Chapter 6: The Multivariate Normal Distribution ~~and Copulas~~

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Agenda

6.1 The Multivariate Normal

6.2 Generating a Multivariate Normal Random Vector

~~6.3 Copulas~~

~~6.4 Generating Variables from Copula Models~~

Symmetric Matrix Factorizations

6.1 The Multivariate Normal-Univariate

In 1-D, we can obtain a random variable x that has a general normal distribution with mean μ and variance σ^2 via the transformation

$$x = \sigma z + \mu \ .$$

The PDF of x can be obtained by

$$f(x \mid \mu, \sigma^2) = f(z(x)) \times |J(z \to x)|$$

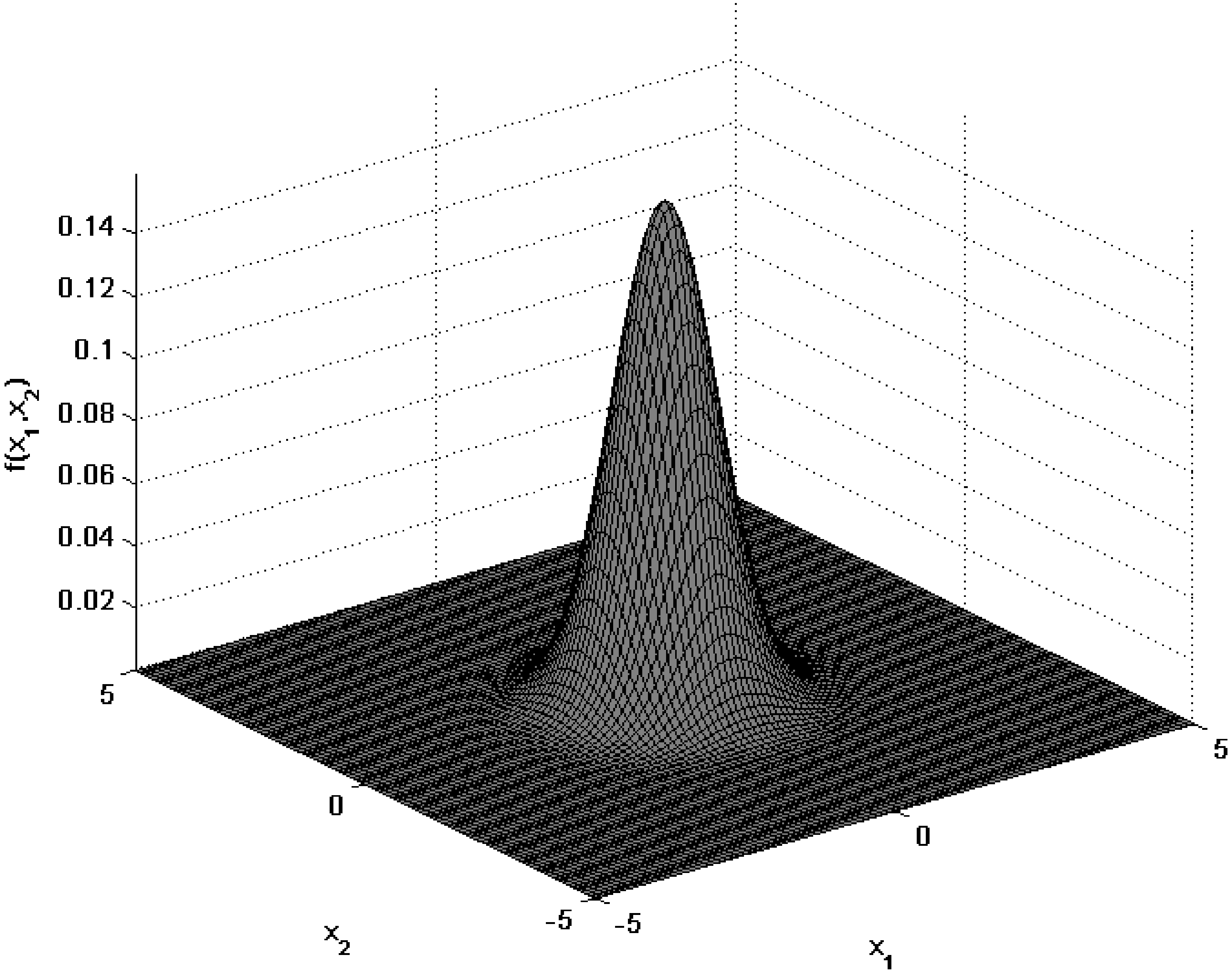
where $z(x)$ is z written in terms of x and $J(\cdot)$ is the Jacobian of the transformation.

