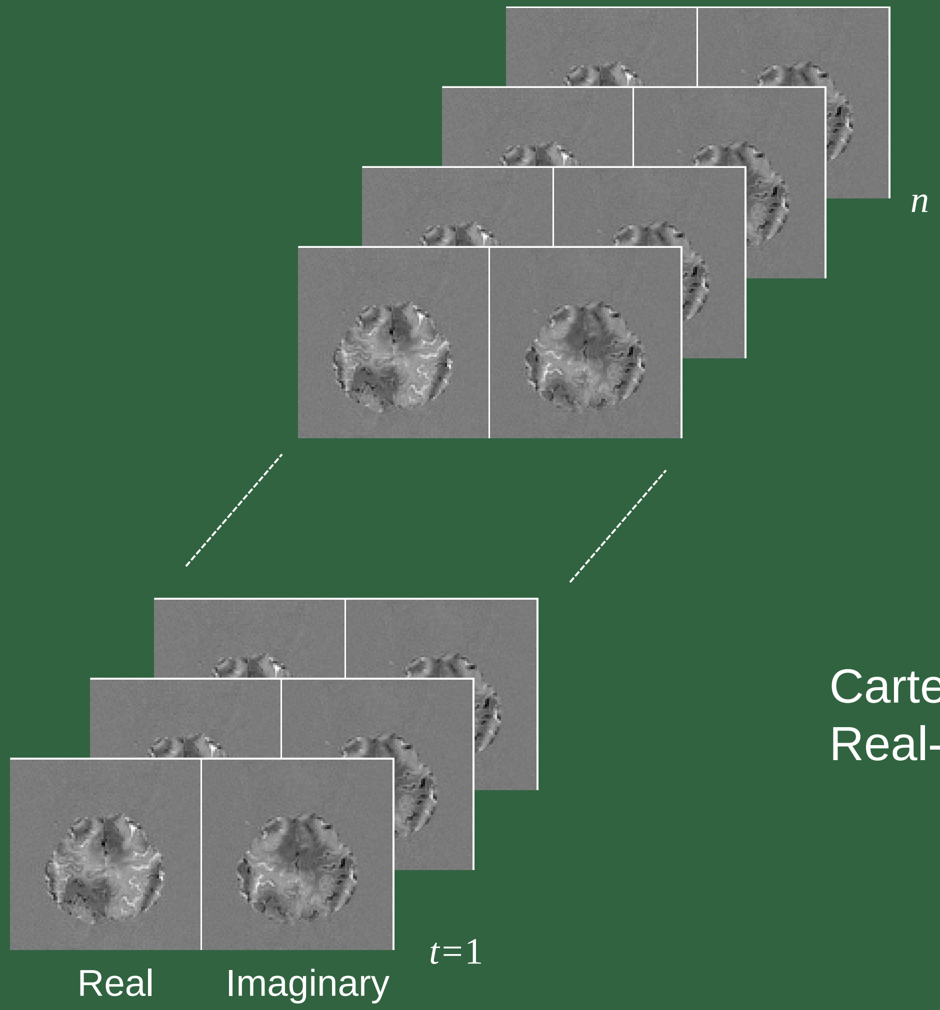
We take n k-space arrays under different signal conditions



Reconstruct each image.

Statistics:

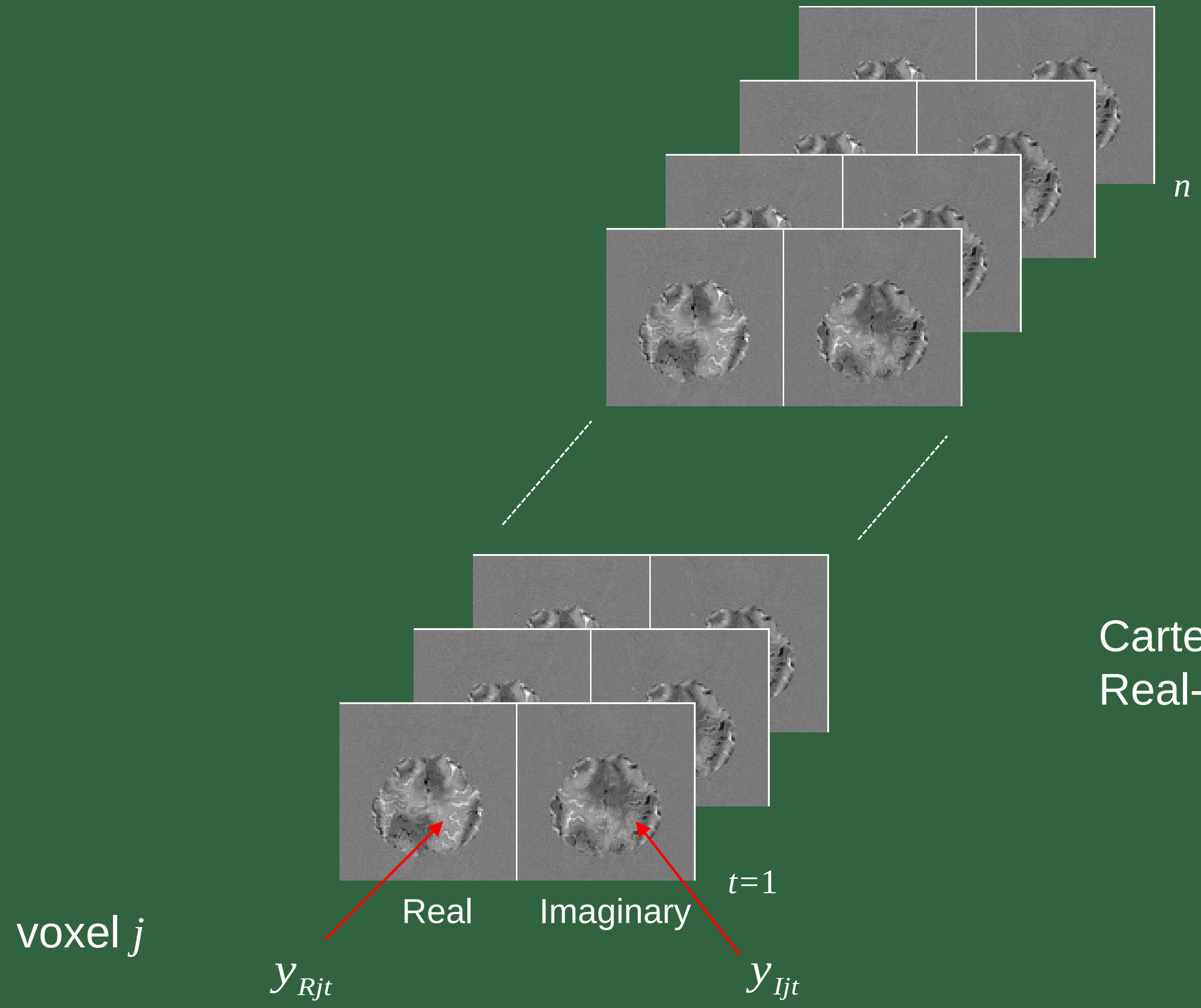
We get n images under different signal conditions



Cartesian coordinates
Real-Imaginary

Statistics:

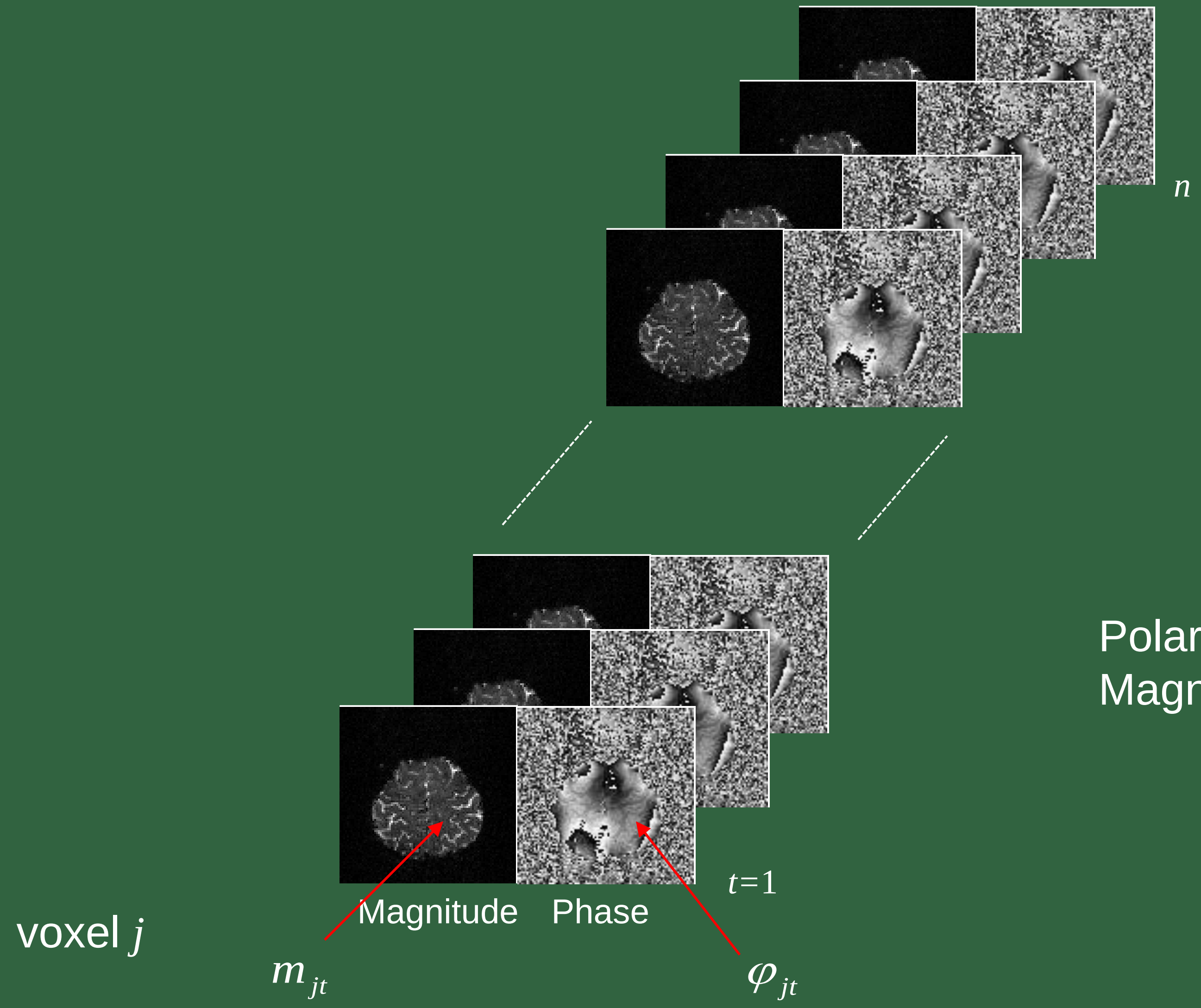
We get n images under different signal conditions



