# Symmetric Matrix PDFs
## (Multivariate Gamma and Inverse Gamma)

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# Outline

**The Wishart PDF (multivariate gamma)**

**The Inverse Wishart PDF (multivariate inverse gamma)**

**Discussion**

**Homework**

# The Wishart PDF

In our first Statistics course we learned that if $x_1,x_2,\ldots,x_n$

are iid $N(\mu,\sigma^2)$ and we calculate

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ , then } y = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \text{ .}$$

That is, the PDF of $y$ is

$$f(y\,|\,\nu) = \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\nu/2}\Gamma\left(\nu/2\right)}$$

where $\nu=n$-1 and $y>0$.

# The Wishart PDF

$$y = \frac{(n-1)s^2}{\sigma^2}$$

If we perform the transformation of variable

$$s^2 = \frac{\sigma^2}{(n-1)} y$$

($s^2$ is treated as one symbol)

$$f(s^2 \mid \alpha, \beta) = \frac{(s^2)^{\alpha-1} e^{-\frac{s^2}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

$$E(s^2 \mid \alpha, \beta) = \alpha\beta$$

$$\mathrm{var}(s^2 \mid \alpha, \beta) = \alpha\beta^2$$

Then $s^2 \sim \Gamma\left(\frac{\nu}{2}, \frac{2\sigma^2}{\nu}\right)$ where $\nu = n-1$

$$\alpha = \frac{\nu}{2} \qquad \beta = \frac{2\sigma^2}{\nu}$$

with $f(s^2 \mid \nu, \sigma^2) = \dfrac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$ and $s^2, \sigma^2 > 0$.

$$E(s^2 \mid \nu, \sigma^2) = \sigma^2 \qquad var(s^2 \mid \nu, \sigma^2) = \frac{2\sigma^4}{\nu}$$

# The Wishart PDF

i.e. $n$=10, $m$=$10^6$

If we were to simulate a large number of samples $m$ each of size $n$ from $N(\mu,\sigma^2)$ then calculate $\bar{x}$ and $s^2$ for each

sample

$$\begin{array}{ccccc}
x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & & x_1^{(m)} \\
x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots & x_2^{(m)} \\
\vdots & \vdots & \vdots & & \vdots \\
x_n^{(1)} & x_n^{(2)} & x_n^{(3)} & & x_n^{(m)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bar{x}_{(1)} & \bar{x}_{(2)} & \bar{x}_{(3)} & \cdots & \bar{x}_{(m)} \\
s_{(1)}^2 & s_{(2)}^2 & s_{(3)}^2 & \cdots & s_{(m)}^2
\end{array}$$

If we made a histogram of $\bar{x}$'s we would see that they are $N(\mu,\sigma^2/n)$ and …

If we made a histogram of $s^2$'s, then we would see that they are $\Gamma\left(\frac{\nu}{2},\frac{2\sigma^2}{\nu}\right)$.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

# The Wishart PDF

We can write the gamma PDF

$$f(s^2 \mid \nu, \sigma^2) = \frac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$$

$$\nu = n-1$$

$$s^2, \sigma^2 > 0$$

as

$$f(s^2 \mid \nu, \sigma^2) = k\left|\frac{\sigma^2}{\nu}\right|^{-\frac{\nu}{2}} \left|s^2\right|^{\frac{\nu-1-1}{2}} e^{-\frac{1}{2}\left(\frac{\sigma^2}{\nu}\right)^{-1} s^2}$$

Note the eccentric way I wrote this.

$$k = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}$$

# The Wishart PDF

$$f(s^2|\nu,\sigma^2)=\frac{(s^2)^{\frac{\nu}{2}-1}e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$$

In bivariate (multivariate) statistics if

$\underset{2\times 1}{x_1}, \underset{2\times 1}{x_2},.., \underset{2\times 1}{x_n}$ are iid $N(\mu,\Sigma)$ and we calculate the covariance matrix

$\underset{2\times 2}{S}=\frac{1}{n-1}\sum_{i=1}^{n}\underbrace{(\underset{2\times 1}{x_i}-\underset{2\times 1}{\bar{x}})}_{2\times 1}\underbrace{(\underset{2\times 1}{x_i}-\underset{2\times 1}{\bar{x}})'}_{1\times 2}$, then the PDF of the covariance matrix

$\underset{2\times 2}{S}$ has the (multivariate) generalization of the gamma distribution

$$f(\underset{2\times 2}{S}|\underset{2\times 2}{\Sigma},\nu)=k_W\,|\underset{2\times 2}{\Sigma}/\nu|^{-\frac{\nu}{2}}\,|\underset{2\times 2}{S}|^{\frac{\nu-2-1}{2}}\,e^{-\frac{1}{2}tr\left((\underset{2\times 2}{\Sigma}/\nu)^{-1}\underset{2\times 2}{S}\right)} .$$

$p$

$S,\Sigma>0$

$\nu=n-1$

$tr()$=trace

normalizing constant

Remember the eccentric way I wrote this.

$p$

$$f(s^2|\nu,\sigma^2)=k\left|\frac{\sigma^2}{\nu}\right|^{-\frac{\nu}{2}}|s^2|^{\frac{\nu-1-1}{2}}e^{-\frac{1}{2}\left(\frac{\sigma^2}{\nu}\right)^{-1}s^2}$$

# The Wishart PDF

i.e. $n$=10, $m$=$10^6$

If we were to simulate a large number of samples $m$ each of size $n$ from $N(\underset{p\times 1}{\mu},\underset{p\times p}{\Sigma})$ then calculate $\underset{p\times 1}{\bar{x}}$ and $\underset{p\times p}{S}$ for each.

sample

$$\begin{array}{ccccc}
x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & & x_1^{(m)} \\
x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots & x_2^{(m)} \\
\vdots & \vdots & \vdots & & \vdots \\
x_n^{(1)} & x_n^{(2)} & x_n^{(3)} & & x_n^{(m)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\underset{p\times 1}{\bar{x}}^{(1)} & \underset{p\times 1}{\bar{x}}^{(2)} & \underset{p\times 1}{\bar{x}}^{(3)} & \cdots & \underset{p\times 1}{\bar{x}}^{(m)} \\
\underset{p\times p}{S}^{(1)} & \underset{p\times p}{S}^{(2)} & \underset{p\times p}{S}^{(3)} & \cdots & \underset{p\times p}{S}^{(m)}
\end{array}$$

If we made histograms of $\bar{x}$'s we would see that they are $N(\mu,\Sigma/n)$ and …

If we made histograms of $S$'s, then we would see that they are $W(\Sigma/\nu,\nu)$.

$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})'$$

# The Wishart PDF

Multivariate version of gamma distribution.