6.1 The Multivariate Normal-Bivariate

A bivariate (2-D) PDF of two continuous random variables (x_1,x_2) depending upon parameters θ satisfies

1) $0 \leq f(x_1,x_2 \mid \theta), \ \ \forall(x_1,x_2)$

2) $\iint\limits_{x_1 x_2} f(x_1,x_2 \mid \theta)dx_1 dx_2 = 1$.



6.1 The Multivariate Normal-Bivariate

Let $\underset{2\times 1}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a 2-dimensional (or p-dimensional) random variable

with PDF of $\underset{2\times 1}{x}$ being $f(x|\theta)$, then

Marginal means.

$$E(x|\theta) = \begin{pmatrix} E(x_1|\theta) \\ E(x_2|\theta) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$= \underset{2\times 1}{\mu}$$

Marginal variances.

$$\mathrm{cov}(x|\theta) = \begin{pmatrix} \mathrm{var}(x_1|\theta) & \mathrm{cov}(x_1,x_2|\theta) \\ \mathrm{cov}(x_1,x_2|\theta) & \mathrm{var}(x_2|\theta) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$= \underset{2\times 2}{\Sigma}$$

Scalars and vectors and are lower case, matrices are upper case.

6.1 The Multivariate Normal-Bivariate

Note the way I have written this.

Recall: If $z \sim N(0,1)$, then

mean mean

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = (2\pi)^{-1/2} (1)^{-1/2} e^{-\frac{1}{2}(z-0)(1)^{-1}(z-0)}$$

variance variance

We can obtain a random variable x that has a

general normal distribution with mean μ and

variance σ^2 via the transformation $\underset{1\times 1}{x} = \underset{1\times 1}{\sigma}\ \underset{1\times 1}{z} + \underset{1\times 1}{\mu}$.

6.1 The Multivariate Normal-Univariate

The PDF of x can be obtained by

$$f(x|\mu,\sigma^2) = f(z(x)) \times |J(z \to x)| \qquad J(z \to x) = \frac{dz(x)}{dx}$$

where $\underset{1\times 1}{z} = z(x)$ and $J(\cdot)$ is the Jacobian of the transformation.

The PDF of $\underset{1\times 1}{x}$ is

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

which can be written as

$$f(x|\mu,\sigma^2) = (2\pi)^{-1/2} \left(\sigma^2\right)^{-1/2} e^{-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)}$$

Note the way I have written this.

6.1 The Multivariate Normal-Univariate

Given two continuous random variables (z_1,z_2), we write them as a 2-dimensional vector $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, and this vector

vector

has the PDF $f_z(z|\theta)$.

vector

If z_1 and z_2 are independent, then

$$f_Z(z\,|\,\theta) = f_{Z_1}(z_1\,|\,\theta_1)\, f_{Z_2}(z_2\,|\,\theta_2) \quad .$$

6.1 The Multivariate Normal-Bivariate

Let $\underset{1\times 1}{z_1}$ and $\underset{1\times 1}{z_2}$ be iid $N(0,1)$ random variables.

Then, $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ has PDF

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= f_{Z_1}(z_1) f_{Z_2}(z_2) \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} \end{aligned}.$$

With vector $\underset{2\times 1}{z}$, this can be rewritten as $f_Z(z) = (2\pi)^{-\frac{2}{2}} e^{-\frac{1}{2}z'z}$.

6.1 The Multivariate Normal-Bivariate

This can also be written as

$$\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$f_Z(z) = \frac{1}{2\pi} e^{-\frac{1}{2}z'z}$$

$$= (2\pi)^{-2/2} |\underbrace{I_2}_{\text{covariance matrix}}|^{-1/2} e^{-\frac{1}{2}(z-\overbrace{0}^{\text{mean vector}})'(\underbrace{I_2}_{\text{covariance matrix}})^{-1}(z-\overbrace{0}^{\text{mean vector}})}$$

Note the way I have written this.

and we write that $\underset{2\times 1}{z} \sim N(\underbrace{0}_{\text{mean vector}},\underbrace{I_2}_{\text{covariance matrix}})$.

I_2 is 2×2 Identity matrix

That is, the 2-dimensional random vector $\underset{2\times 1}{z}$ has a

mean vector of zero and identity variance-covariance matrix.