Cartesian coordinates
Real-Imaginary

Statistics:

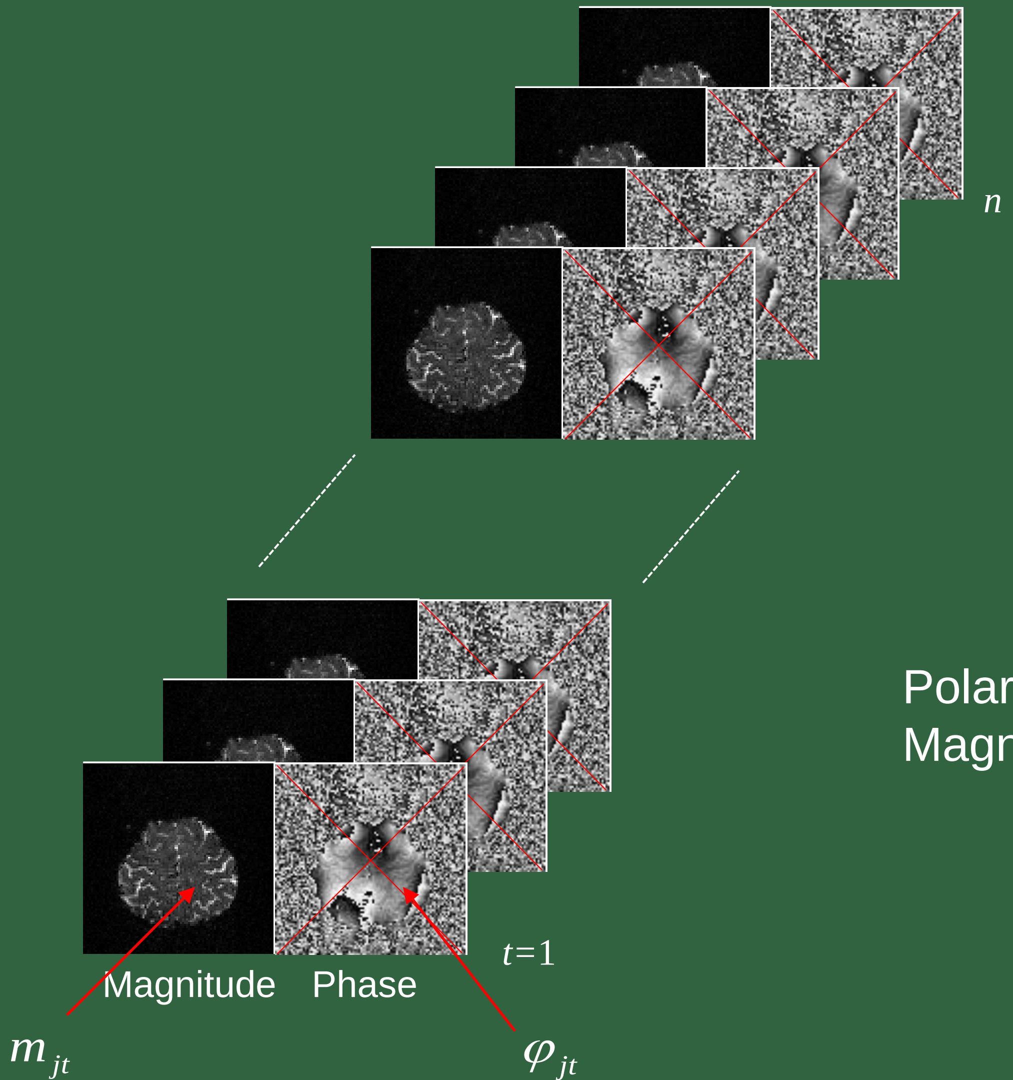
We get n images under different signal conditions



Polar Coordinates
Magnitude-Phase

Statistics:

We get n images under different signal conditions



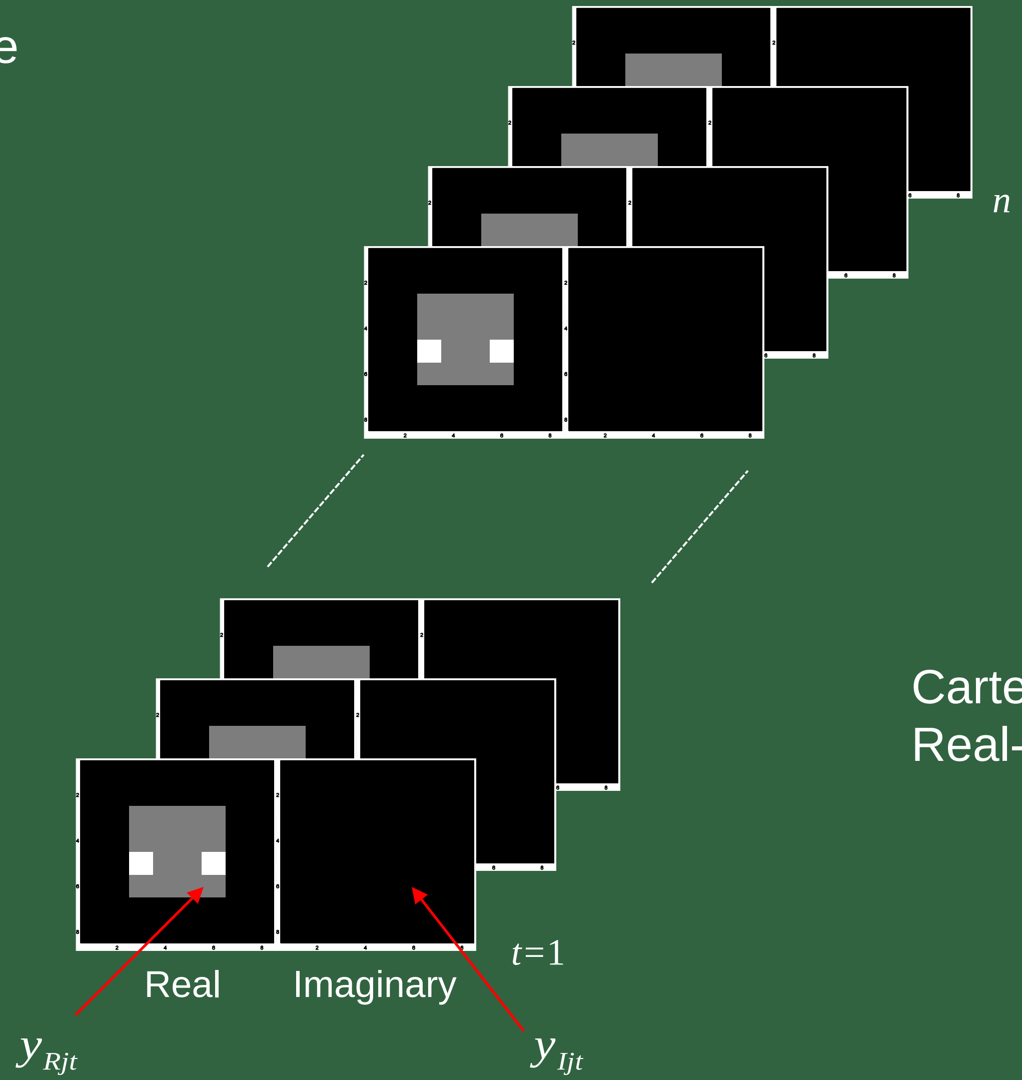
Phase discarded in a lot of MRI especially fMRI