$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$$

A random $p \times p$ matrix variate $\underset{p\times p}{S}$ follows the Wishart

distribution with scale matrix $\underset{p\times p}{\Sigma/\nu}$ and $\nu$ $df$ denoted $\underset{p\times p}{S} \sim W(\underset{p\times p}{\Sigma/\nu},\nu)$

iff $f(\underset{p\times p}{S} \mid \underset{p\times p}{\Sigma},\nu) = k_W \left|\underset{p\times p}{\Sigma / \nu}\right|^{-\frac{\nu}{2}} \left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma/\nu})^{-1}\underset{p\times p}{S}}$

where $k_W^{-1} = 2^{\frac{\nu p}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(\frac{\nu+1-j}{2}\right)$

If $p$=1, $f(s^2 \mid \nu,\sigma^2) = \dfrac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$ is

Gamma distribution by $\alpha=\nu/2$ and $\beta=2\sigma^2/\nu$.

# The Wishart PDF

The Wishart matrix PDF is

$$f(\underset{p\times p}{S}\,|\,\Sigma,\nu)=k_W\left|\underset{p\times p}{\Sigma}/\nu\right|^{-\frac{\nu}{2}}\left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}}e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma}/\nu)^{-1}\underset{p\times p}{S}}$$

$$k_W^{-1}=2^{\frac{\nu p}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma\left(\frac{\nu+1-j}{2}\right)$$

The mean, variance, and covariance of its elements are

$$E(\underset{p\times p}{S}\,|\,\Sigma,\nu)=\underset{p\times p}{\Sigma}$$

$$\mathrm{var}(S_{ij}\,|\,\Sigma,\nu)=(\Sigma_{ij}^2+\Sigma_{ii}\Sigma_{jj})/\nu$$

$$\mathrm{cov}(S_{ij}S_{kl}\,|\,\Sigma,\nu)=(\Sigma_{ik}\Sigma_{jl}+\Sigma_{il}\Sigma_{jk})/\nu$$

If $p$=1

$$E(s\,|\,\sigma^2,\nu)=\sigma^2$$

$$var(s^2\,|\,\sigma^2,\nu)=\sigma^4/\nu$$

# The Wishart PDF

**Theorem:**

If $S$ is a $p \times p$ random matrix variable from $f(S|\Sigma,\nu)$, with

$$f(\underset{p\times p}{S}|\Sigma,\nu) = k_W \left|\underset{p\times p}{\Sigma}/\nu\right|^{-\frac{\nu}{2}} \left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma}/\nu)^{-1}\underset{p\times p}{S}}$$

then if we form $\underset{r\times r}{Q} = \underset{r\times p}{A}\;\underset{p\times p}{S}\;\underset{p\times r}{A'}$ where dimensions match

and $A$ full row rank ($A$: $r \times p$, $r \leq p$), then $\underset{r\times r}{Q} \sim W(\underset{r\times r}{\Delta} = A\,\Sigma\,A'/\nu,\nu)$

$$E(\underset{r\times r}{Q}|\Delta,\nu) = \underset{r\times r}{\Delta}$$

$$\text{var}(Q_{ij}|\Delta,\nu) = (\Delta_{ij}^2 + \Delta_{ii}\Delta_{jj})/\nu$$

$$\text{cov}(Q_{ij}Q_{kl}|\Delta,\nu) = (\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk})/\nu \quad .$$

# The Wishart PDF

$$\underset{2\times 1}{\mu} = \begin{pmatrix} 67 \\ 150 \end{pmatrix} \quad \underset{2\times 2}{\Sigma} = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix}$$

$$\nu = 9$$

Took $10^6$ sets of $n$=10 variates $\underset{2\times 1}{x}$, subtracted mean $\underset{2\times 1}{\bar{x}}$ from each set, transpose multiplied each value, added the 10 values in set and divided by 9 to form each $\underset{2\times 2}{S}$. The $\underset{2\times 2}{S}$'s are now $\underset{2\times 2}{W}(\Sigma/\nu,\nu=n\text{-}1)$.

$$E(\underset{p\times p}{S} \mid \Sigma,\nu) = \underset{p\times p}{\Sigma}$$

$$\mathrm{var}(S_{ij} \mid \Sigma,\nu) = (\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj}) / \nu$$

$$\mathrm{cov}(S_{ij}S_{kl} \mid \Sigma,\nu) = (\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) / \nu$$

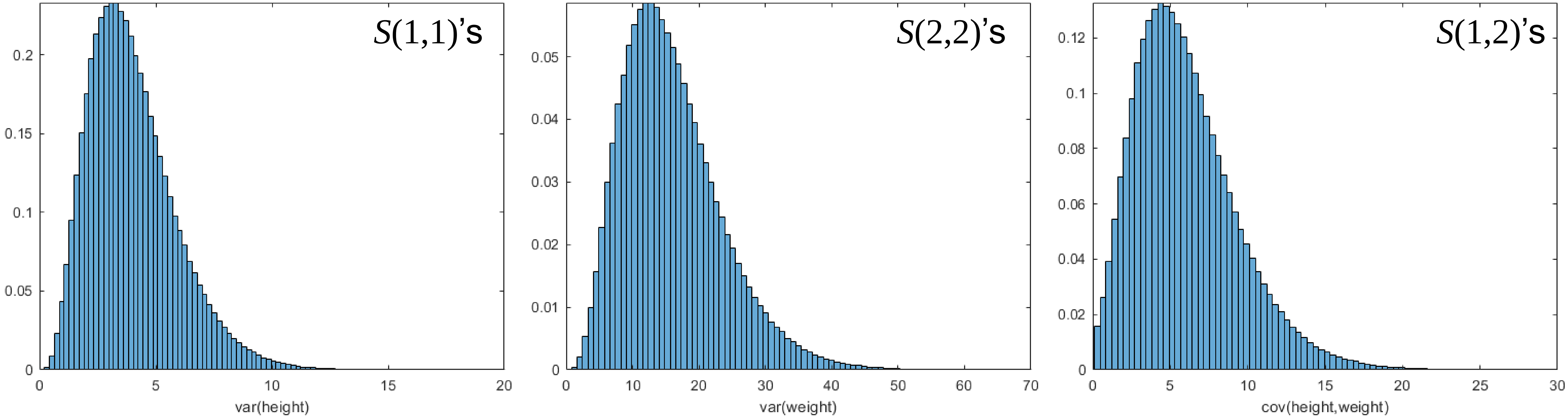
$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)$$

# The Wishart PDF

$$\underset{2\times 1}{\mu}=\begin{pmatrix}67\\150\end{pmatrix}\quad \underset{2\times 2}{\Sigma}=\begin{pmatrix}4 & 6\\6 & 16\end{pmatrix}$$

$$\nu=9$$

The $S$'s, $S=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})$ are now $W(\Sigma/\nu,\nu=n\text{-}1)$.