6.1 The Multivariate Normal-Bivariate

This means that

$$\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$f_Z(z) = (2\pi)^{-2/2} \, | I_2 |^{-1/2} \, e^{-\frac{1}{2}(z-0)'(I_2)^{-1}(z-0)}$$

$$E(z) = \begin{pmatrix} E(z_1) \\ E(z_2) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \underset{2\times 1}{\mu} = 0$$

2 × 2 Identity matrix

$$\text{cov}(z) = \begin{pmatrix} \text{var}(z_1) & \text{cov}(z_1, z_2) \\ \text{cov}(z_1, z_2) & \text{var}(z_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \underset{2\times 2}{\Sigma} = I_2$$

6.1 The Multivariate Normal-Bivariate

If $\underset{2\times 1}{z} \sim N(0, I_2)$, then

$$\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$f(z) = (2\pi)^{-2/2} |I_2|^{-1/2} e^{-\frac{1}{2}(z-0)'(I_2)^{-1}(z-0)}$$

We can obtain a random variable $\underset{2\times 1}{x}$ that has a general normal distribution with mean vector $\underset{2\times 1}{\mu}$ and variance-covariance matrix $\underset{2\times 2}{\Sigma}$ via the transformation $\underset{2\times 1}{x} = \underset{2\times 2}{A}\,\underset{2\times 1}{z} + \underset{2\times 1}{\mu}$ where $\underset{2\times 2}{\Sigma} = \underset{2\times 2}{A}\,\underset{2\times 2}{A'}$, is a factorization (i.e. Cholesky or Eigen).

$$\underset{2\times 1}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underset{2\times 2}{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

6.1 The Multivariate Normal-Bivariate

If a random variable $\underset{2\times 1}{x}$ has a normal distribution with mean vector $\underset{2\times 1}{\mu}$ and variance-covariance matrix $\underset{2\times 2}{\Sigma}$, then

$$f(\underset{2\times 1}{x}|\mu,\Sigma) = (2\pi)^{-p/2}|\Sigma|^{-1/2} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

mean vector

mean vector

covariance matrix

covariance matrix

$$x,\mu \in \mathbb{R}^p$$
$$p = 2$$
$$\Sigma > 0$$

set of pos def matrices

mean vector

covariance matrix

and we write $\underset{2\times 1}{x} \sim N(\underset{2\times 1}{\mu}, \underset{2\times 2}{\Sigma})$. The covariance matrix $\underset{2\times 2}{\Sigma}$, has to be of full rank (there is an inverse in PDF).

make sure you know what this means

6.1 The Multivariate Normal-Bivariate

Let's take a closer look at this bivariate transformation.

$$\underset{2\times1}{x} = \underset{2\times2}{A}\,\underset{2\times1}{z} + \underset{2\times1}{\mu}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$x_1 = a_{11}z_1 + a_{12}z_2 + \mu_1$$

$$x_2 = a_{21}z_1 + a_{22}z_2 + \mu_2$$

We can solve for $\underset{1\times1}{z_1}$ and $\underset{1\times1}{z_2}$ in terms of $\underset{1\times1}{x_1}$ and $\underset{1\times1}{x_2}$.

6.1 The Multivariate Normal-Bivariate

This will give us

$$\underset{2\times1}{x} = \underset{2\times2}{A}\,\underset{2\times1}{z} + \underset{2\times1}{\mu}$$

$$\underset{2\times2}{A^{-1}} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\underset{2\times1}{z} = \underset{2\times2}{A^{-1}} \underset{2\times1}{(x-\mu)}$$

A invertible

$$\underset{2\times2}{B} = \underset{2\times2}{A^{-1}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\underset{2\times1}{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \underset{2\times2}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}} \underset{2\times1}{\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}}$$

$$z_1 = b_{11}(x_1 - \mu_1) + b_{12}(x_2 - \mu_2)$$

$$z_2 = b_{21}(x_1 - \mu_1) + b_{22}(x_2 - \mu_2)$$

6.1 The Multivariate Normal-Bivariate

$$z_1 = b_{11}(x_1 - \mu_1) + b_{12}(x_2 - \mu_2)$$

$$z_2 = b_{21}(x_1 - \mu_1) + b_{22}(x_2 - \mu_2)$$

Continuing on, this leads to

$$J(z_1, z_2 \to x_1, x_2) = \begin{vmatrix} \dfrac{\partial z_1(x_1, x_2)}{\partial x_1} & \dfrac{\partial z_1(x_1, x_2)}{\partial x_2} \\ \dfrac{\partial z_2(x_1, x_2)}{\partial x_1} & \dfrac{\partial z_2(x_1, x_2)}{\partial x_2} \end{vmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \underset{2\times 2}{B}$$

i.e. with $\underset{2\times 1}{z} = B(x-\mu)$, the vector derivative is $J = \dfrac{\partial z}{\partial x} = \underset{2\times 2}{B}$.

