# Chapter 5: Generating Continuous Random Variables

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# Agenda

5.1 The Inverse Transform Algorithm

5.2 The Rejection Method ⟵

5.3 The Polar Method for Generating Normal Random Variables

5.4 Generating a Poisson Process

# 5.1 The Inverse (CDF) Transform Method

## Proposition

Let $U$ be a uniform(0,1) random variable. For any continuous distribution function $F$, the random variable $X$ defined by

$$X = F^{-1}(U)$$

has a distribution function $F$.

[$F^{-1}(u)$ is defined to be that value of $x$ such that $F(x)=u$.]

# 5.1 The Inverse (CDF) Transform Method

## Proof

Let $F_X$ denote the distribution function of $X$=$F^{-1}(U)$. Then

$$\begin{aligned} F_X(x) &= P\{X \le x\} \\ &= P\{F^{-1}(U) \le x\} \end{aligned}$$

Since $F$ is a CDF

$$\begin{aligned} F_X(x) &= P\{F(F^{-1}(U)) \le F(x)\} \\ &= P\{U \le F(x)\} \\ &= F(x) \end{aligned}$$

And thus $X$ is generated by $X$=$F^{-1}(U)$.

# 5.1 The Inverse (CDF) Transform Method

## Example

Generate random variables from the exponential distribution.

$f(x) = e^{-x}, \ \ x > 0$ $\qquad$ $F(x) = 1 - e^{-x}, \ \ x > 0$ $\qquad$ $\lambda = 1$

Let $x=F^{-1}(u)$, then $x=F^{-1}(U)=-\log(1-u)$.

Thus we can generate an exponential $\lambda=1$ by generating a random number $U$ and setting

$$X = F^{-1}(U) = -\log(1-U)$$

# 5.1 The Inverse (CDF) Transform Method

$$F(x) = 1 - e^{-x}$$

**Example**

Generate exponential RVs.