$$E(S\mid\Sigma,\nu)=\Sigma \quad =\begin{pmatrix}4 & 6\\6 & 16\end{pmatrix}\qquad \mathrm{var}(S_{ij}\mid\Sigma,\nu)=(\Sigma_{ij}^2+\Sigma_{ii}\Sigma_{jj})/\nu=\begin{pmatrix}3.56 & 11.11\\11.11 & 56.89\end{pmatrix}$$

$$\Sigma=AA'$$

$$\mathrm{cov}(S_{ij}S_{kl}\mid\Sigma,\nu)=(\Sigma_{ik}\Sigma_{jl}+\Sigma_{il}\Sigma_{jk})/\nu=\underset{11,22}{5.33},\underset{11,12}{8.00},\underset{22,12}{21.33}\leftarrow ij,kl$$

$$A=\begin{pmatrix}2 & 0\\3 & \sqrt{7}\end{pmatrix}$$

# The Wishart PDF

```
rng('default')
% define dimensions
p=2;     % dimension of vectors
n=10;    % sample size
m=10^6;  % repeated samples
% specify the mean vector
mu=[67;150]
% specify the covariance matrix
Sigma=[2^2,2*4*.75;2*4*.75,4^2]
% Cholesky factorize
A=chol(Sigma)'; %A=[2,0;3,sqrt(7)]

% Wishart Distribution %%%%%%%%%%%%
% Generate simulated observations
XX=A*randn(p,n*m)+repmat(mu,[1,n*m]);
X=zeros(p,n,m);
for j=1:m
    X(:,:,j)=XX(:,(j-1)*n+1:n*j);
end
clear j
```

```
% compute the mean in each sample
meanX=squeeze(mean(X,2));
mean(meanX,2)
% compute the covariance in each sample
S=zeros(p,p,m);
for j=1:m
   S(:,:,j)=cov(squeeze(X(:,:,j)'));
end


figure;
histogram(meanX(1,:),100)
xlabel('mean(height)'), axis tight
figure;
histogram(meanX(2,:),100)
xlabel('mean(weight)'), axis tight
```

# The Wishart PDF

```
figure;
histogram(squeeze(S(1,1,:)),100,'normalization','pdf')
xlabel('var(height)'), axis tight, xlim([0,20])
figure;
histogram(squeeze(S(2,2,:)),100,'normalization','pdf')
xlabel('var(weight)'), axis tight, xlim([0,70])
figure;
histogram(squeeze(S(1,2,:)),100,'normalization','pdf')
xlabel('cov(height,weight)'), axis tight, xlim([0,30])

% covariance
[Sigma,mean(S,3)]
% true variances
i=1; j=1;
vS11=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
i=1; j=2;
vS12=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
i=2; j=2;
vS22=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
V=[vS11,vS12;vS12,vS22];
[V,var(S,1,3)]
```

# The Wishart PDF

```
% true covariances
i=1; j=1; k=1; l=2;
cS1112=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=1; k=2; l=2;
cS1122=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=2; k=2; l=2;
cS1222=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
[cS1112,cS1122,cS1222]
% sample covariances
covS1112=cov(squeeze(S(1,1,:)),squeeze(S(1,2,:)));
covS1122=cov(squeeze(S(1,1,:)),squeeze(S(2,2,:)));
covS1222=cov(squeeze(S(1,2,:)),squeeze(S(2,2,:)));
[covS1112(1,2),covS1122(1,2),covS1222(1,2)]
```

# The Wishart PDF

We can generate random matrix variate observations directly from the Wishart PDF via the Matlab function wishrnd(Sigma/nu,nu)

```
% define dimensions
p=2;     % dimension of vectors
n=10;    % sample size
m=10^6; % repeated samples
nu=n-1; % degrees of freedom
% specify the covariance matrix
Sigma=[2^2,2*4*.75;2*4*.75,4^2]
% use matlab function %%%%%%%%%%%%%%%%%%%%%%%
Wmat=zeros(p,p,m); D=chol(Sigma/nu); Hmat=zeros(p,p,m);
for count=1:m
    Wmat(:,:,count) = wishrnd(Sigma,nu,D);
    Hmat(:,:,count) = inv(squeeze(Wmat(:,:,count))); % use later
end
```

and compare sample to theoretical values!

# The Wishart PDF

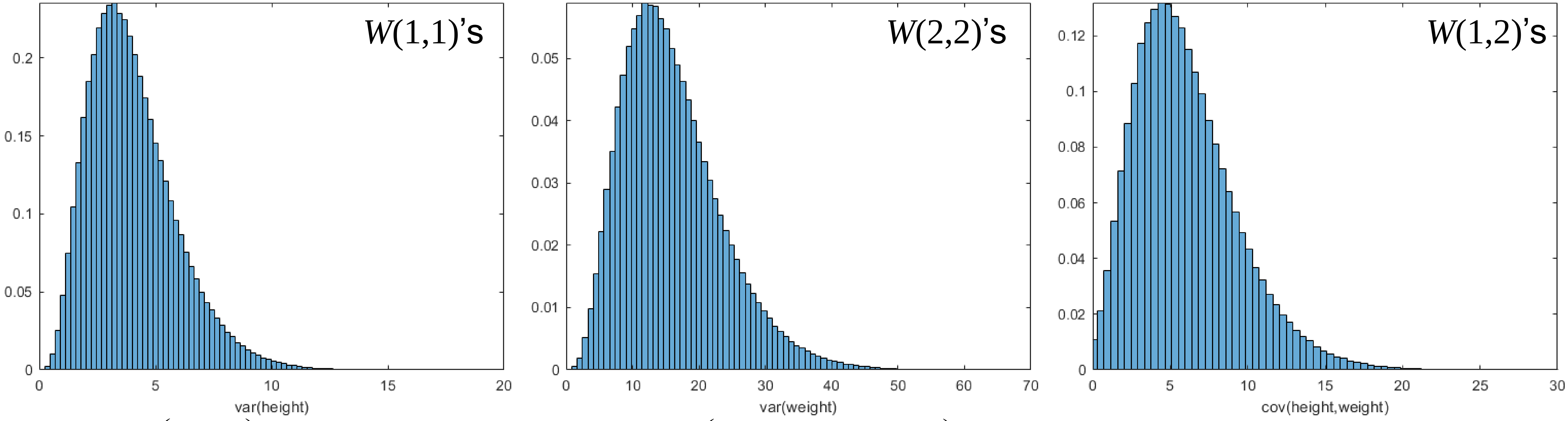
```
% mean and variances
[Sigma,mean(Wmat,3)]
[(Sigma(1,1)^2+Sigma(1,1)*Sigma(1,1)),
(Sigma(1,2)^2+Sigma(1,1)*Sigma(2,2)),...
(Sigma(2,2)^2+Sigma(2,2)*Sigma(2,2))]/nu
varWmat=var(Wmat,[],3);
[varWmat(1,1),varWmat(1,2),varWmat(2,2)]
% covariances
i=1; j=1; k=1; l=2;
cG1112=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=1; k=2; l=2;
cG1122=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=2; k=2; l=2;
cG1222=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
[cG1112,cG1122,cG1222]
covG1112=cov(squeeze(Wmat(1,1,:)),squeeze(Wmat(1,2,:)));
covG1122=cov(squeeze(Wmat(1,1,:)),squeeze(Wmat(2,2,:)));
covG1222=cov(squeeze(Wmat(1,2,:)),squeeze(Wmat(2,2,:)));
[covG1112(1,2),covG1122(1,2),covG1222(1,2)]
```