6.1 The Multivariate Normal-Bivariate

$$z = B(x-\mu)$$

The distribution of the vector variable $\underset{2\times 1}{x}$ (joint of x_1 and x_2) is

$$f_X(x\,|\,\theta) = f_Z(z(x)) \times |J(z \to x)|$$

$$J = \frac{\partial z}{\partial x} = \underset{2\times 2}{B}$$

$$f(z) = (2\pi)^{-p/2}\,|I_p|^{-1/2}\, e^{-\frac{1}{2}(z-0)'(I_p)^{-1}(z-0)}$$

$$f_X(x\,|\,\mu,\Sigma) = (2\pi)^{-2/2}\,|I_p|^{-1/2}\, e^{-\frac{1}{2}\left(B(x-\mu)-0\right)'(I_p)^{-1}\left(B(x-\mu)-0\right)}\,|B|$$

$$\Sigma = AA' \;, |\Sigma| = |A||A'| = |A|^2 \;, \;\; |\Sigma|^{1/2} = |A| \;, B = A^{-1} \;\; |\Sigma|^{-1/2} = |B|$$

$$f_X(x\,|\,\mu,\Sigma) = (2\pi)^{-p/2}\,|\Sigma|^{-1/2}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)} \qquad x,\mu \in \mathbb{R}^p \quad \Sigma > 0$$

6.1 The Multivariate Normal-Bivariate

This form may be more familiar

$$f_X(x_1,x_2\,|\,\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2}Q}$$

$$Q=\frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$\sigma_1>0,\sigma_2>0,-1<\rho<1$$

$$\rho=\sigma_{12}/(\sigma_1\sigma_2) \qquad \sigma_{12}=\text{cov}(x_1,x_2)$$

6.1 The Multivariate Normal

Theorem:

If x is a 2-D (or p-D) random variable from $f(x|\mu,\Sigma)$, with

$$E(x|\mu,\Sigma) = \underset{p\times 1}{\mu}$$ think of $p=2$

$$\operatorname{cov}(x|\mu,\Sigma) = \underset{p\times p}{\Sigma}$$

then we form $\underset{r\times 1}{y} = \underset{r\times p}{A}\,\underset{p\times 1}{x} + \underset{r\times 1}{\delta}$ where dimensions match

and $\underset{r\times p}{A}$ full column rank (A: $r\times p$, $r\leq p$), then

$$E(y|\mu,\Sigma,\delta,A) = \underset{r\times p}{A}\,\underset{p\times 1}{\mu} + \underset{r\times 1}{\delta}$$ and $$\operatorname{var}(y|\mu,\Sigma,A) = \underset{r\times p}{A}\,\underset{p\times p}{\Sigma}\,\underset{p\times r}{A'}\,.$$

6.2 Generating a Multivariate Normal Random Vector

Recall: Let u_1~uniform(0,1) and u_2~uniform(0,1).

Let $z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$ and $z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$

then $u_1(z_{1,} z_2) = e^{-\frac{1}{2}(z_1^2+z_2^2)}$ and $u_2(z_1, z_2) = \frac{1}{2\pi}\text{atan}\left(\frac{z_2}{z_1}\right)$.

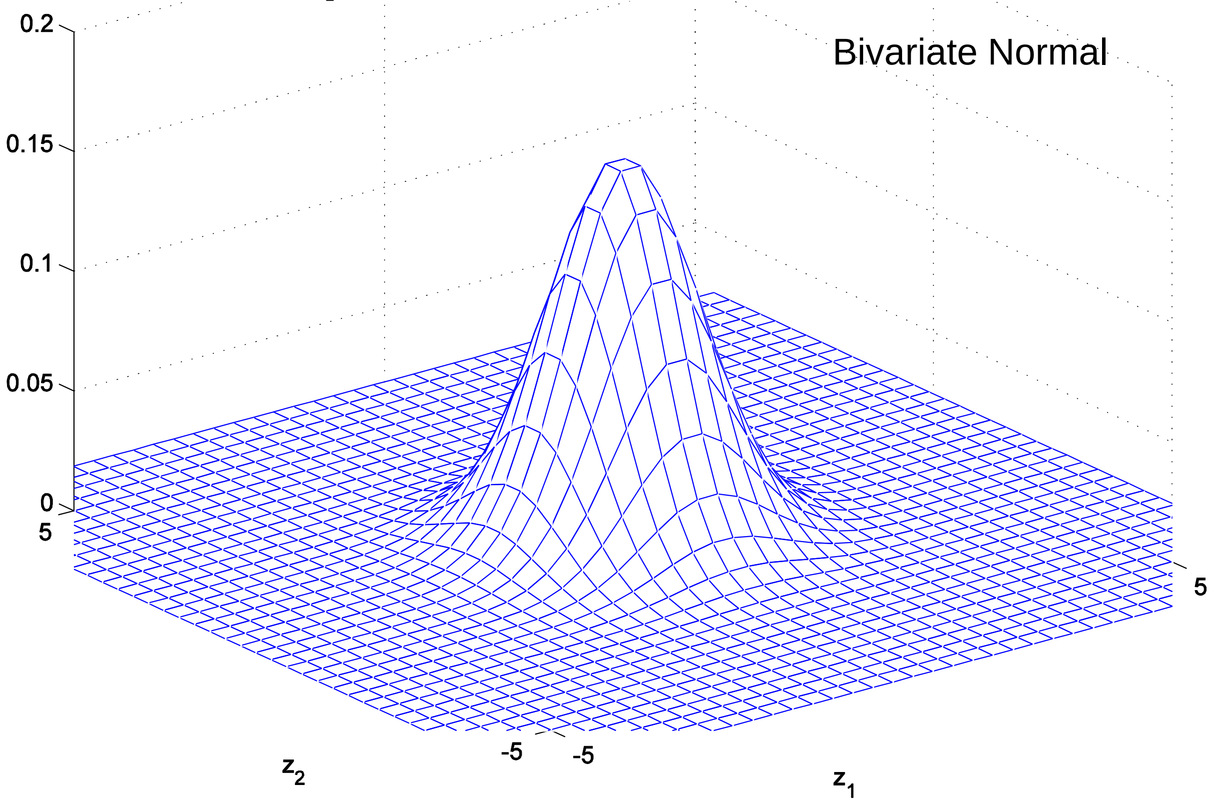
$$f_{Z_1,Z_2}(z_1, z_2 \mid \theta) = f_{U_1,U_2}\left(u_1(z_1, z_2), u_2(z_1, z_2) \mid \theta\right) \times \mid J(u_1, u_2 \to z_1, z_2) \mid$$