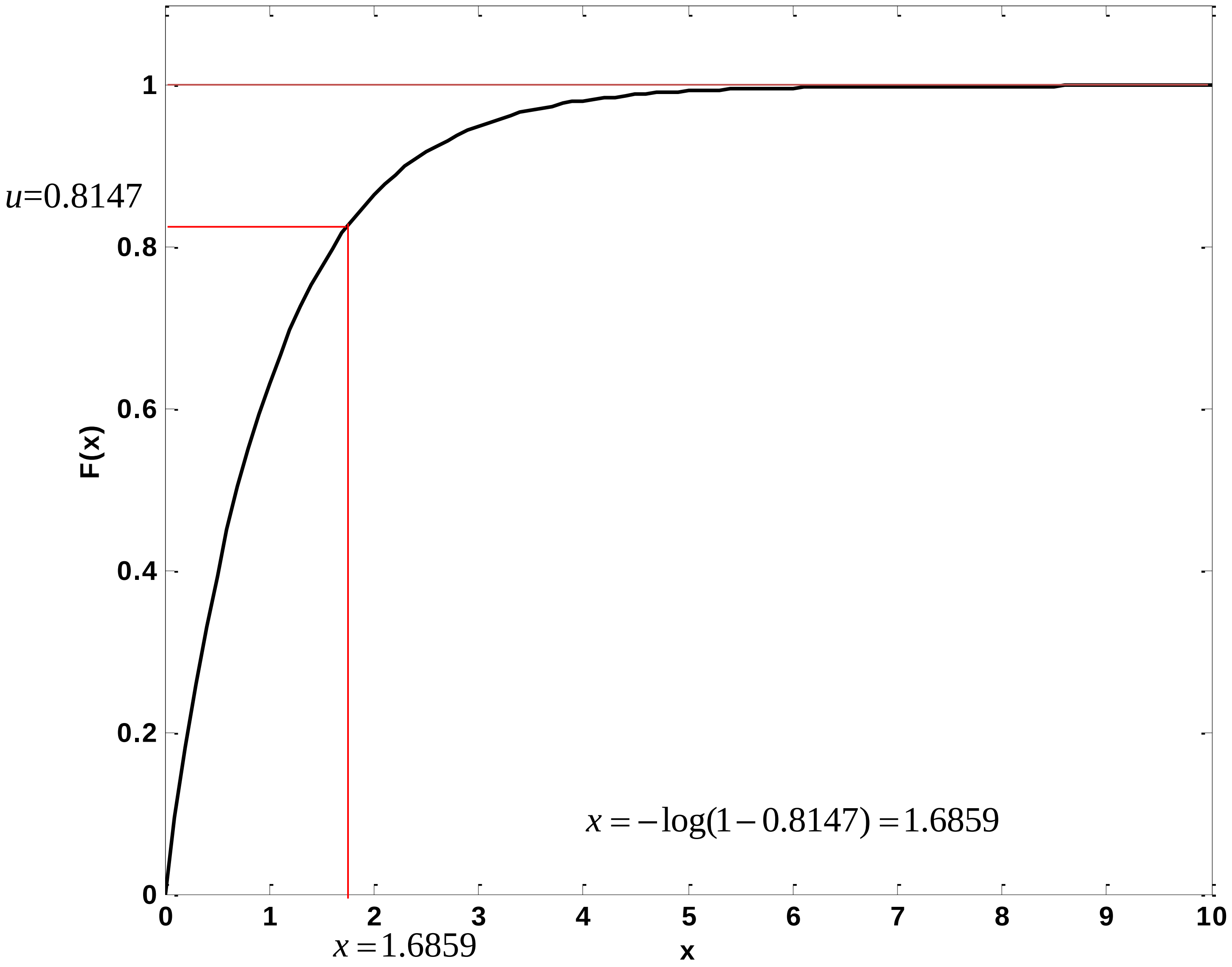
Generate random $u$, then

Calculate $x = -\log(1-u)$ .

$x$ is now a RV that has an exponential distribution.

$u$ = rand;

$x$ = -log(1-$u$);



# 5.2 The Rejection Method

Sometimes it is difficult to generate $X$ from PDF $f(x)$.

If we have a technique to generate $Y$ from PDF $g(y)$.

Then we can use $Y$ to generate an $X$ with PDF $f(x)$.

Let $c$ be a constant such $\dfrac{f(y)}{g(y)} \leq c$ for all $y$.

Then we accept the random $X=Y$ if $U < f(y)/(c\, g(y))$

and $X$ has the desired PDF $f(x)$.

# 5.2 The Rejection Method

**Rejection Method**

STEP 1: Simulate the value of $Y$, having PDF $g(y)$.

STEP 2: Generate a random $U$.

STEP 3: If $U \leq f(y)/(cg(y))$, set $X=Y$. Otherwise go to 1.

## Theorem

(i) The random variable generated by the rejection method has density $f$.

(ii) The number of iterations of the algorithm that are needed is a geometric RV with mean $c$.