# The Wishart PDF

$$\mu = \begin{pmatrix} 67 \\ 150 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix}$$

$$p = 2 \quad \nu = 9$$

Matlab: Wmat=wishrnd(Sigma/nu,nu)




$$E(W) = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix} \qquad \mathrm{var}(W) = \begin{pmatrix} 3.5556 & 11.1111 \\ 11.1111 & 56.8889 \end{pmatrix} \qquad \mathrm{cov}(W_{ij}) = (5.3333, 8.0000, 21.3333)$$

$$\bar{W} = \begin{pmatrix} 3.9978 & 5.9952 \\ 5.9952 & 15.9916 \end{pmatrix} \qquad s_W^2 = \begin{pmatrix} 3.5426 & 11.0882 \\ 11.0882 & 56.7960 \end{pmatrix} \qquad s_{W_{ij}} = (5.3152, 7.9784, 21.2961)$$

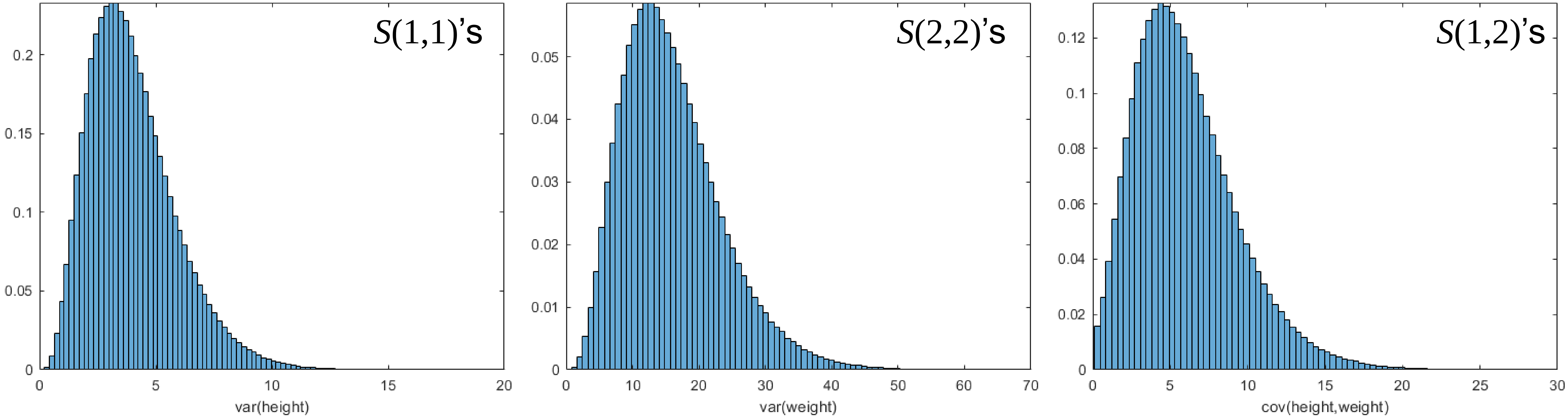
toggle forward

# The Wishart PDF

$$\mu=\begin{pmatrix}67\\150\end{pmatrix}\quad \Sigma=\begin{pmatrix}4&6\\6&16\end{pmatrix}$$

$$p=2\quad \nu=9$$

From $x$'s: $S=\dfrac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})$




$$E(S)=\begin{pmatrix}4&6\\6&16\end{pmatrix}\qquad \mathrm{var}(S)=\begin{pmatrix}3.5556&11.1111\\11.1111&56.8889\end{pmatrix}\qquad \mathrm{cov}(S_{ij})=(5.3333,8.0000,21.3333)$$

$$\bar{S}=\begin{pmatrix}4.0010&6.0007\\6.0007&15.9936\end{pmatrix}\qquad s_W^2=\begin{pmatrix}3.5647&11.1342\\11.1342&56.9372\end{pmatrix}\qquad s_{S_{ij}}=(5.3477,8.0200,21.3676)$$

toggle backward

# The Wishart PDF

```
figure;
histogram(squeeze(Wmat(1,1,:)),100,'normalization','pdf')
xlabel('var(height)'), axis tight, xlim([0,20])
figure;
histogram(squeeze(Wmat(2,2,:)),100,'normalization','pdf')
xlabel('var(weight)'), axis tight, xlim([0,70])
figure;
histogram(squeeze(Wmat(1,2,:)),100,'normalization','pdf')
xlabel('cov(height,weight)'),axis tight, xlim([0,30])
```