Polar Coordinates
Magnitude-Phase

voxel j

Statistics:
We take n images under different signal conditions

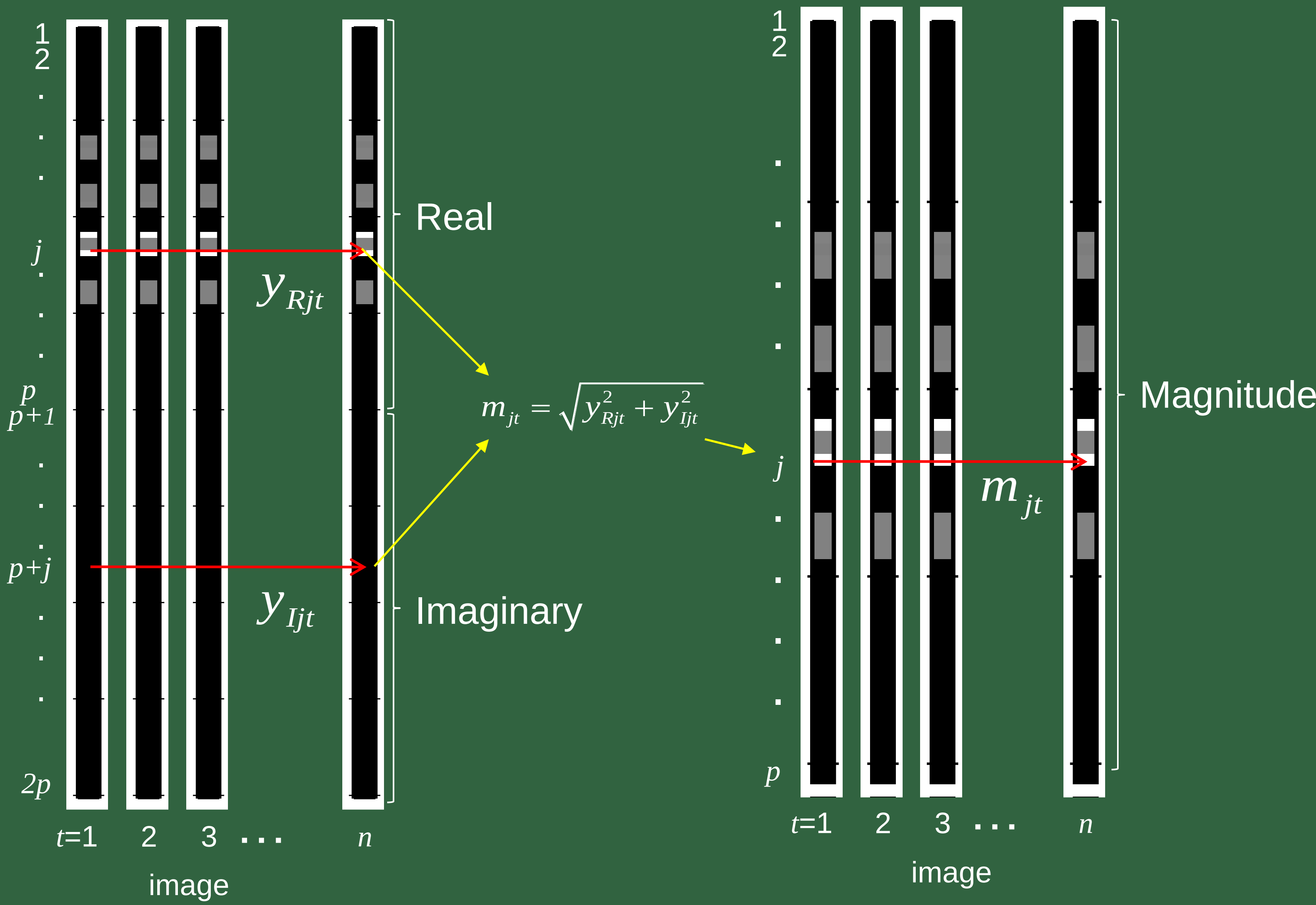
Toy Example



Cartesian coordinates
Real-Imaginary

Statistics: $y_t = \Omega s_t$

We take n images under different signal conditions



Statistics: Bivariate Normal to Ricean

The distribution of measurement t in voxel j is:

$$p(y_{Rjt}, y_{Ijt}) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2} \left[(y_{Rjt} - \rho_{jt}\cos\theta_{jt})^2 + (y_{Ijt} - \rho_{jt}\sin\theta_{jt})^2 \right] \right\}$$

$$p(m_{jt}) = \frac{m_{jt}}{\sigma^2} \exp\left\{ -\frac{m_{jt}^2 + \rho_{jt}^2}{2\sigma^2} \right\} I_0\left(\frac{\rho_{jt} m_{jt}}{\sigma^2} \right) \qquad t = 1, \ldots, n$$

The goal is to estimate a functional form $\rho_{jt}=f(x_t|\beta_j)$ for the magnitude and possibly $\theta_{jt}=g(u_t|\gamma_j)$ for the phase from the data $y_{Cj1}, \ldots, y_{Cjn}$ or $m_{j1}, \ldots, m_{jn}$ in each voxel.

$$\mu_{Rjt} = \omega_j s_{0t} = \rho_{jt}\cos\theta_{jt}$$

$$\mu_{Ijt} = \omega_{p+j} s_{0t} = \rho_{jt}\sin\theta_{jt}$$

ω_j is j^{th} row of Ω

ω_{p+j} is $(p+j)^{th}$ row of Ω

x is a vector of known “dial” settings

β is a vector of unknown parameters.

Estimation:

Types of functions to estimate:

Data $y_{C1},\ldots,y_{Cn}$ or $m_1,\ldots,m_n$ in each voxel. No j subscript.

	$f(x\mid\beta)$	x	β
1.	ρ	1	ρ
2.	$\rho\exp(-TE/T_2)$	TE	ρ, T_2
3.	$S_0\exp(-br'Dr)$	b, r	S_0, D
4.	$\rho(1-2\exp(t/T_1))$	t	ρ, T_1
5.	$x'\beta$	x'	β

Estimation: Ricean

Estimate parameters of function from magnitude data:

$$p(m_t) = \frac{m_t}{\sigma^2} \exp\left\{-\frac{m_t^2 + (f(x_t \mid \beta))^2}{2\sigma^2}\right\} I_0\left(\frac{f(x_t \mid \beta) m_t}{\sigma^2}\right)$$

$$t = 1, \ldots, n$$

$$L = \frac{\prod_{t=1}^{n} m_t}{\sigma^{2n}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[m_t^2 + (f(x_t \mid \beta))^2\right]\right\} \prod_{t=1}^{n} I_0\left(\frac{f(x_t \mid \beta) m_t}{\sigma^2}\right)$$

$$LL = -n \log(\sigma^2) + \sum_{t=1}^{n} \log(m_t)$$

$$-\frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[m_t^2 + (f(x_t \mid \beta))^2\right] + \sum_{t=1}^{n} \log\left[I_0\left(\frac{f(x_t \mid \beta) m_t}{\sigma^2}\right)\right]$$

Maximize LL: $\frac{\partial LL}{\partial \beta} = 0$ and $\frac{\partial LL}{\partial \sigma^2} = 0$ Under H_1 and H_0

Estimation: Large SNR Normal

Ricean

$$p(m_t) = \frac{m_t}{\sigma^2} \exp\left\{-\frac{m_t^2 + (f(x_t \mid \beta))^2}{2\sigma^2}\right\} I_0\left(\frac{f(x_t \mid \beta) m_t}{\sigma^2}\right)$$

Normal as SNR ↑

$$p(m_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}[m_t - f(x_t \mid \beta)]^2\right\}$$

Then use usual least squares estimation.

Maximize LL: $\frac{\partial LL}{\partial \beta} = 0$ and $\frac{\partial LL}{\partial \sigma^2} = 0$ Under H_1 and H_0

$$LL = -2n \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} [m_t - f(x_t \mid \beta)]^2$$

Estimation: Large SNR Normal

$$LL = -2n\log(\sigma^2) + -\frac{1}{2\sigma^2}\sum_{t=1}^{n}[m_t - f(x_t \mid \beta)]^2$$

Under H_1:

$$\frac{\partial LL}{\partial \beta} = -\frac{2}{\sigma^2}\sum_{t=1}^{n}[m_t - f(x_t \mid \beta)]\frac{\partial f(x_t \mid \beta)}{\partial \beta}$$

Under H_0: add Lagrange constraint $h(\beta,\sigma^2)$ to LL

$$\frac{\partial LL}{\partial \beta} = -\frac{2}{\sigma^2}\sum_{t=1}^{n}[m_t - f(x_t \mid \beta)]\frac{\partial f(x_t \mid \beta)}{\partial \beta} + \frac{\partial h(\beta,\sigma^2)}{\partial \beta}$$

Under H_0 and H_1:

$$\frac{\partial LL}{\partial \sigma^2} = -\frac{2n}{\sigma^2} - \frac{1}{\sigma^4}\sum_{t=1}^{n}[m_t - f(x_t \mid \beta)]^2 \quad \left(+ \frac{\partial h(\beta,\sigma^2)}{\partial \sigma^2} \right)$$

May require numerical maximization depending on $f(x_t|\beta)$.

Estimation: Large SNR Normal

GLM: Does not require numerical maximization. X known

Under H_1: $\hat{\beta} = (X'X)^{-1}X'm$ $\hat{\sigma}^2 = (y - X\hat{\beta})'(y_j - X\hat{\beta})/n$

Under H_0: $h(\beta,\sigma^2) = 2\psi'C\beta/\sigma^2$

$$\tilde{\beta} = \Psi(X'X)^{-1}X'm \qquad \tilde{\sigma}^2 = (y - X\tilde{\beta})'(y - X\tilde{\beta})/n$$

$$\Psi = I - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C$$

Insert back into likelihoods and take ratio.

$$\lambda = L(\tilde{\beta},\tilde{\sigma}^2)/L(\hat{\beta},\hat{\sigma}^2)$$

This is how we get our usual t and F statistics.

Estimation: Large SNR Normal

DTI: Requires numerical maximization. b and r_t known

$$LL = -2n\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}[m_t - S_0\exp(-br_t'Dr_t)]^2$$

Under H_1:

$$\frac{\partial LL}{\partial S_0} = -\frac{2}{\sigma^2}\sum_{t=1}^{n}[m_t - S_0\exp(-br_t'Dr_t)]\frac{\partial S_0\exp(-br_t'Dr_t)}{\partial S_0}$$

$$\hat{S}_0 \mid \hat{D} = \left[\sum_{t=1}^{n} m_t\exp(-br_t'\hat{D}r_t)\right] / \left[\sum_{t=1}^{n}\exp(-2br_t'\hat{D}r_t)\right]$$

$$\frac{\partial LL}{\partial D} = 0$$ Does not yield a closed form solution.

Need numerical maximization with say Newton-Raphson or Levenberg-Marquardt.

Estimation: Large SNR Normal

Numerical maximization.

$$LL = -\frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t - \sqrt{f(x_t \mid \beta)^2 + \sigma^2}\right]^2$$

$\beta^{(0)}: \quad \sum_{t=1}^{n}[m_t - f(x_t \mid \beta)]^2$ Minimized by Levenberg-Marquardt

$$(\sigma^2)^{(0)} = \sum_{t=1}^{n}\left[m_t - f(x_t \mid \beta^{(0)})\right]^2 / n$$

$\beta^{(r+1)}: \sum_{t=1}^{n}\left[m_t - \sqrt{f(x_t \mid \beta)^2 + (\sigma^2)^{(r)}}\right]^2$ Minimize by Levenberg-Marquardt

$(\sigma^2)^{(0)}: \quad LL = -\frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t - \sqrt{f(x_t \mid \beta)^2 + \sigma^2}\right]^2$

Minimized by Newton-Raphson

$\beta^{(0)}, (\sigma^2)^{(0)}, \beta^{(1)}, (\sigma^2)^{(1)}, \ldots. \beta^{(r+1)}, (\sigma^2)^{(r+1)}$ sequence converges to MLE!