$$P\{X=n\} = p(1-p)^{n-1}, \quad n \geqslant 1$$

$$E[X] = \sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{p}$$

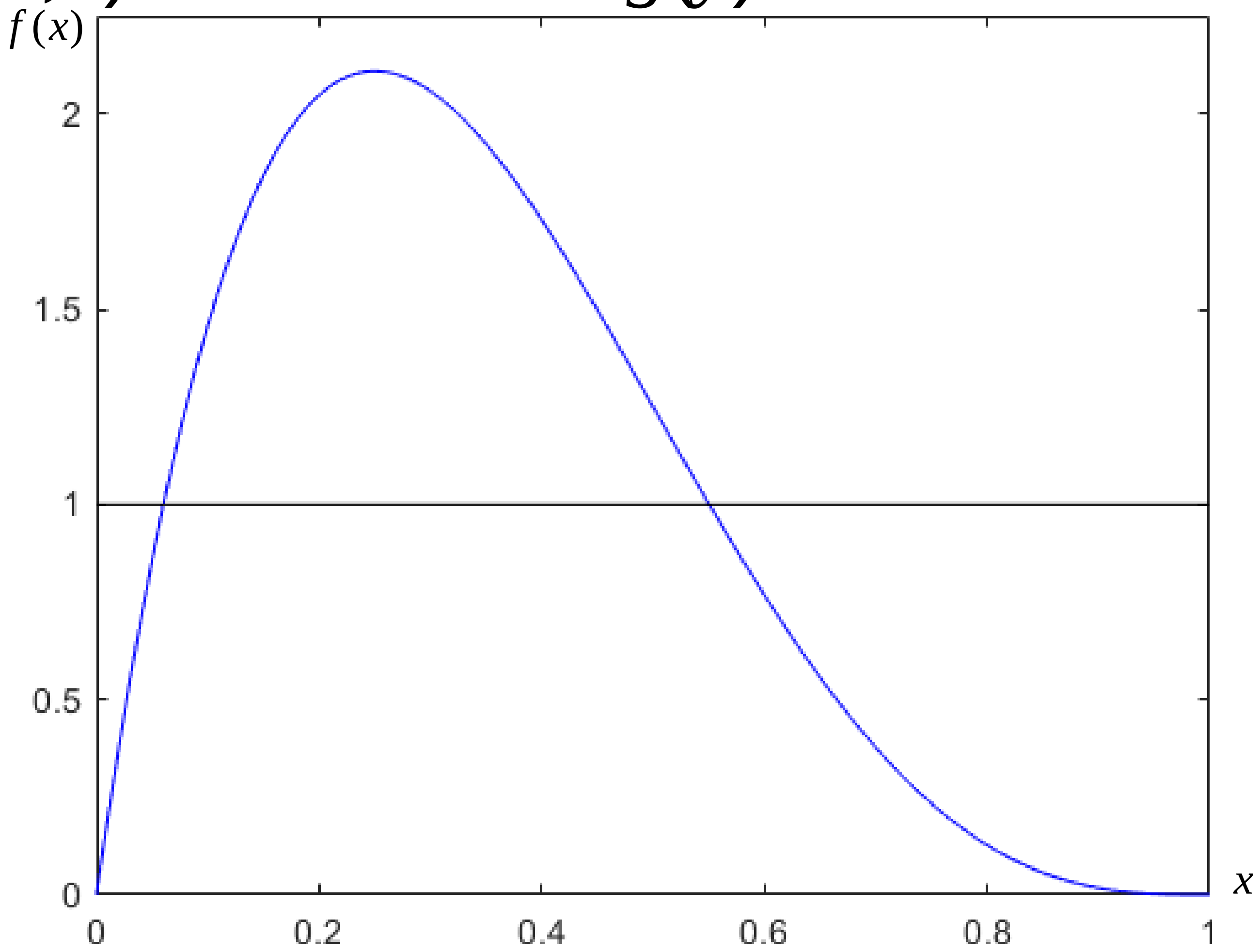
# 5.2 The Rejection Method

## Example

Use rejection sampling to generate random variables from

$f(x) = 20x(1-x)^3,\ 0 \le x \le 1$. Use uniform(0,1) instrumental $g(y)$.

$g(y) = 1,\ 0 \le y \le 1$



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## Example

Use rejection sampling to generate random variables from $f(x)=20x(1-x)^3,\ 0\le x\le 1$. Use uniform(0,1) instrumental $g(y)$.