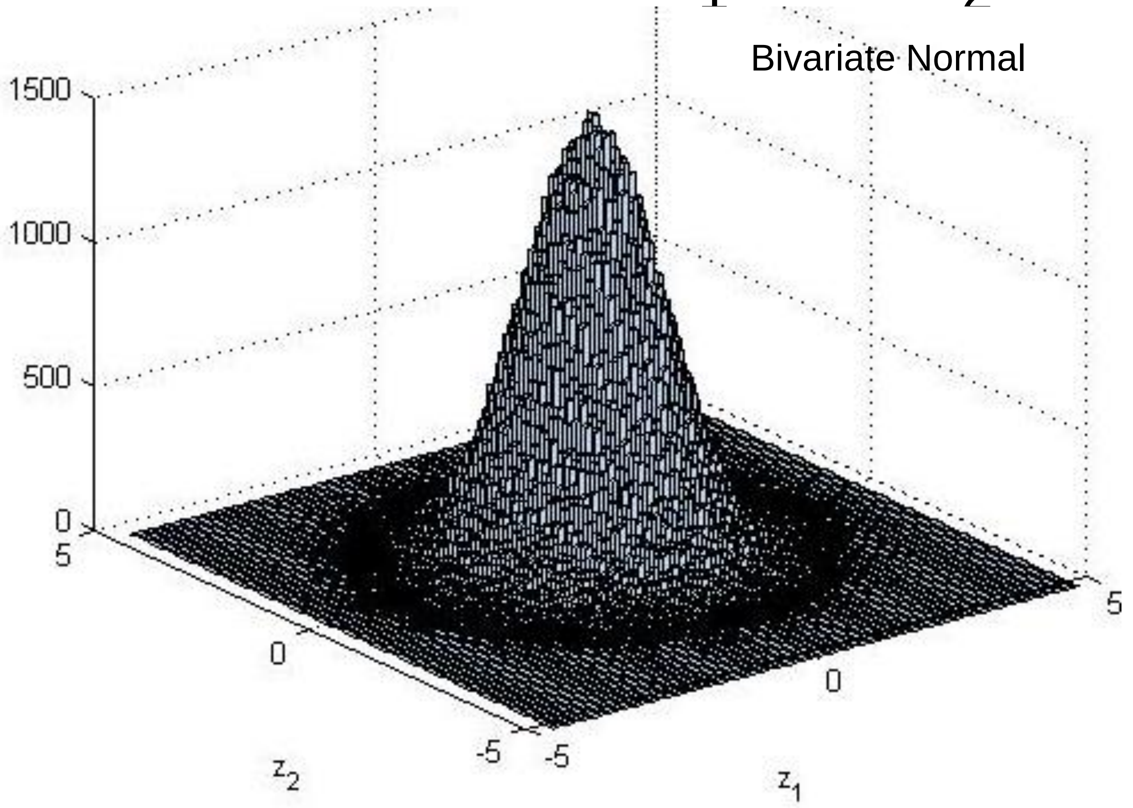
$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2}$$

This means z_1~$N(0,1)$, z_2~$N(0,1)$, z_1 and z_2 are independent.

6.2 Generating a Multivariate Normal Random Vector

Obtain 2-D standard normal variates $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by transforming two independent standard uniform random variates u_1 and u_2.