Estimation: Small SNR Ricean

$$LL = -n\log(\sigma^2) + \sum_{t=1}^{n}\log(m_t) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t^2 + (f(x_t\,|\,\beta))^2\right] + \sum_{t=1}^{n}\log\left[I_0\left(f(x_t\,|\,\beta)m_t/\sigma^2\right)\right]$$

Under H_1: $A_t = f(x_t\,|\,\beta)m_t/\sigma^2$

$$\frac{\partial LL}{\partial\beta} = \frac{1}{\sigma^2}\sum_{t=1}^{n}\left[m_t I_1(A_t)/I_0(A_t) - f(x_t\,|\,\beta)\right]\frac{\partial f(x_t\,|\,\beta)}{\partial\beta}$$

Under H_0:

$$\frac{\partial LL}{\partial\beta} = \frac{1}{\sigma^2}\sum_{t=1}^{n}\left[m_t I_1(A_t)/I_0(A_t) - f(x_t\,|\,\beta)\right]\frac{\partial f(x_t\,|\,\beta)}{\partial\beta} + \frac{\partial h(\beta,\sigma^2)}{\partial\beta}$$

Under H_1 and H_0

$$\frac{\partial LL}{\partial\sigma^2} = \frac{1}{2\sigma^4}[m_t^2 + (f(x_t\,|\,\beta))^2 - 2m_tA_t f(x_t\,|\,\beta) - 2n\sigma^2] \quad \left(+\frac{\partial h(\beta,\sigma^2)}{\partial\sigma^2}\right)$$

No closed form solution. Requires numerical maximization!

Estimation: Small SNR Ricean

EM Algorithm. Easier and convenient. Does not need phase.

Take magnitude variates $m_1,\ldots,m_n$ that are Ricean distributed

$$p(m_t)=\frac{m_t}{\sigma^2}\exp\left\{-\frac{m_t^2+f(x_t\mid\beta)^2}{2\sigma^2}\right\}I_0\left(\frac{f(x_t\mid\beta)m_t}{\sigma^2}\right)$$

Introduce latent phase variables $\phi_1,\ldots,\phi_n$ such that

$$p(m_t,\phi_t) = \frac{m_t}{2\pi\sigma^2}\exp[-(m_t^2+f(x_t\mid\beta)^2 - 2m_t f(x_t\mid\beta)\cos\phi_t)/2\sigma^2]$$

and

$$LL=-n\log(2\pi\sigma^2)+\sum_{t=1}^{n}\log(m_t) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t^2+f(x_t\mid\beta)^2-2m_t f(x_t\mid\beta)\cos(\phi_t)\right]$$

Zhu, H. et al., JASA, in press, 2009.
Dempster, Laird, Rubin. JRSS, 1977.

Estimation: Small SNR
EM Algorithm. Iterative.

$$LL = -n\log(2\pi\sigma^2) + \sum_{t=1}^{n}\log m_t - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t^2 + f(x_t\,|\,\beta)^2 - 2m_t f(x_t\,|\,\beta)\cos\phi_t\right]$$

E Step: Let $\mathrm{Y_m}=(m_1,\ldots,m_n)$, $\mathrm{Y}_\phi=(\phi_1,\ldots,\phi_n)$, $Y_x=(x_1,\ldots,x_n)$

given $\beta^{(r)},(\sigma^2)^{(r)}$: Initial values from normal GLM

$$E[L_c(\beta,\sigma^2\,|\,Y_m,Y_\phi,Y_x)\,|\,Y_m,Y_x,\beta^{(r)},(\sigma^2)^{(r)}] =$$

$$-n\log(\sigma^2)^{(r)} - \frac{1}{2(\sigma^2)^{(r)}}\sum_{t=1}^{n}\left[m_t^2 + f(x_t\,|\,\beta^{(r)})^2 - 2m_t f(x_t\,|\,\beta^{(r)})A_t^{(r)}\right]$$

$$A_t^{(r)} = f(x_t\,|\,\beta^{(r)})m_t\big/(\sigma^2)^{(r)}$$

with respect to $p(Y_\phi\,|\,Y_m,Y_x,\beta^{(r)},(\sigma^2)^{(r)}) = \prod_{t=1}^{n} p(\phi_t\,|\,m_t,\beta^{(r)},(\sigma^2)^{(r)})$

Zhu, H. et al., JASA, in press, 2009.
Dempster, Laird, Rubin. JRSS, 1977.

Estimation: Small SNR
EM Algorithm. Iterative.

M Step:

given $\beta^{(r)}, (\sigma^2)^{(r)}$:

$$(\sigma^2)^{(r+1)} = \frac{1}{2n}\sum_{t=1}^{n}\left[m_t^2 + f(x_t \mid \beta^{(r)})^2 - 2m_t f(x_t \mid \beta^{(r)})A_t^{(r)}\right]$$

$$A_t^{(r)} = f(x_t \mid \beta^{(r)})m_t \Big/ (\sigma^2)^{(r)}$$

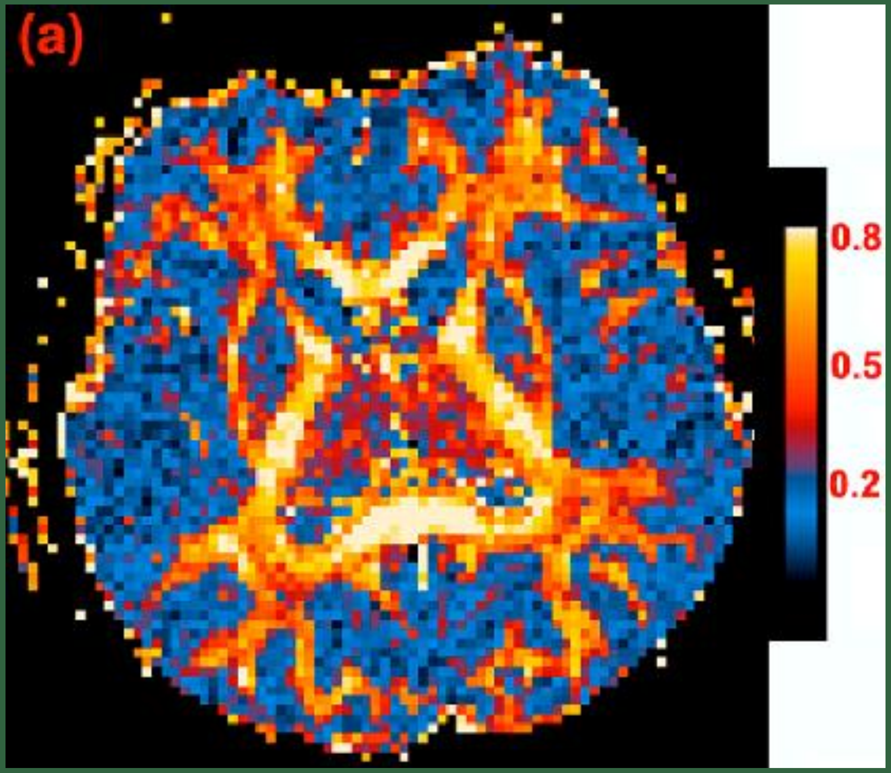
$\beta^{(r+1)}$: minimize $\sum_{t=1}^{n}\left[f(x_t \mid \beta)^2 - m_t A_t^{(r)}\right]^2$ given $(\sigma^2)^{(r+1)}$

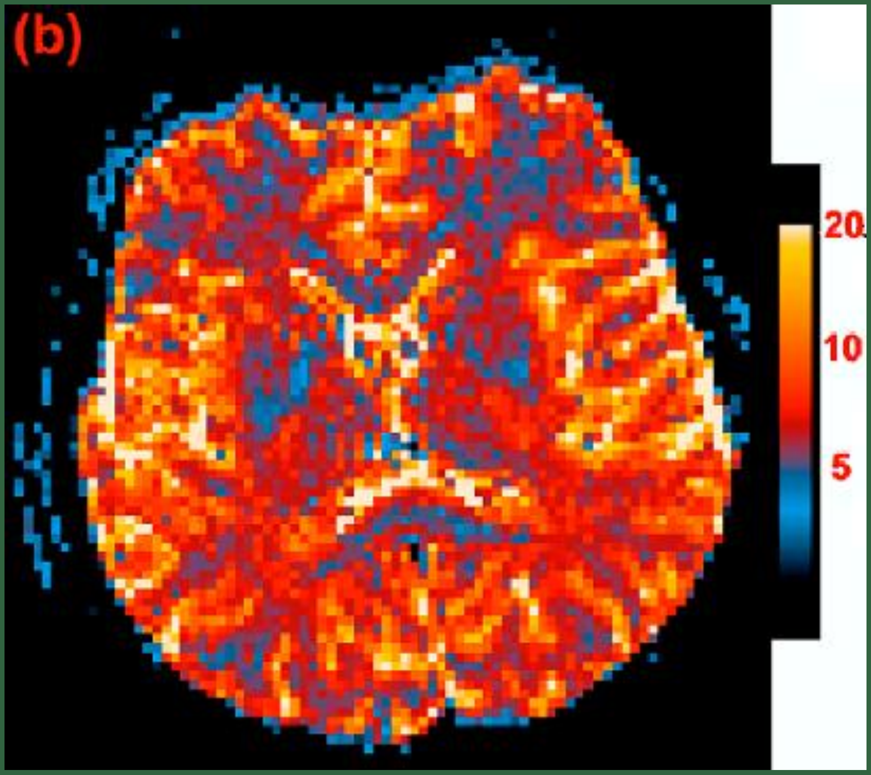
$\beta^{(0)}, (\sigma^2)^{(0)}, \beta^{(1)}, (\sigma^2)^{(1)}, \ldots. \beta^{(r+1)}, (\sigma^2)^{(r+1)}$ sequence converges to MLE!

Zhu, H. et al., JASA, in press, 2009.
Dempster, Laird, Rubin. JRSS, 1977.