$g(y)=1,\ 0\le y\le 1$

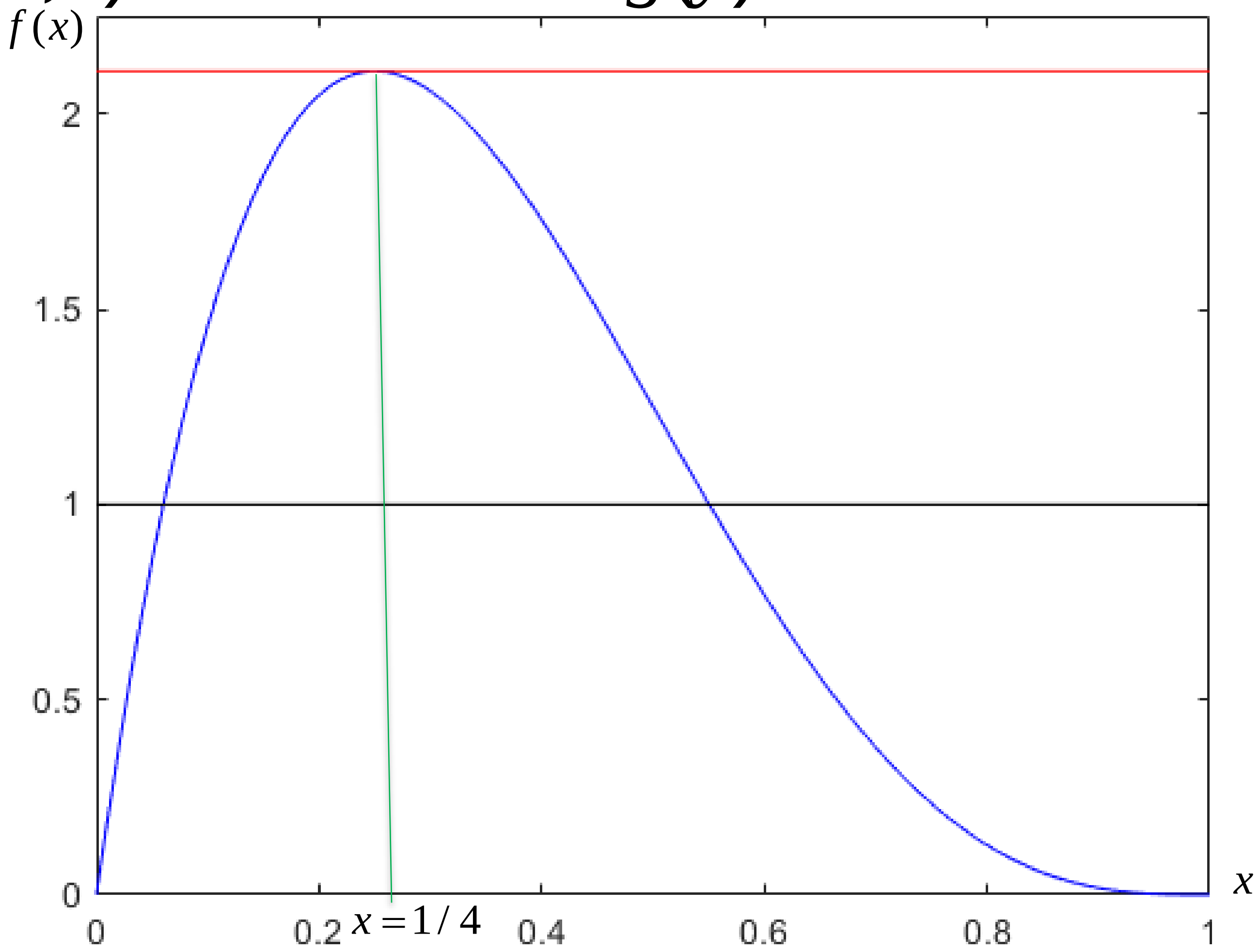
Get envelope "distribution"

$cg(y)$, where $c$=max($f(x)/g(x)$).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{d}{dx}20x(1-x)^3$$

$$0=20\left[(1-x)^3-3x(1-x)^2\right]$$

$$x=1/4$$



# 5.2 The Rejection Method

**Example**

Use rejection sampling to generate random variables from $f(x)=20x(1-x)^3,\ 0\le x\le 1$. Use uniform(0,1) instrumental $g(y)$.