# The Inverse Wishart PDF

A random variable $h$ has a continuous

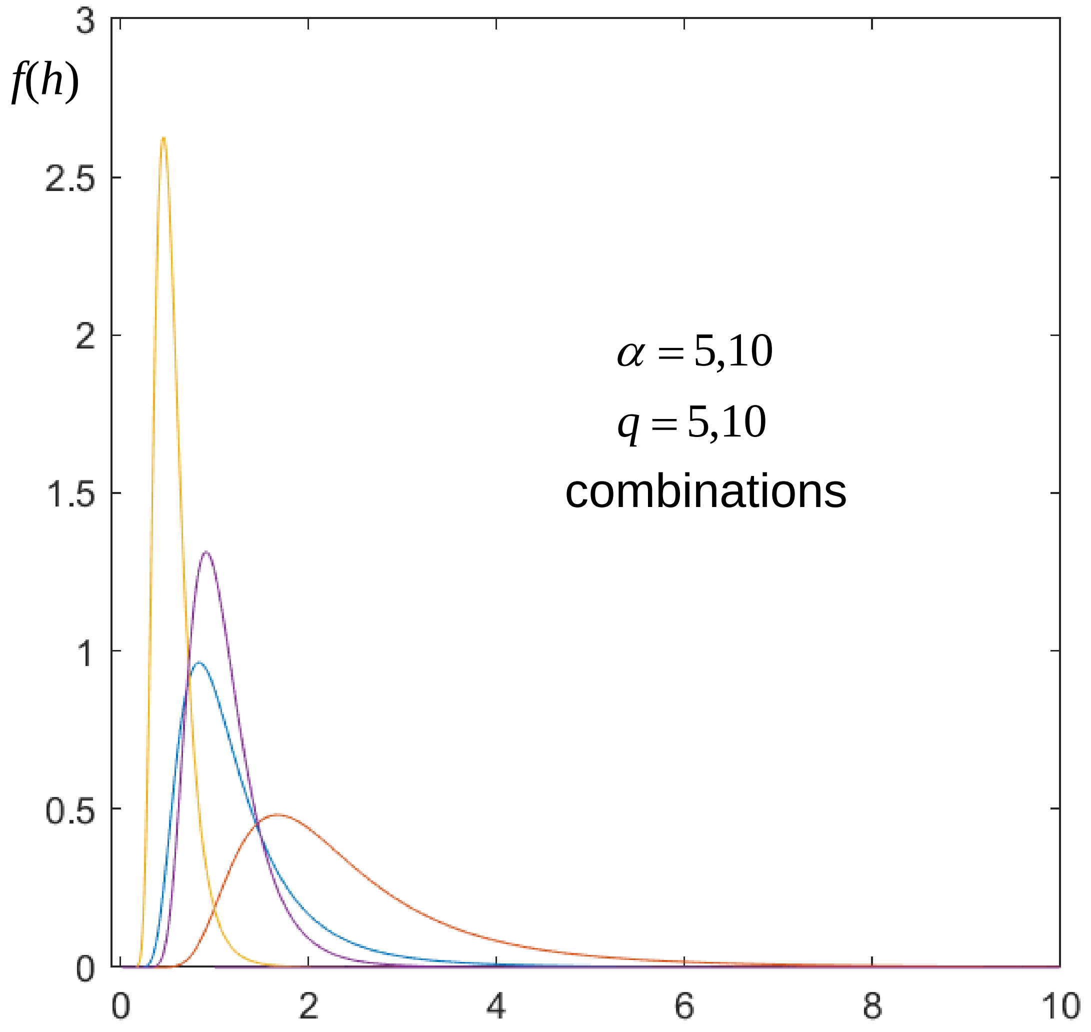
Inverse Gamma distribution, $h \sim \Gamma^{-1}(\alpha,q)$ if

$$f(h|\alpha,q)=\frac{q^{\alpha}}{\Gamma(\alpha)}h^{-\alpha-1}e^{-q/h} \quad ,$$

where $\alpha,q>0$ and $h>0$.

And can take on many shapes.

$\Gamma(a)=(a-1)!$ for integer $a$.




$$E(h|\alpha,q)=\frac{q}{\alpha-1}$$

$$\mathrm{var}(h|\alpha,q)=\frac{q^2}{(\alpha-1)^2(\alpha-2)}$$

# The Inverse Wishart PDF

The inverse gamma PDF can be arrived at by starting with

$$f(s^2|\alpha,\beta)=\frac{(s^2)^{\alpha-1}e^{-\frac{s^2}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} \qquad \alpha=\frac{\nu}{2} \quad \beta=\frac{2\sigma^2}{\nu}$$

defining

$$h=s^{-2} \qquad J(s^2\to h)=h^{-2} \qquad q=1/\beta$$

and performing a transformation of variable resulting in

$$f(h|\alpha,q)=\frac{q^{\alpha}}{\Gamma(\alpha)}h^{-\alpha-1}e^{-qh^{-1}} \qquad \begin{array}{r} h>0 \\ q,\nu>0 \end{array}$$

for the univariate case.

# The Inverse Wishart PDF

$$f(h|\nu,q)=\frac{q^{\nu/2}}{\Gamma(\nu/2)}h^{-\nu/2+1}e^{-qh^{-1}}$$

And upon generalizing to higher dimensions, we obtain the inverse Wishart PDF.

$$f(\underset{p\times p}{H}|Q,\nu)=k_{IW}Q^{\nu/2}|\underset{p\times p}{H}|^{-(\nu+p+1)/2}e^{-\frac{1}{2}trQ_{p\times p}H_{p\times p}^{-1}}$$

$$H,Q>0$$
$$\nu>0$$

$$k_{IW}^{-1}=2^{\nu p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left(\frac{\nu-1-j}{2}\right)$$

$$E(\underset{p\times p}{H}|\underset{p\times p}{Q},\nu)=\frac{Q}{\nu-p-1}$$

$$\mathrm{var}(H_{ii}|Q,\nu)=\frac{2Q_{ii}^2}{(\nu-p-1)^2(\nu-p-3)}$$

$$\mathrm{var}(H_{ij}|Q,\nu)=\frac{(\nu-p+1)Q_{ij}^2+(\nu-p-1)Q_{ii}Q_{jj}}{(\nu-p)(\nu-p-1)^2(\nu-p-3)}$$

$$\mathrm{cov}(H_{ij}|Q,\nu)=\frac{2Q_{ij}Q_{kl}+(\nu-p-1)(Q_{ik}Q_{jl}+Q_{il}Q_{kj})}{(\nu-p)(\nu-p-1)^2(\nu-p-3)}$$

# The Inverse Wishart PDF

The inverse Wishart PDF can be arrived at by starting with

$$f(S|V,\nu) = k_W |V|^{-\frac{\nu}{2}} |S|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2} tr V^{-1} S}$$

defining

$$\underset{p\times p}{H} = \underset{p\times p}{S}^{-1} \qquad \underset{p\times p}{Q} = \underset{p\times p}{V}^{-1} = \nu\Sigma^{-1} \qquad J(S \to H) = H^{-(p+1)}$$

and performing a transformation of variable resulting in

$$f(\underset{p\times p}{H}|\underset{p\times p}{Q},\nu) = k_{IW} |\underset{p\times p}{Q}|^{\nu/2} |\underset{p\times p}{H}|^{-(\nu+p+1)/2} e^{-\frac{1}{2} tr(\underset{p\times p}{Q} \underset{p\times p}{H}^{-1})} \qquad \begin{matrix} H, Q > 0 \\ \nu > 0 \end{matrix}$$

$$k_{IW}^{-1} = 2^{\nu p/2} \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left(\tfrac{\nu-p-j}{2}\right)$$

for the multivariate case.

$$f(h|\nu,q) = \frac{q^{\nu/2}}{\Gamma(\nu/2)} h^{-\nu/2-1} e^{-\frac{1}{2} q h^{-1}}$$