Estimation: Small SNR EM Algorithm.

$$f(S_0, D \mid r, b) = S_0 \exp(-br'Dr)$$





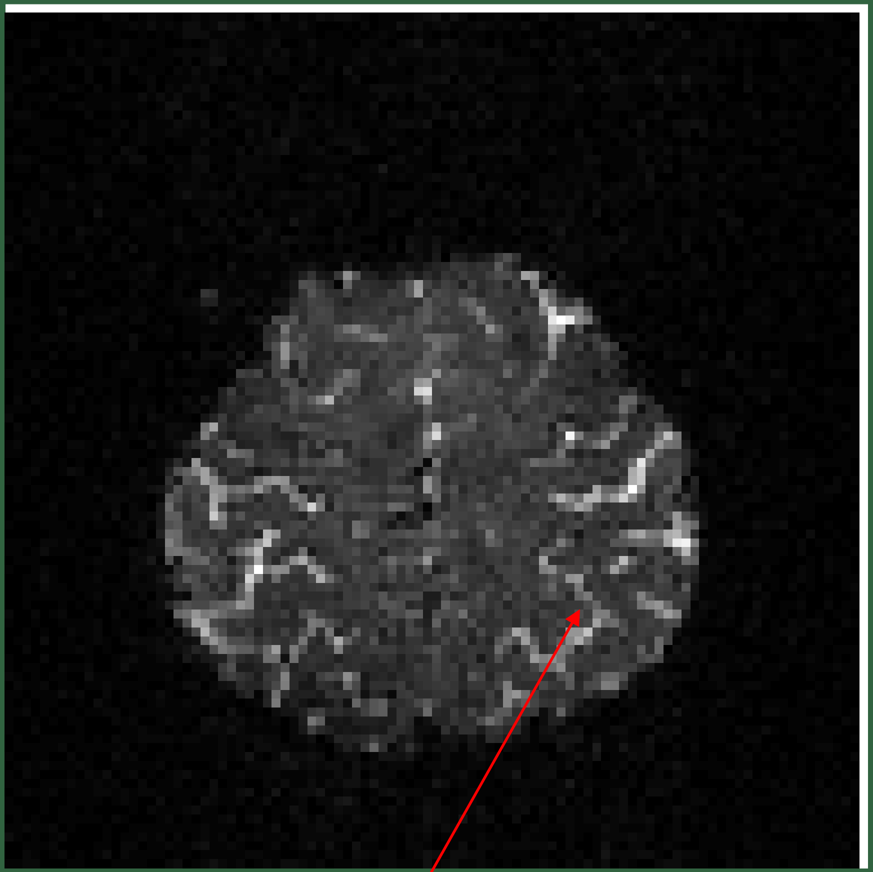
Fractional Anisotropy, FA

Signal-to-Noise Ratio, S_0/σ^2

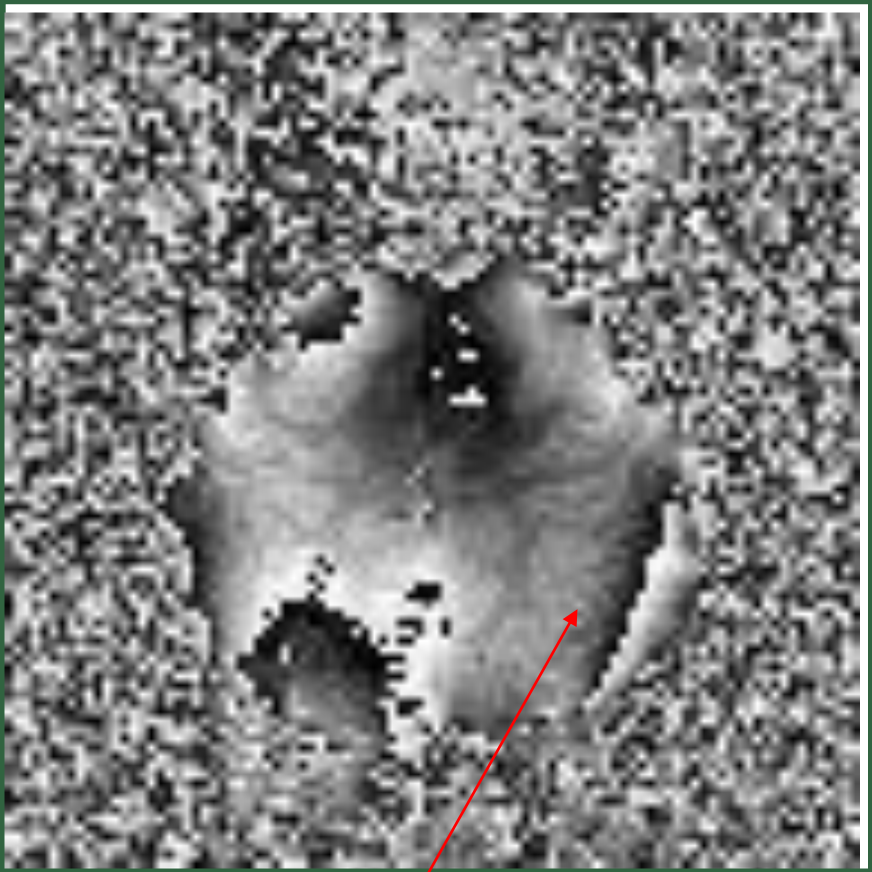
Zhu, H. et al., JASA, in press, 2009.

Estimation: Bivariate Normal

Magnitude Image



Phase Image



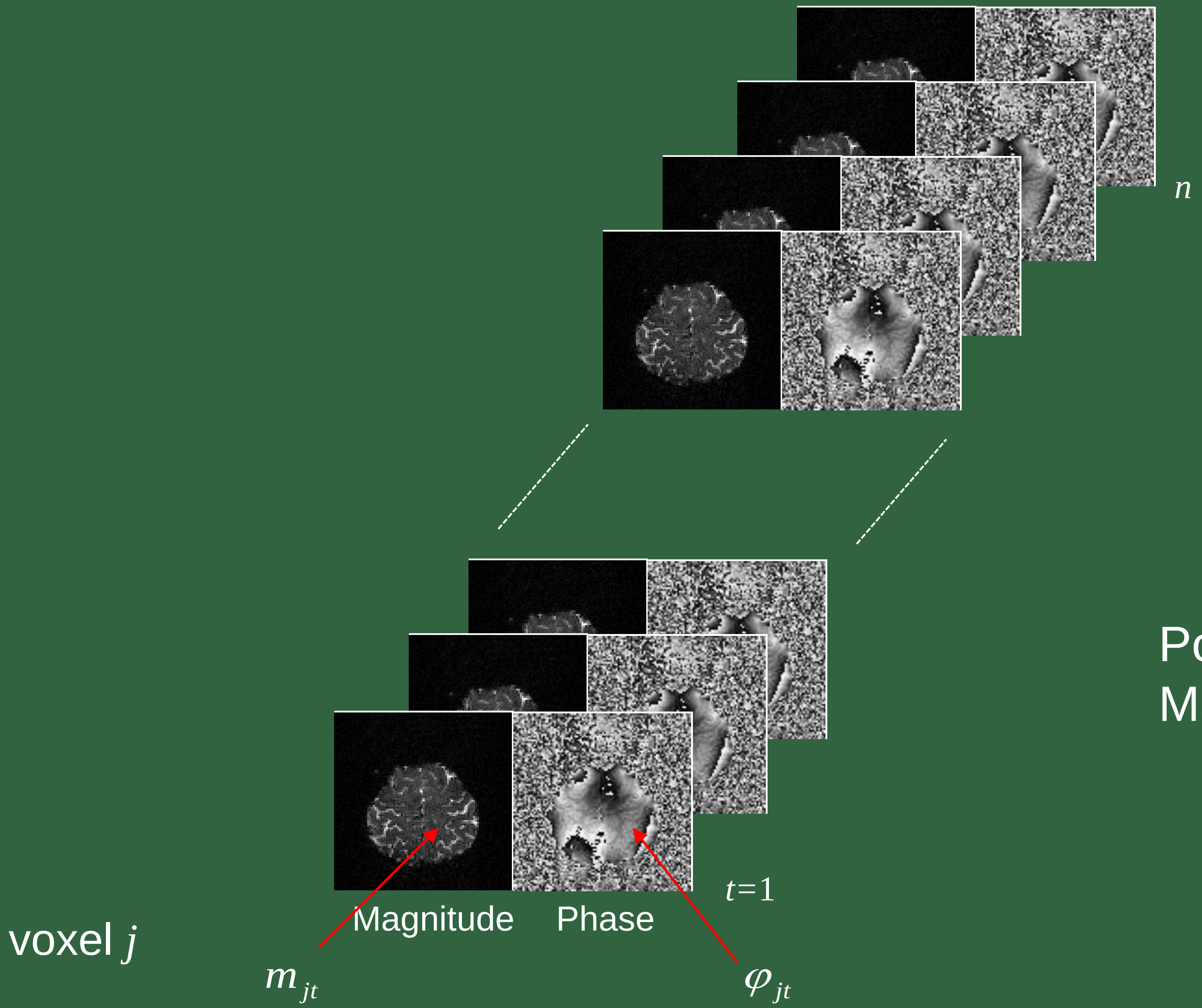
voxel j

$$m_j = \sqrt{y_{Rj}^2 + y_{Ij}^2}$$

$$\varphi_j = \tan^{-1}(y_{Ij} / y_{Rj})$$

Statistics:

We get n images under different signal conditions



Polar Coordinates
Magnitude-Phase

Estimation: All SNRs Bivariate Normal

$$\begin{pmatrix} y_{Rt} \\ y_{It} \end{pmatrix} = \begin{pmatrix} \rho_t \cos\theta_t \\ \rho_t \sin\theta_t \end{pmatrix} + \begin{pmatrix} \eta_{Rt} \\ \eta_{It} \end{pmatrix}, \quad \begin{pmatrix} \eta_{Rt} \\ \eta_{It} \end{pmatrix} \sim N\left(0, \Sigma\right)$$

$$p(y_{Rt}, y_{It}) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2} \left[(y_{Rt} - \rho_t \cos\theta_t)^2 + (y_{It} - \rho_t \sin\theta_t)^2 \right] \right\}$$

$$p(m_t, \varphi_t) = \frac{m_t}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2} \left[m_t^2 + \rho_t^2 - 2 m_t \rho_t \cos(\varphi_t - \theta_t) \right] \right\}$$

$\rho_t = f(x_t \mid \beta)$ and $\theta_t = g(u_t \mid \gamma)$

$$LL = -n \log(2\pi\sigma^2) + \sum_{t=1}^{n} \log(m_t)$$
$$- \frac{1}{2\sigma^2} \sum_{t=1}^{n} \left[m_t^2 + f(x_t \mid \beta)^2 - 2 m_t f(x_t \mid \beta) \cos(\varphi_t - g(u_t \mid \gamma)) \right]$$

Estimation: All SNR Bivariate Normal

$$LL = -n\log(2\pi\sigma^2) + \sum_{t=1}^{n}\log(m_t)$$
$$-\frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t^2 + f(x_t\,|\,\beta)^2 - 2m_t f(x_t\,|\,\beta)\cos(\varphi_t - g(u_t\,|\,\gamma))\right]$$

$$\frac{\partial LL}{\partial \beta} = \frac{1}{\sigma^2}\sum_{t=1}^{n}\left[m_t\cos(\varphi_t - g(u_t\,|\,\gamma)) - f(x_t\,|\,\beta)\right]\frac{\partial f(x_t\,|\,\beta)}{\partial \beta}$$