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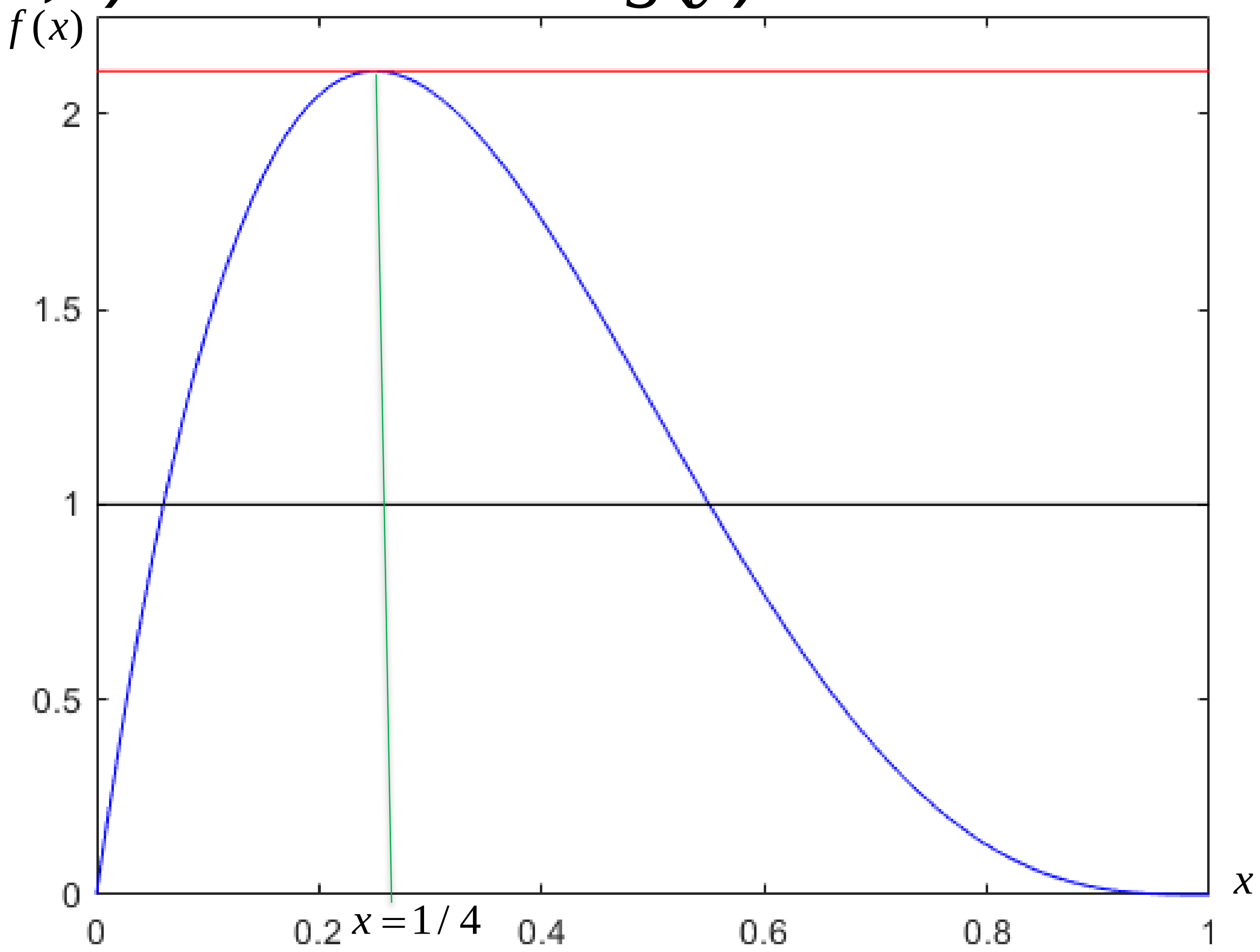
Get envelope "distribution"

$cg(y)$, where $c$=max($f(x)/g(x)$).