# The Inverse Wishart PDF

We can invert each of the Wishart distributed matrix observations, $H=S^{-1}$, then the $H$ has an inverse Wishart PDF **or** we can generate random matrix variate observations from the inverse Wishart PDF via the Matlab function iwishrnd(Sigma/nu,nu)

```
% Matlab function
IWmat=zeros(p,p,m); Q=inv(Sigma/nu); DI=chol(inv(Q));
for count=1:m
    IWmat(:,:,count) = iwishrnd(Q,nu,DI);
end
```

and compare sample to theoretical values

# The Inverse Wishart PDF

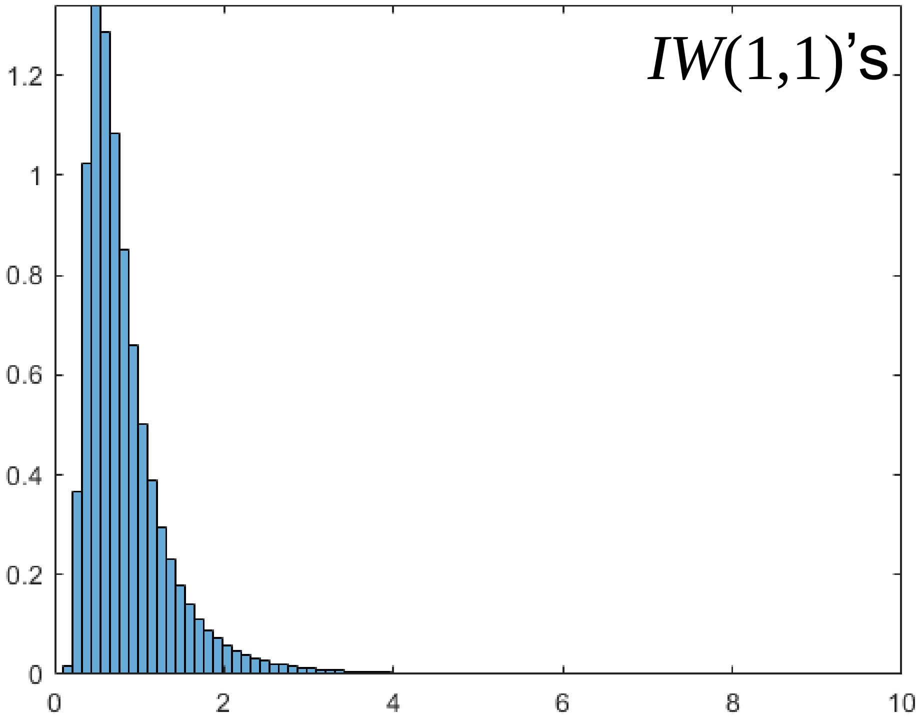
```
% Inverse Wishart Distribution %%%%%%%%%%%%%%%%%%%%%%%%%%%
Q=inv(Sigma/nu);
% means and variances
[Q/(nu-p-1),mean(Hmat,3)]
[2*Q(1,1)^2/((nu-p-1)^2*(nu-p-3)),...
    ((nu-p+1)*Q(1,2)^2+(nu-p-1)*Q(1,1)*Q(2,2))/((nu-p)*(nu-p-1)^2*(nu-p-3)),...
    2*Q(2,2)^2/((nu-p-1)^2*(nu-p-3))]
varHmat=var(Hmat,[],3);
[varHmat(1,1),varHmat(1,2),varHmat(2,2)]
% covariances
i=1;,j=1;,k=1;,l=2;
cH1112=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
i=1;,j=1;,k=2;,l=2;
cH1122=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
i=1;,j=2;,k=2;,l=2;
cH1222=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
[cH1112,cH1122,cH1222]
covH1112=cov(squeeze(Hmat(1,1,:)),squeeze(Hmat(1,2,:)));
covH1122=cov(squeeze(Hmat(1,1,:)),squeeze(Hmat(2,2,:)));
covH1222=cov(squeeze(Hmat(1,2,:)),squeeze(Hmat(2,2,:)));
[covH1112(1,2),covH1122(1,2),covH1222(1,2)]
```