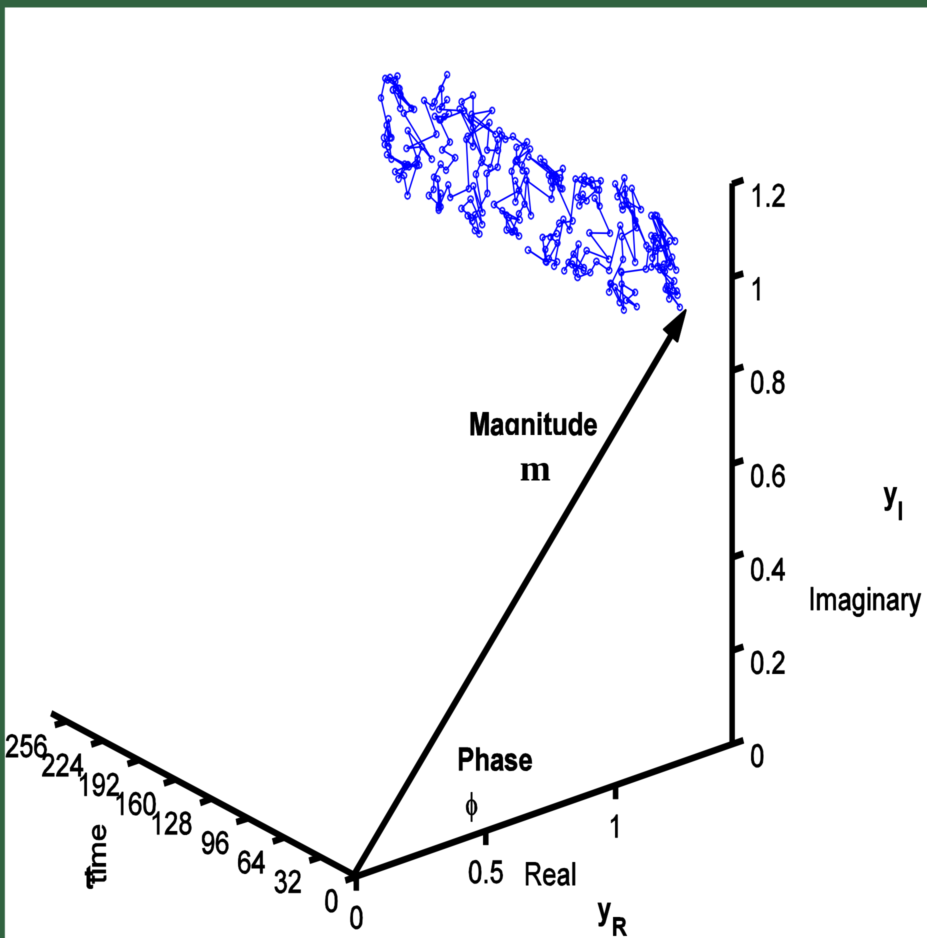
$$\frac{\partial LL}{\partial \gamma} = \frac{1}{\sigma^2}\sum_{t=1}^{n}\left[m_t f(x_t\,|\,\beta)\sin(\varphi_t - g(u_t\,|\,\gamma))\right]\frac{\partial g(u_t\,|\,\gamma)}{\partial \gamma}$$

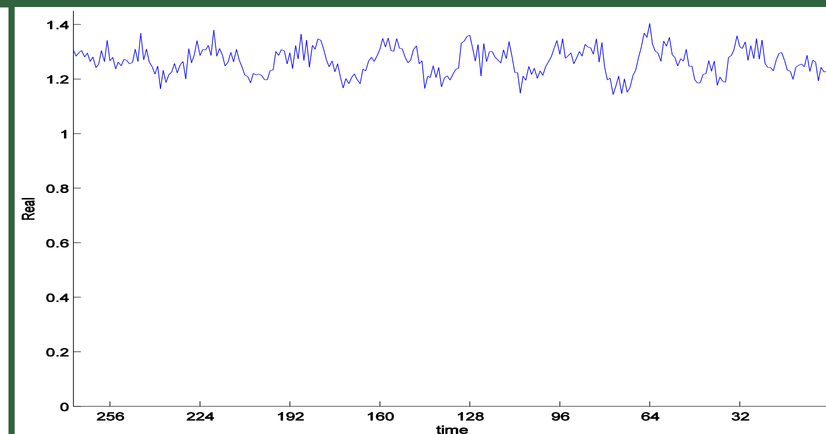
$$\frac{\partial LL}{\partial \sigma^2} = \frac{n}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^{n}\begin{bmatrix} m_t^2 + f(x_t\,|\,\beta)^2 \\ \quad - 2m_t f(x_t\,|\,\beta)\cos(\varphi_t - g(u_t\,|\,\gamma)) \end{bmatrix}$$

σ^2 can be uniquely solved for given β, γ

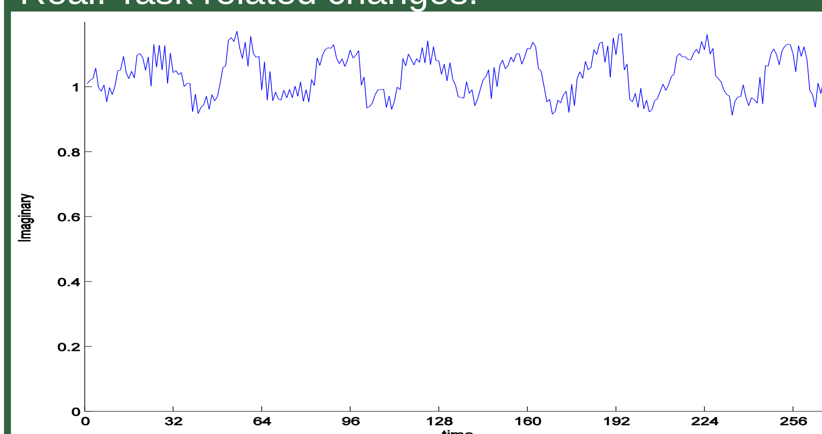
Estimation:

Time series are complex, bivariate with phase coupled means.





Real: Task related changes!

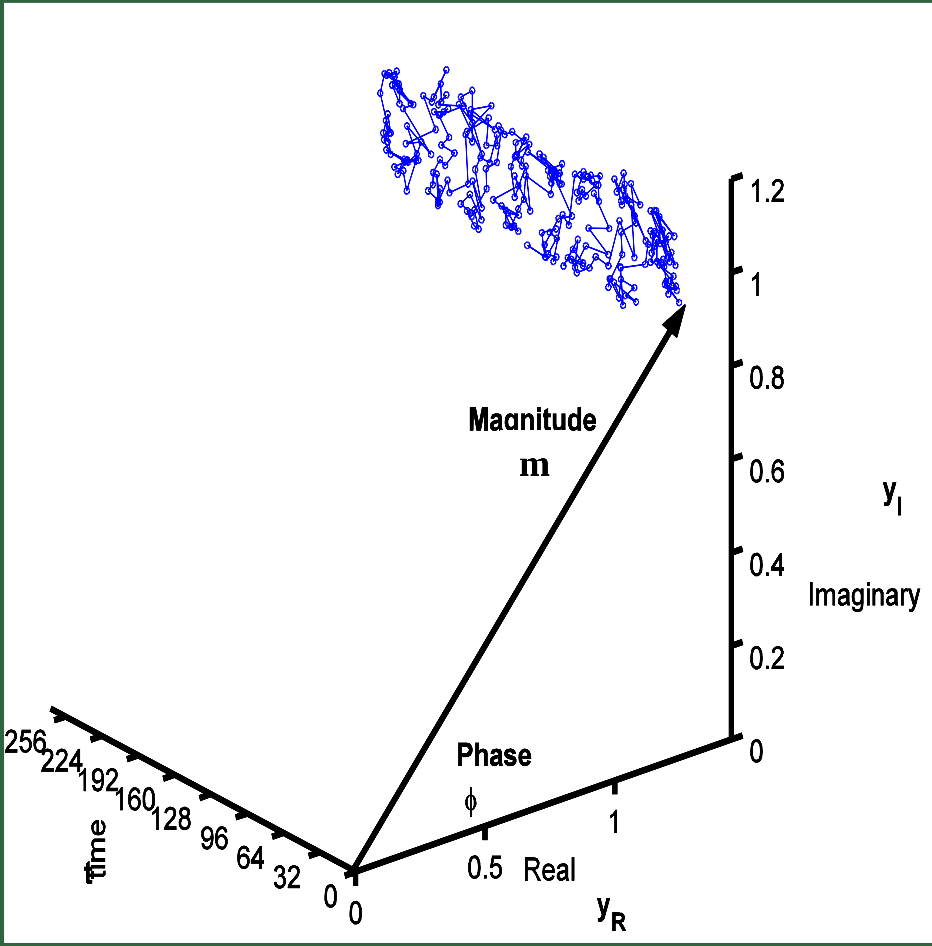


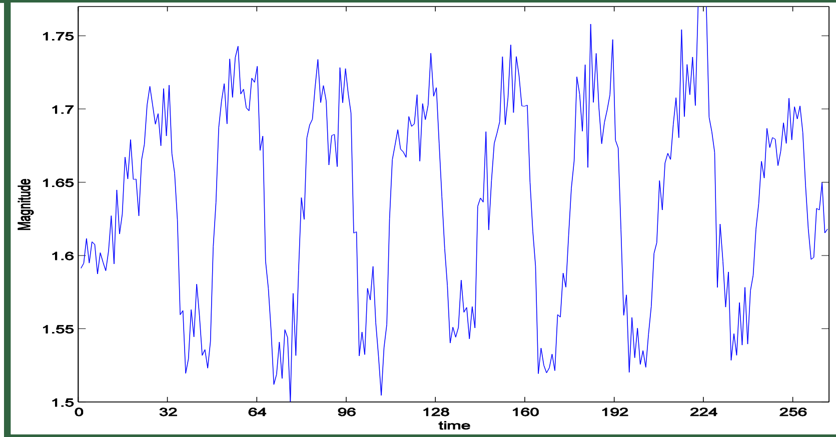
Imaginary: Task related changes!