First half of 10^6 standard normal variates as z_1's and second half as z_2's. Produce 5×10^5 statistically independent z's

6.2 Generating a Multivariate Normal Random Vector

$$\underset{2\times1}{\mu}=\begin{pmatrix}3\\5\end{pmatrix} \qquad \underset{2\times2}{\Sigma}=\begin{pmatrix}3 & 2\\2 & 4\end{pmatrix}$$

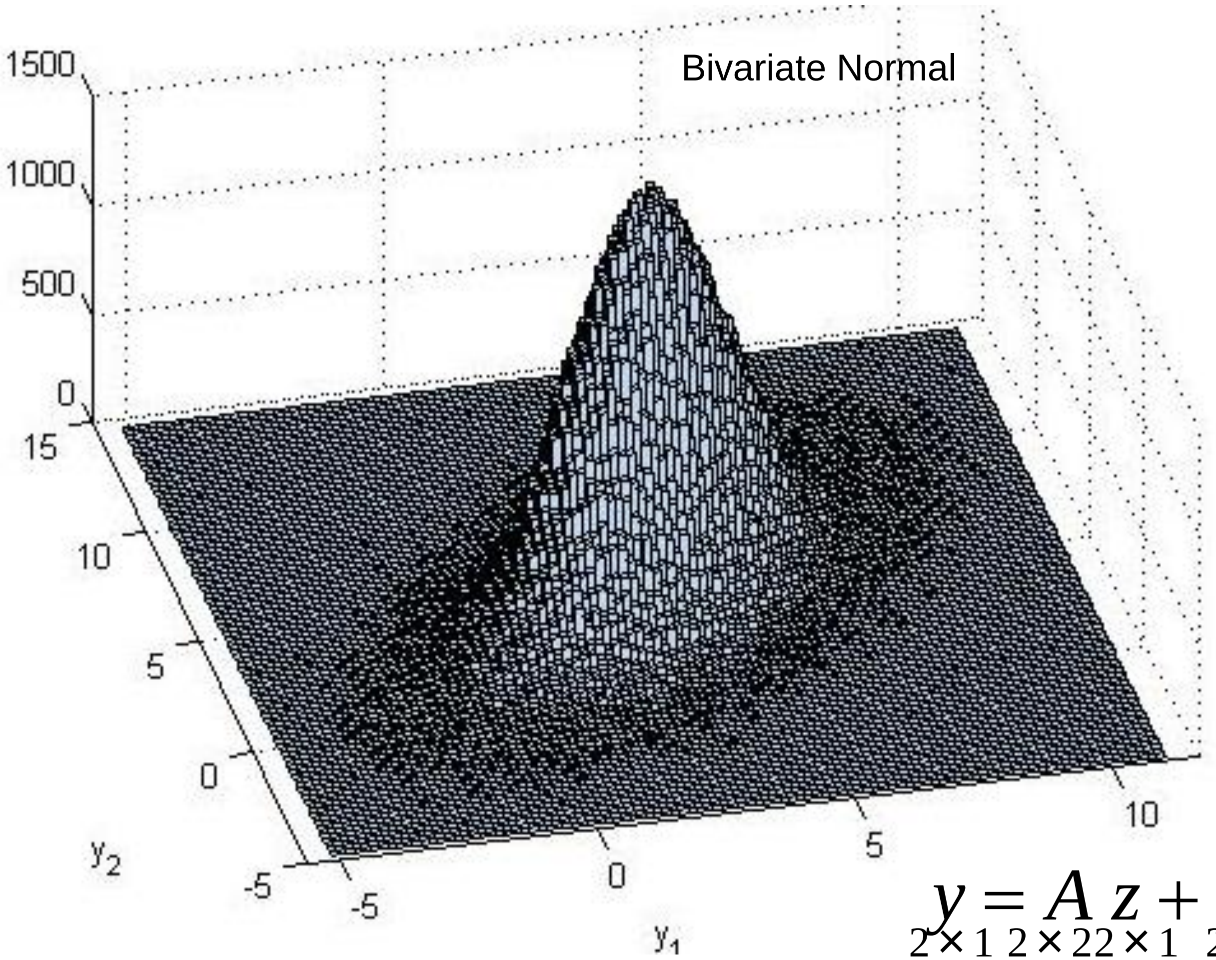
Multiplied 5×10^5 simulated z's by $\underset{2\times2}{A}$ and added $\underset{2\times1}{\mu}$.

The y's are now $N(\mu,\Sigma=AA')$.