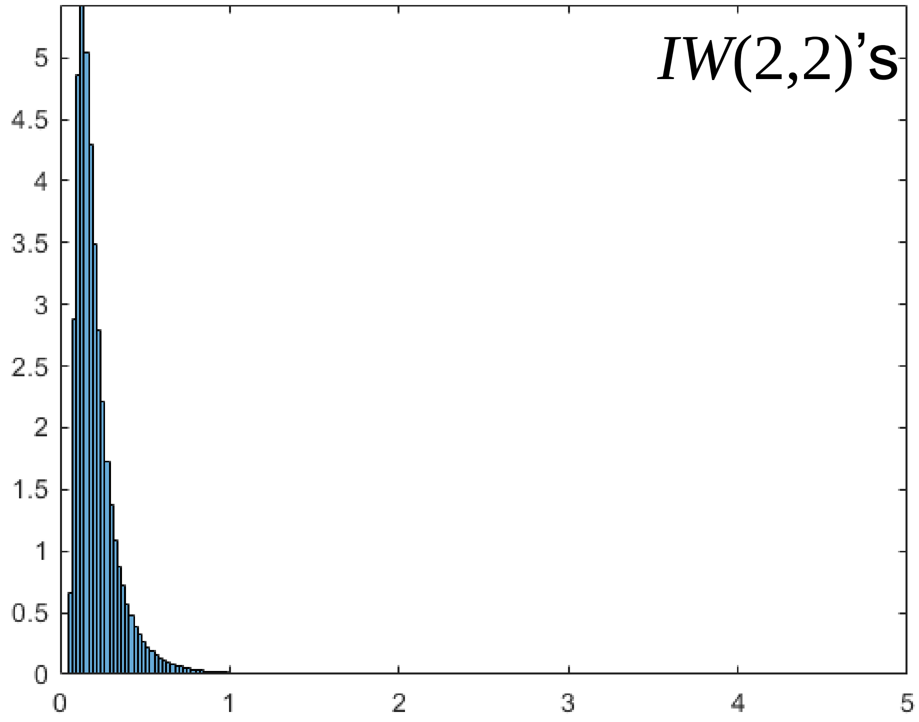
# The Wishart PDF

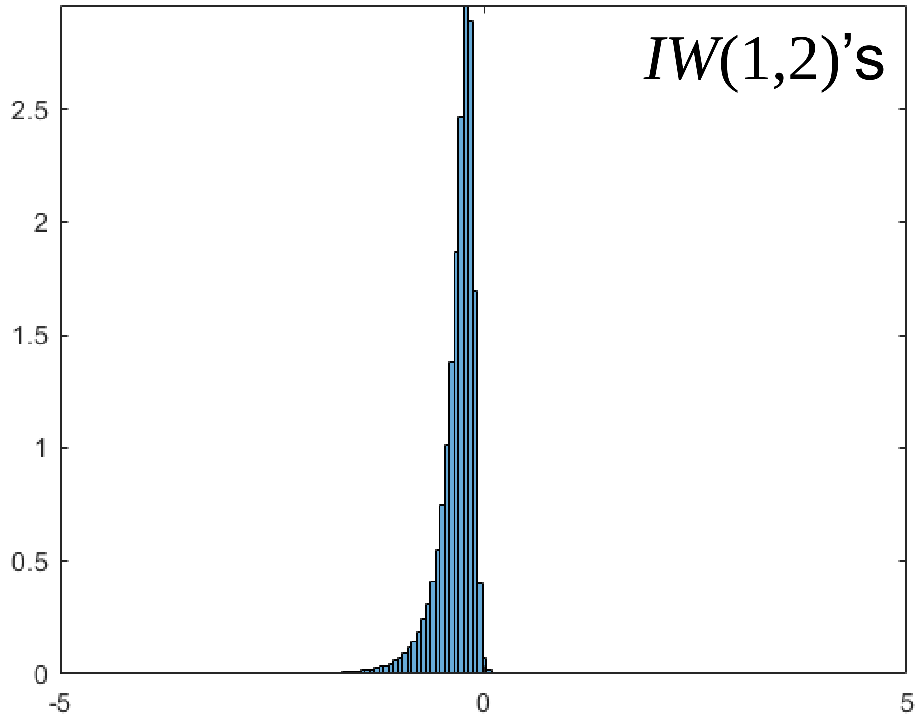
$$E(H \mid Q,\nu) = \frac{Q}{\nu - p - 1} \qquad Q = \begin{pmatrix} 5.1429 & -1.9286 \\ -1.9286 & 1.2857 \end{pmatrix}$$

Matlab: IWmat = iwishrnd(Q,nu,DI);

$$p = 2 \quad \nu = 9$$




$$E(IW) = \begin{pmatrix} 0.8571 & -0.3214 \\ -0.3214 & 0.2143 \end{pmatrix} \qquad \mathrm{var}(IW) = \begin{pmatrix} 0.3673 & 0.0689 \\ 0.0689 & 0.0230 \end{pmatrix} \qquad \mathrm{cov}(IW_{ij}) = (-0.1378, 0.0574, -0.0344)$$

$$\overline{IW} = \begin{pmatrix} 0.8568 & -0.3216 \\ -0.3216 & 0.2144 \end{pmatrix} \qquad s_{IW}^2 = \begin{pmatrix} 0.3723 & 0.0721 \\ 0.0721 & 0.0245 \end{pmatrix} \qquad s_{IW_{ij}} = (-0.1400, 0.0583, -0.0348)$$