The y_R and y_I time courses have related info! From actual human data!

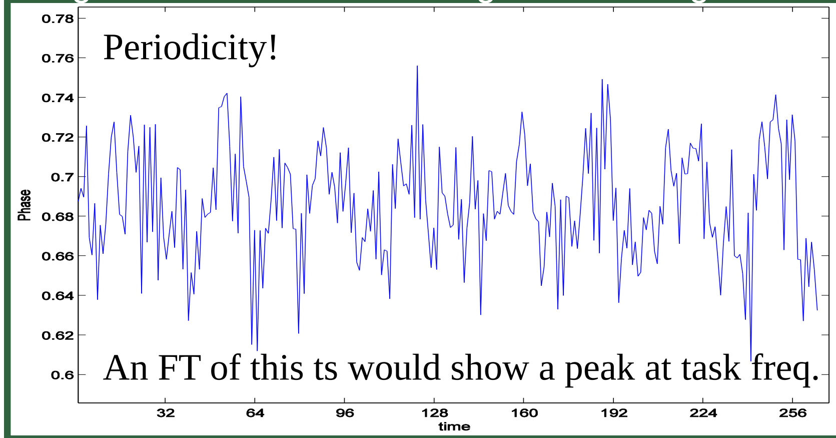
Estimation:

Time series are complex, bivariate with phase coupled means.





Magnitude: Task related magnitude changes!

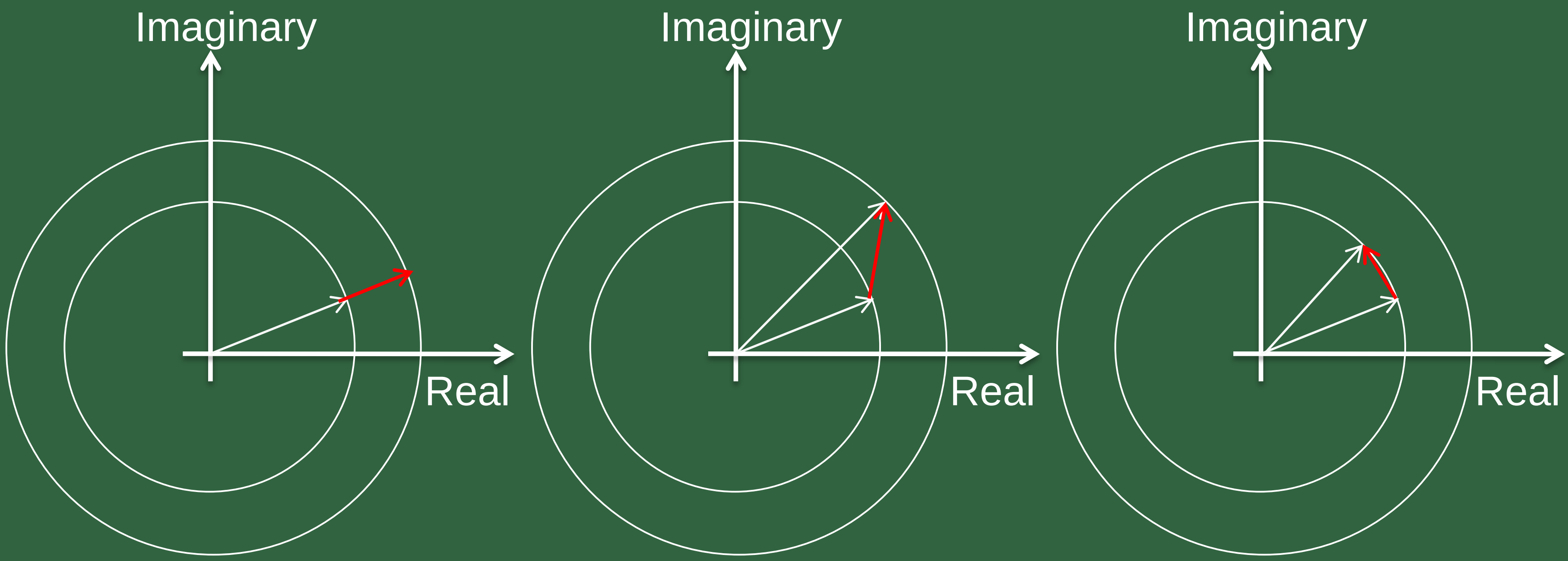


Phase: Task related phase changes!

Hoogenrad et al. 1998, Menon 2002, Nencka et al. 2007, Bodurka et al. 1999, Chow et al. 2006.

Estimation:

Magnitude and or phase change.

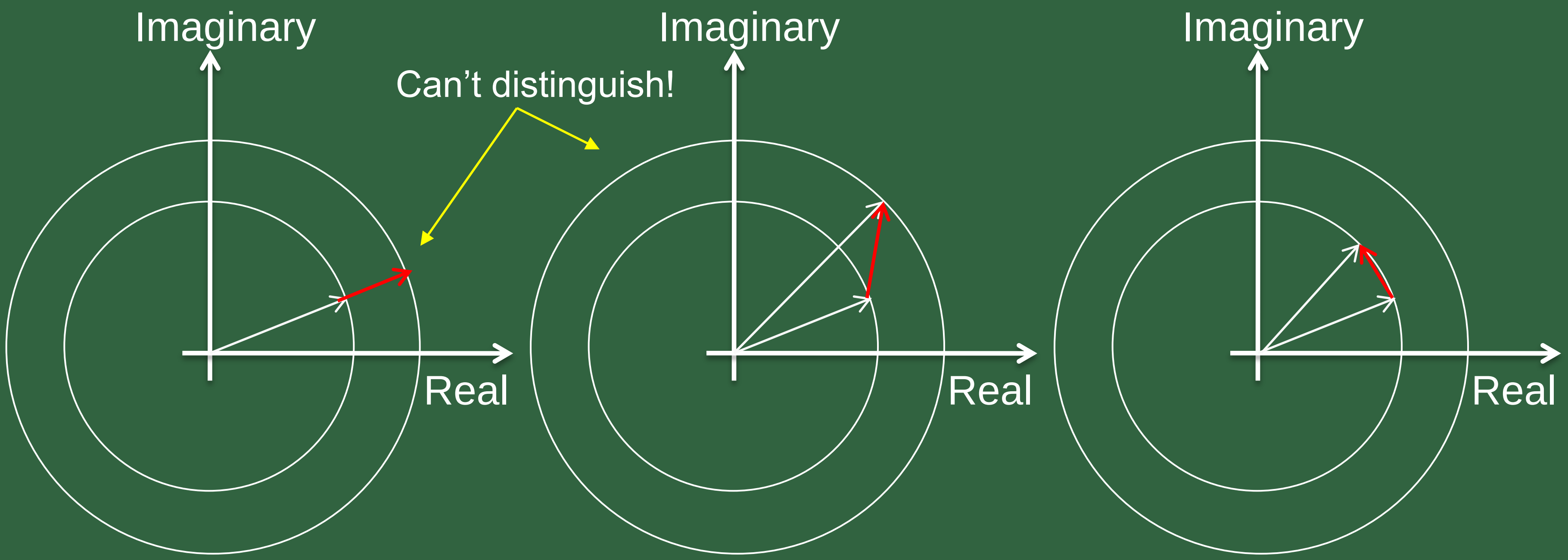


$\rho_t = x_t'\beta$ and $\theta_t = u_t'\gamma$

[1]Rowe and Logan: NIMG, 23:1078-1092, 2004.
[2]Rowe: NIMG 25:1124-1132, 2005a.
[3]Rowe: NIMG, 25:1310-1324, 2005b.
[4]Bandettini et al.: MRM, 30:161-173, 1993.
[5]Rowe and Logan: NIMG 24:603-606, 2005.
[6]Rowe, et al.: JNeuroSciMeth, 161:331-341, 2007.
[7]Rowe: MRM, to appear, 2009.

Estimation:

Magnitude and or phase change.



$\rho_t = x_t'\beta$ and $\theta_t = u_t'\gamma$

[1]Rowe and Logan: NIMG, 23:1078-1092, 2004.
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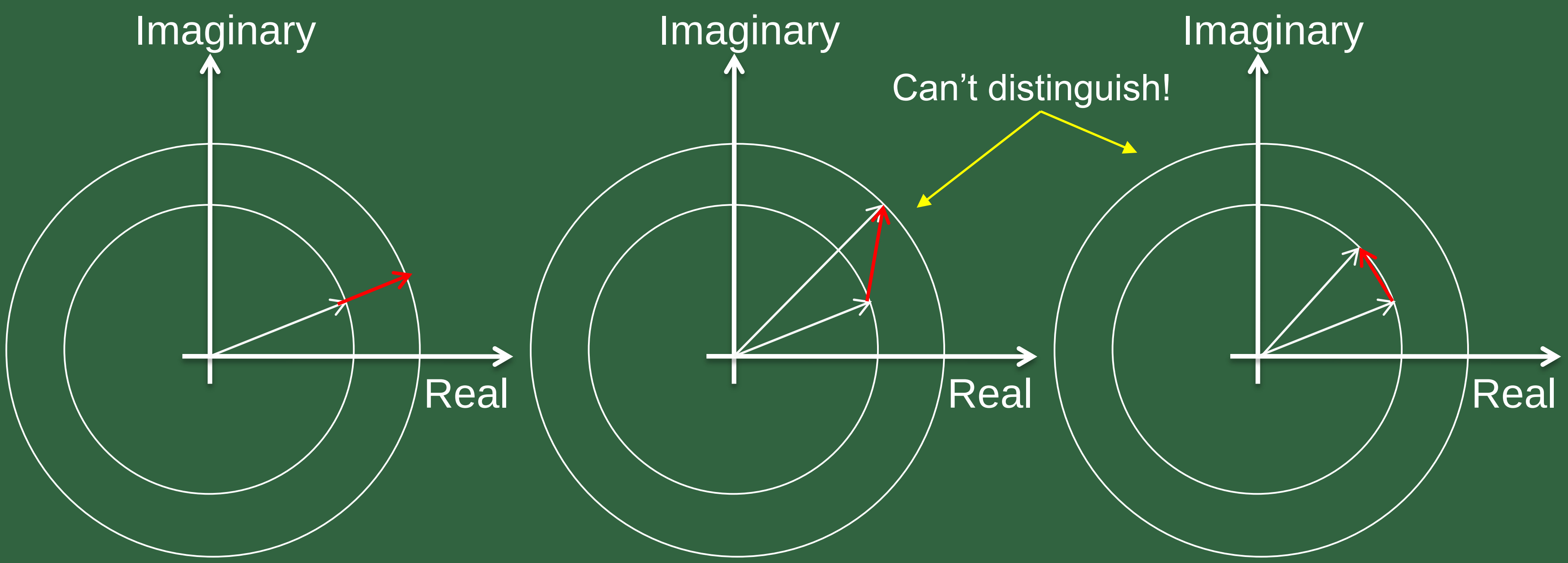
Estimation:

Magnitude and or phase change.