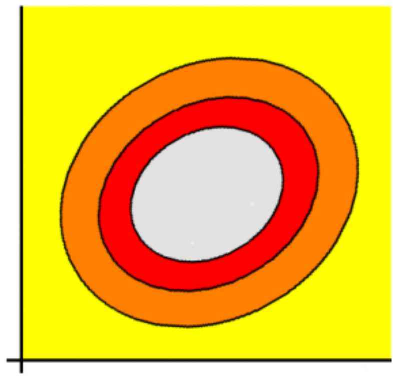
Cholesky $\qquad \rho=0.58$

$$\underset{2\times2}{A}=\begin{pmatrix}1.7321 & 0\\1.1547 & 1.6330\end{pmatrix}$$

2×1

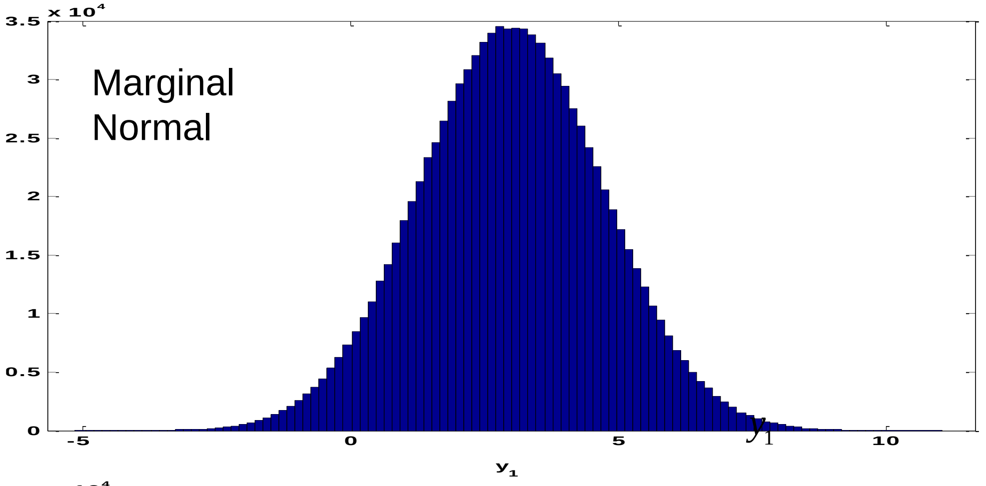


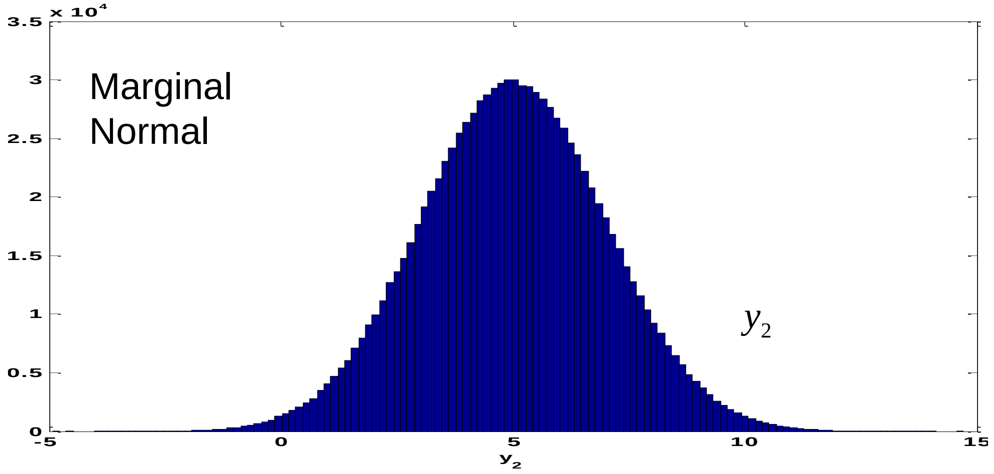
$$\underset{2\times1}{y}=\underset{2\times2}{A}\,\underset{2\times1}{z}+\underset{2\times1}{\mu}$$



cross section

A=chol(Sigma)'





Symmetric Matrix Factorizations-Cholesky

6.1.6.1 Cholesky Factorization

Cholesky's method for factorizing a symmetric positive definite matrix Σ of dimension p is very straightforward. This factorization $\Sigma = A_\Sigma A_\Sigma'$ has the property that A_Σ be a lower triangular matrix. Denote the ij^{th} element of Σ and A_Σ to be σ_{ij} and a_{ij} respectively. Simple formulas [32] for the method are

$$a_{11} = \sqrt{\sigma_{11}} \qquad (6.1.38)$$

$$a_{ii} = \sqrt{\sigma_{ii} - \sum_{k=1}^{i-1} a_{ik}^2} \qquad i = 2, \ldots, n \qquad (6.1.39)$$

$$a_{i1} = \frac{\sigma_{i1}}{a_{11}} \qquad i = 1, \ldots, n \qquad (6.1.40)$$

$$a_{ij} = \frac{1}{a_{jj}} \left(\sigma_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{jk} \right) \qquad i = j+1, \ldots, n; j \geq 2. \qquad (6.1.41)$$