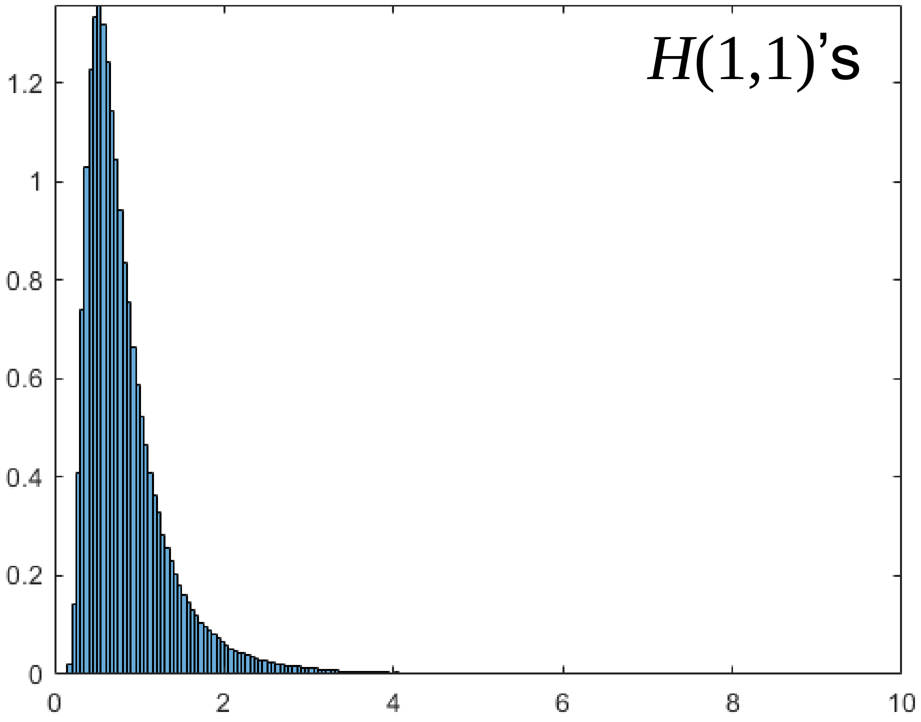
toggle forward

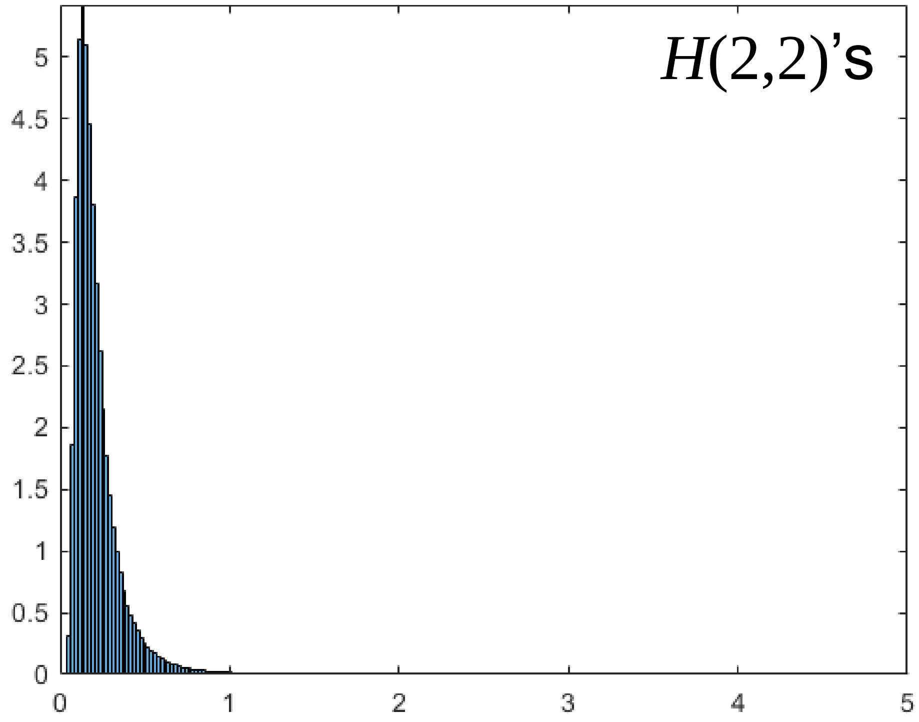
# The Wishart PDF

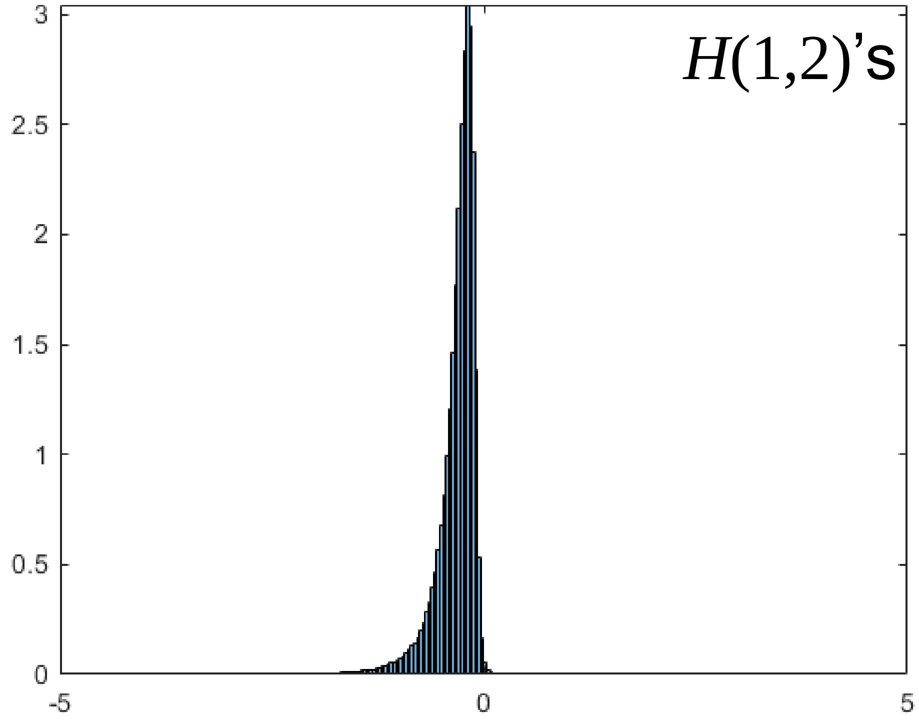
$$E(H \mid Q,\nu) = \frac{Q}{\nu - p - 1} \qquad Q = \begin{pmatrix} 5.1429 & -1.9286 \\ -1.9286 & 1.2857 \end{pmatrix}$$

From $x$'s: $H = S^{-1}$

$$p = 2 \quad \nu = 9$$




$$E(H) = \begin{pmatrix} 0.8571 & -0.3214 \\ -0.3214 & 0.2143 \end{pmatrix} \qquad \text{var}(H) = \begin{pmatrix} 0.3673 & 0.0689 \\ 0.0689 & 0.0230 \end{pmatrix} \qquad \text{cov}(H_{ij}) = (-0.1378, 0.0574, -0.0344)$$

$$\bar{H} = \begin{pmatrix} 0.8568 & -0.3216 \\ -0.3216 & 0.2144 \end{pmatrix} \qquad s_H^2 = \begin{pmatrix} 0.3723 & 0.0721 \\ 0.0721 & 0.0245 \end{pmatrix} \qquad s_{H_{ij}} = (-0.1400, 0.0583, -0.0348)$$

toggle backward

# The Inverse Wishart PDF

```
% make histograms of the inv S elements
figure;
histogram(squeeze(IWmat(1,1,:)),500,'normalization','pdf'), axis tight,xlim([0,10])
figure;
histogram(squeeze(IWmat(2,2,:)),500,'normalization','pdf'), axis tight,xlim([0,5])
figure;
histogram(squeeze(IWmat(1,2,:)),500,'normalization','pdf'), axis tight,xlim([-5,5])

% make histograms of the inv S elements
figure;
histogram(squeeze(Hmat(1,1,:)),500,'normalization','pdf'), axis tight,xlim([0,10])
figure;
histogram(squeeze(Hmat(2,2,:)),500,'normalization','pdf'), axis tight,xlim([0,5])
figure;
histogram(squeeze(Hmat(1,2,:)),500,'normalization','pdf'), axis tight,xlim([-5,5])
```

# Discussion

## Questions?

# Homework 5

$$
\begin{array}{ccccc}
x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & & x_1^{(10^6)} \\
x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots & x_2^{(10^6)} \\
\vdots & \vdots & \vdots & & \vdots \\
x_{10}^{(1)} & x_{10}^{(2)} & x_{10}^{(3)} & & x_{10}^{(10^6)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bar{x}_{(1)} & \bar{x}_{(2)} & \bar{x}_{(3)} & \cdots & \bar{x}_{(10^6)} \\
S_{(1)} & S_{(2)} & S_{(3)} & \cdots & S_{(10^6)}
\end{array}
$$

1. Assume Marquette Undergrads heights have $\mu_h$=70 *in* and $\sigma_h$=3 *in* while their weights have $\mu_w$=160 *lbs* and $\sigma_w$=4 *lbs* with $\rho$=.75.
   - a) Generate $10^7$ h/w $2 \times 1$ vectors using the Cholesky, $\Sigma = AA'$.
   - b) Divide the $10^7$ random vectors into samples (sets) of $10$.
   - c) Calculate means, variances, and covariance from each $10$.
   - d) Calculate the mean of: means, variances, and covariance.
   - e) Calculate the variance of: means, variances, and covariance.
   - f) Calculate the covariance between: means, variances & covariances.
   - g) Make histograms of everything. Do your results match theory?

# Homework 5

2. With pencil and paper sketch out the transformation from the Wishart RV $S$ to the inverse Wishart RV $H$ where $H$=$S^{-1}$, use $J(S \rightarrow H)=|H|^{-(p+1)}$.

   This is a similar transformation as $h=s^{-2}$ using $J(s^2 \rightarrow h)=h^{-2}$ for gamma to inverse gamma. (Do this one first.)

# Homework 5

3. Compute the inverse of each of your $S$ matrices from 1 to form $H$ matrices.

   Make histograms of $11$, $22$, and $12$ elements of $H$. Calculate means, variances, and covariances of the elements of $H$. Compare to the theoretical values.

# Homework 5

4*.Develop a method to generate matrix observation from the $2 \times 2$ inverse Wishart PDF using rejection sampling.

$$f(S_{11}, S_{22}, S_{12} \mid V, \nu) = k_W \begin{vmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{vmatrix}^{-\frac{\nu}{2}} \begin{vmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{vmatrix}^{\frac{\nu-p-1}{2}} \exp\left\{ -\frac{1}{2} tr\left[ \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}^{-1} \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \right] \right\}$$

* Only for students that have had MSSC 5790 Bayesian Statistics. Excused from 1, 2, and 3.