$\rho_t = x_t'\beta$ and $\theta_t = u_t'\gamma$

[1]Rowe and Logan: NIMG, 23:1078-1092, 2004.
[2]Rowe: NIMG 25:1124-1132, 2005a.
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[4]Bandettini et al.: MRM, 30:161-173, 1993.
[5]Rowe and Logan: NIMG 24:603-606, 2005.
[6]Rowe, et al.: JNeuroSciMeth, 161:331-341, 2007.
[7]Rowe: MRM, to appear, 2009.

Estimation: All SNR

GLM:

$\rho_t = x_t'\beta$ and $\theta_t = u_t'\gamma$

$$H_0: \mathcal{C}\beta = 0 \quad vs. \quad H_1: \mathcal{C}\beta \neq 0$$
$$\mathcal{D}\gamma = 0 \qquad\qquad \mathcal{D}\gamma \neq 0$$

$$LL = -n\log(2\pi\sigma^2) + \sum_{t=1}^{n}\log(m_t)$$
$$-\frac{1}{2\sigma^2}\sum_{t=1}^{n}\left[m_t^2 + (x_t'\beta)^2 - 2m_t x_t'\beta\cos(\varphi - u_t'\gamma)\right]$$

Maximize LL: under H_1 and H_0

$\beta^{(0)}$: initial value

$\varphi_*^{(r)}$ has elements $\varphi_t\sqrt{m_t x_t'\hat{\beta}^{(r)}}$

$\hat{Z}_{(r)}'$ has rows $u_t'\sqrt{m_t x_t'\hat{\beta}^{(r)}}$

$m_*^{(r)}$ has elements $m_t\cos(\varphi_t - u_t'\hat{\gamma}^{(r)})$

$$\hat{\gamma}^{(r)} = (\hat{Z}_{(r)}'\hat{Z}_{(r)})^{-1}\hat{Z}_{(r)}'\hat{\varphi}_*^{(r)}$$

$$\hat{\beta}^{(r+1)} = (X'X)^{-1}X'm_*^{(r)}$$

$$(\hat{\sigma}^2)^{(r+1)} = \frac{1}{2n}\sum_{t=1}^{n}\left[\begin{array}{l}(m - X\hat{\beta}^{(r+1)})'(m - X\hat{\beta}^{(r+1)}) \\ +2(m - \hat{m}_*^{(r+1)})'X\hat{\beta}^{(r+1)}\end{array}\right]$$

Rowe: NIMG, 25:1310-1324, 2005.
Rowe: MRM, to appear, 2009.

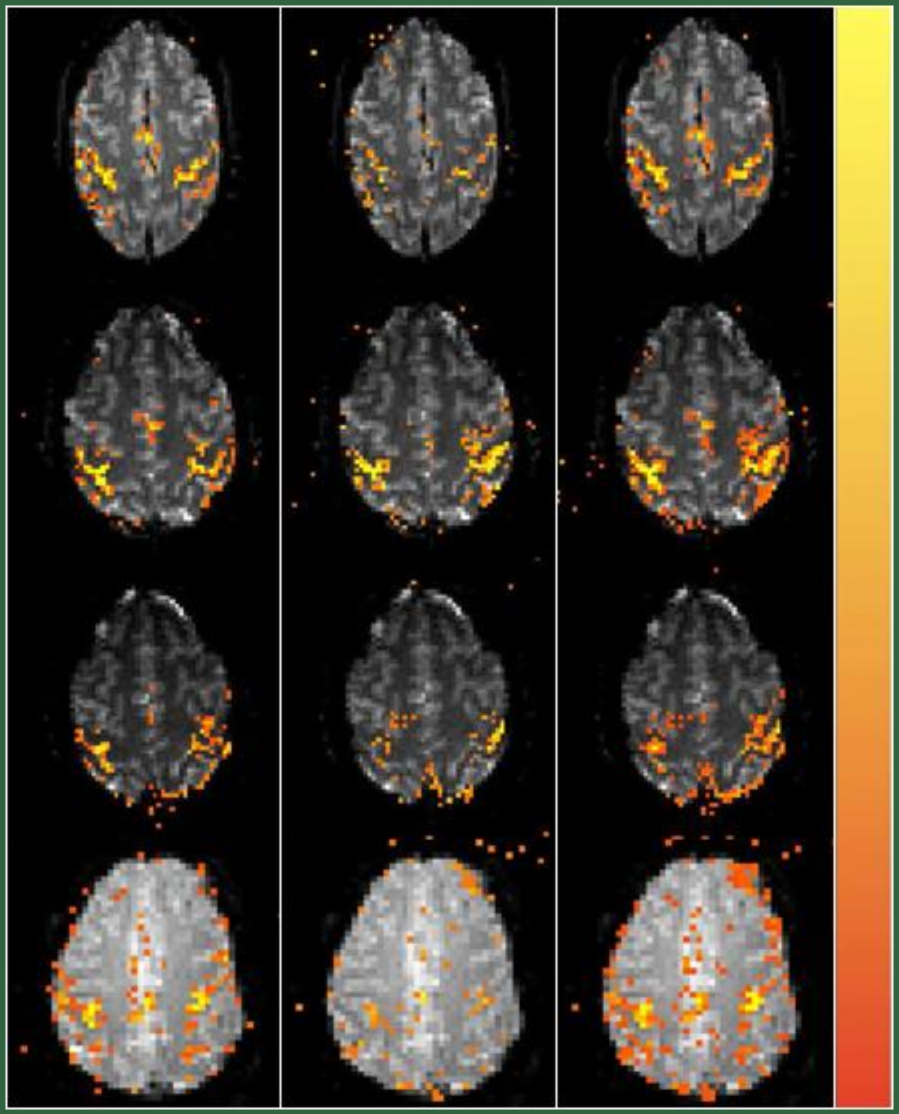
Estimation: GLM:

20s off+16×(8 s on 8 s off), 276 TRs
12 axial slices, 96 × 96, FOV = 24 cm
TH = 2.5 mm, TR = 1 s, TE = 34.6 ms
FA = 45◦, BW = 125 kHz, ES = .708 ms

20s off+16×(8 s on 8 s off), 276 TRs
10 axial slices, 96 × 96, FOV = 24 cm
TH = 2.5 mm, TR = 1 s, TE = 42.8 ms
FA = 45◦, BW = 125 kHz, ES = .768 ms

20s off+16×(8 s on 8 s off), 276 TRs
10 axial slices, 96 × 96, FOV = 24 cm,
TH = 2.5 mm, TR = 1 s, TE = 42.8 ms
FA = 45◦, BW = 125 kHz, ES = . 768 ms

20s off+10×(8 s on 8 s off), 180 TRs
9 axial slices, 64 × 64, FOV = 24 cm
TH = 3.8 mm, TR = 1 s, TE = 26.0 ms
FA = 45◦, BW = 125 kHz, ES = .680 ms



Rowe: NIMG, 25:1310-1324, 2005.
Rowe: MRM, to appear, 2009.

Hahn, Nencka, Rowe: NIMG, 742-752, 2009.
Hahn, Nencka, Rowe: In progress.

$$\Delta B_t = \frac{\arg\left(I_t \sum_{j=1}^{n} \left(\frac{I_j^*}{|I_j|} \right) \right)}{\gamma \mathrm{TE}}$$

Discussion:

- Not clear how much improvement from Ricean distribution.
- Improvements will show below SNR=5. High b-values.
- Other factors hinder it.
  - Dynamic field changes
  - Image Warping
  - Motion
  - Image Processing
- Should also use phase for complete data model.
- More biological info extracted with use of phase.

Discussion:

1. Image Reconstruction

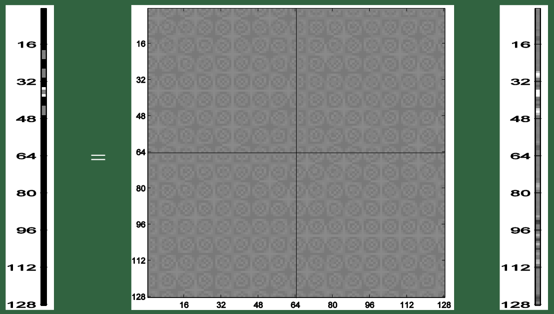
2. Statistics-Ricean & Normal

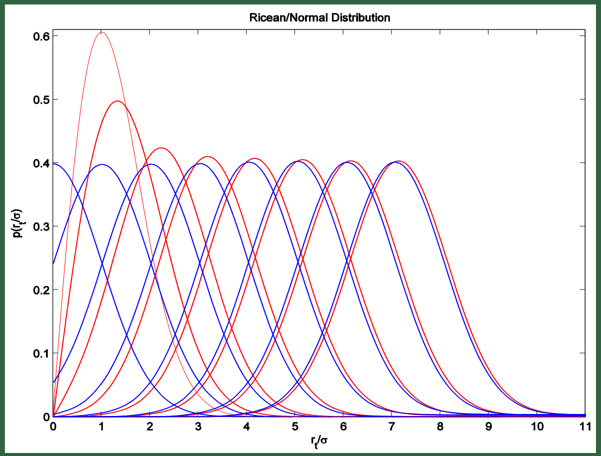
3. Estimation-Ricean & Normal

4. Estimation-Bivariate Normal

5. Discussion

Further research is needed ....





$$\begin{pmatrix} y_{Rt} \\ y_{It} \end{pmatrix} = \begin{pmatrix} \rho_t \cos\theta_t \\ \rho_t \sin\theta_t \end{pmatrix} + \begin{pmatrix} \eta_{Rt} \\ \eta_{It} \end{pmatrix}$$

$$\rho_t = f(x_t \mid \beta)$$

$$\theta_t = g(u_t \mid \gamma)$$

Thank You

Questions?