$x=1/4$

$c=20(1/4)(1-1/4)^3$

$c$=135/64=2.1094



# 5.2 The Rejection Method

## Example

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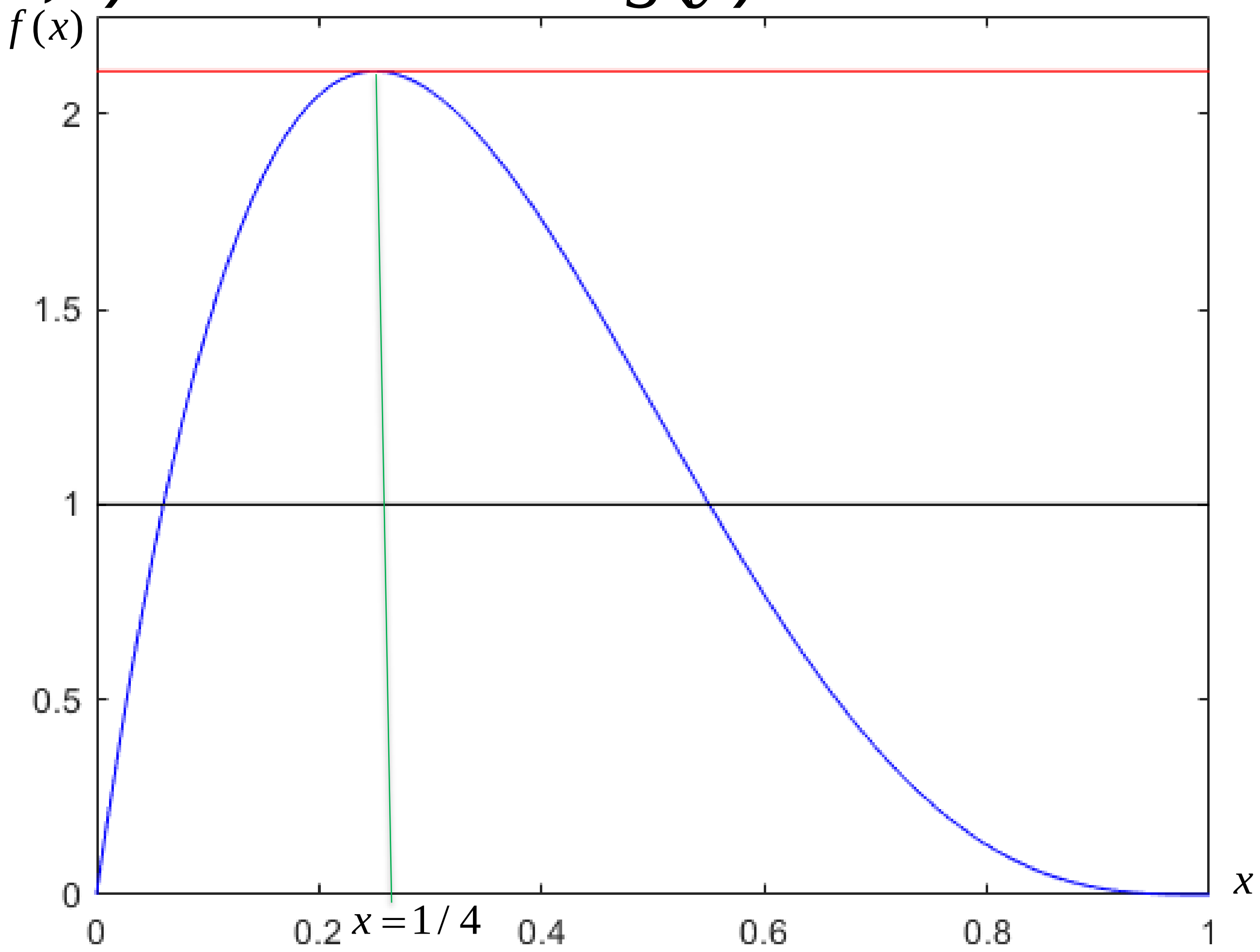


Get envelope "distribution"

$cg(y)$, where $c$=max($f(x)/g(x)$).

$c$=135/64=2.1094

STEP 1: Generate $Y$ from $g(y)$.
STEP 2: Generate a $U$.
STEP 3: If $U < f(y)/(cg(y))$,
set $X$=$Y$ and stop.
Otherwise go to 1.

# 5.2 The Rejection Method

If can't compute $E(x)$ from $f(x)$ I can simply average the $x$'s in the histogram!

## Example

Use rejection sampling to generate random variables from $f(x) = 20x(1-x)^3,\ 0 \leq x \leq 1$. Use uniform(0,1) instrumental $g(y)$.

$g(y) = 1,\ 0 \leq y \leq 1$

Get envelope "distribution"

$cg(y)$, where $c$=max($f(x)/g(x)$).