Rowe, 2003

Symmetric Matrix Factorizations-Cholesky

$$\Sigma = \begin{bmatrix} 4 & .75 \cdot 2 \cdot 4 \\ .75 \cdot 2 \cdot 4 & 16 \end{bmatrix}$$

$$a_{11} = \sqrt{4} = 2$$

$$a_{21} = \frac{.75 \cdot 2 \cdot 4}{\sqrt{4}} = 3$$

$$a_{22} = \sqrt{16 - a_{21}^2} = 2.6458$$

$$A = \begin{bmatrix} 2.0000 & 0 \\ 3.0000 & 2.6458 \end{bmatrix}$$

$$a_{11} = \sqrt{\sigma_{11}}$$

$$a_{ii} = \sqrt{\sigma_{ii} - \sum_{k=1}^{i-1} a_{ik}^2} \qquad i = 2, \ldots, n$$

$$a_{i1} = \frac{\sigma_{i1}}{a_{11}} \qquad i = 1, \ldots, n$$

$$a_{ij} = \frac{1}{a_{jj}} \left(\sigma_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{jk} \right) \qquad i = j+1, \ldots, n; j \geq 2.$$

```
Sigma=[4,2*4*.75;2*4*.75,16];
a11=sqrt(4)
a21=.75*sqrt(4*16)/sqrt(4)
a22=sqrt(16-a21^2)
A=[a11,0;a21,a22]
```

Symmetric Matrix Factorizations-Eigenvector

Given a matrix Σ, it can be factorized symmetrically into $\Sigma=AA'$, where $A=UD^{1/2}$ with D being a diagonal matrix of eigenvalues and U being a matrix of eigenvectors.

Solve $|\Sigma-\lambda I|=0$ to get $\lambda_1>....>\lambda_p$

Then solve $\Sigma u_i=\lambda_i u_i$ to get $U=[u_1,\ldots,u_p]$, $i=1,\ldots,p$.

Form A as $A=UD^{1/2}$.

Symmetric Matrix Factorizations-Eigenvector

$$\Sigma = \begin{bmatrix} 4 & .75 \cdot 2 \cdot 4 \\ .75 \cdot 2 \cdot 4 & 16 \end{bmatrix}$$

$$\left| \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\lambda^2 - 20 + 28 = 0$$

$$\lambda_1 = 10 + 6\sqrt{2}$$

$$\lambda_2 = 10 - 6\sqrt{2}$$

$$U = \begin{bmatrix} 0.3827 & -0.9239 \\ 0.9239 & 0.3827 \end{bmatrix}$$

$$D = \begin{bmatrix} 18.4853 & 0 \\ 0 & 1.5147 \end{bmatrix}$$

$$A = \begin{bmatrix} 1.6453 & -1.1371 \\ 3.9722 & 0.4710 \end{bmatrix}$$

$$\Sigma = AA'$$

```
Sigma=[4,6;6,16]
[U,D]=eigs(Sigma,2,'largestabs')
A=U*sqrt(D)
A*A'
```

Cholesky

$$A = \begin{bmatrix} 2.0000 & 0 \\ 3.0000 & 2.6458 \end{bmatrix}$$

Homework 4

Chapter 6: # 4, 5, 6.

A. Assume Marquette Undergrads heights h have
μ_h=67 in and σ_h=2 in while their weights w have
μ_w=150 lbs and σ_w=4 lbs with ρ=.75.

a) Present an algorithm for generating random height & weights using both Cholesky and Eigen factorizations.

b) Generate 10,000 using each factorizations.
Compute summary means, variances, covariance and correlation.
Make 2-D & 3-D histograms. Comment?

Homework 4

B. Generate 1000 bivariate observations (h,w) from the bivariate normal PDF $f(x\,|\,\mu,\Sigma)=(2\pi)^{-p/2}\,|\Sigma|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$ where $\underset{2\times 1}{x}=\begin{pmatrix} h \\ w \end{pmatrix}$, $\underset{2\times 1}{\mu}=\begin{pmatrix} \mu_h \\ \mu_w \end{pmatrix}=\begin{pmatrix} 67 \\ 150 \end{pmatrix}$

and $\underset{2\times 2}{\Sigma}=\begin{pmatrix} \sigma_h^2 & \sigma_{hw} \\ \sigma_{hw} & \sigma_w^2 \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix}$

For this you will need an instrumental PDF $g(y_1,y_2)$ that you shift c for your “envelope” distribution.

Assume:
$59<h<75$
$134<w<166$

Accept the bivariate random $(h,w)=(y_1,y_2)$ if $U < f((y_1,y_2)\,/(c\,g(y_1,y_2))$.

Then (h,w) has the desired PDF.

Make a histogram of the bivariate (h,w)‘s. In Matlab use hist3.

Calculate means, variances, and correlation. Comment.