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STEP 1: Generate $Y$ from $g(y)$.
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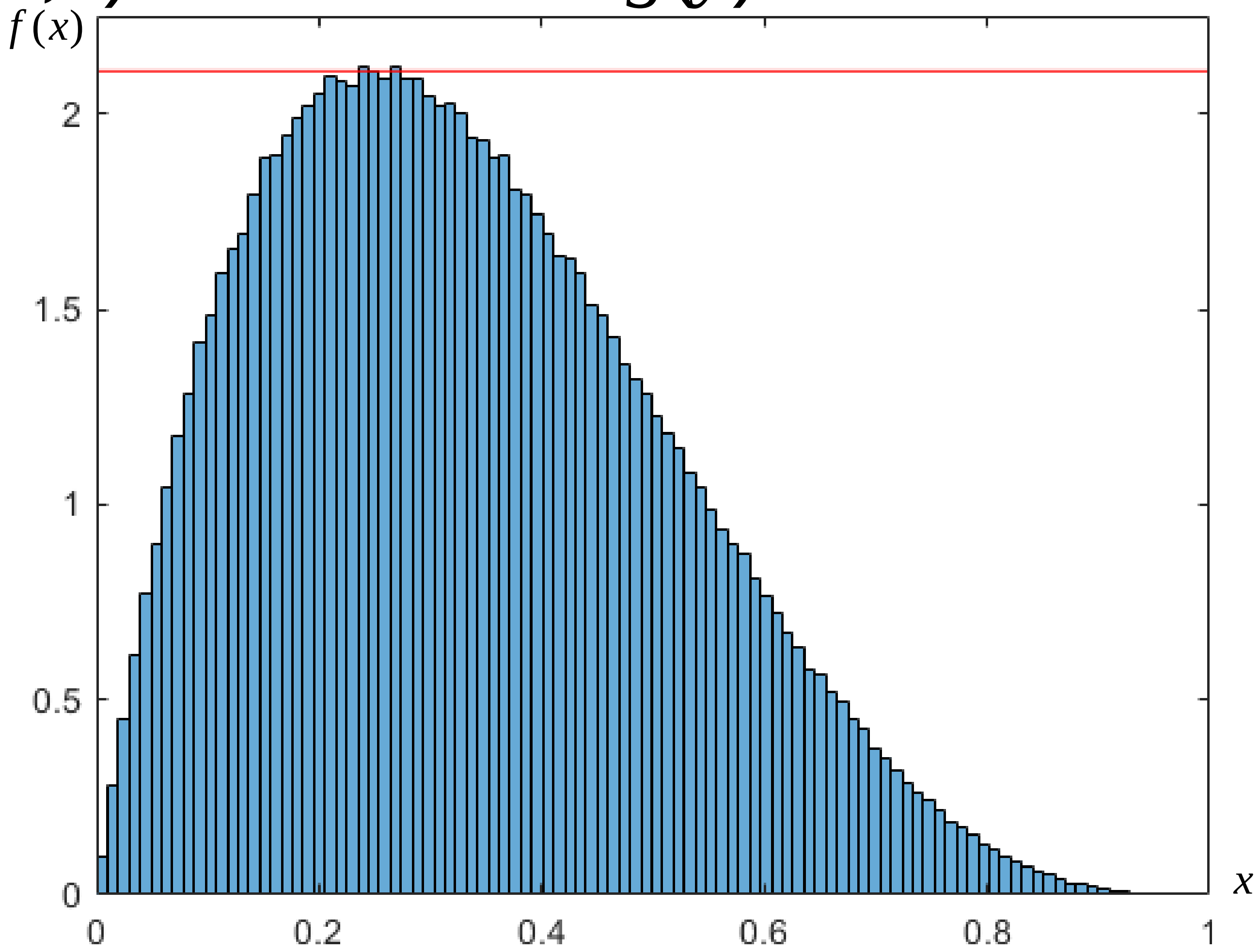


# 5.2 The Rejection Method

```
clear all
close all
rng('default')

x = (0:.01:1)';
f=20*x.*(1-x).^3;
g=ones(length(x),1);

figure;
plot(x,f,'b'), xlim([0,1]), ylim([0 2.25])
hold on
plot(x,g,'k')
c=135/64;
plot(x,c*g,'r')

figure;
plot(x,f./(c*g),'m')
hold on
plot(x,ones(length(x),1),'r')
xlim([0,1]), ylim([0 2.25/c])
```

```
n=10^6;, nn=10^8; X=zeros(n,1); count=0;
for j=1:nn
 Y=rand;
 U=rand;
 if (U<=(20*Y*(1-Y)^3/(c*1)))
  count=count+1;
  X(count,1)=Y;
 end
 if (count==n)
  disp(['Done after ',num2str(j),'.'])
  return;
 end
end

figure;
histogram(X,100,'Normalization','pdf')
hold on
plot(x,c*g,'r')
xlim([0,1]),ylim([0,2.25])
```

# 5.3 The Polar Method for Generating Normal Random Variables

Let $u_1$~uniform(0,1) and $u_2$~uniform(0,1).

The joint PDF of $(u_1,u_2)$ is

$$f(u_1,u_2) = \begin{cases} 1 & \text{if} \quad u_1 \in [0,1] \text{ and } u_2 \in [0,1] \\ 0 & \text{if} \quad u_1 \notin [0,1] \text{ or } u_2 \notin [0,1] \end{cases}$$

If $x = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$ , $y = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$ ,

the PDF of $(x,y)$ is $f(x,y) = \dfrac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$ .

# 5.3 The Polar Method for Generating Normal Random Variables

Generate $10^6$ independent uniform(0,1)'s.

The first half of the $10^6$ standard uniform random variates were used as $u_1$'s and the second half used as $u_2$'s.

Take each $(u_1,u_2)$ pair to produce an $(x,y)$ pair.

$$x = \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \qquad y = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$$

are independent normally distributed.

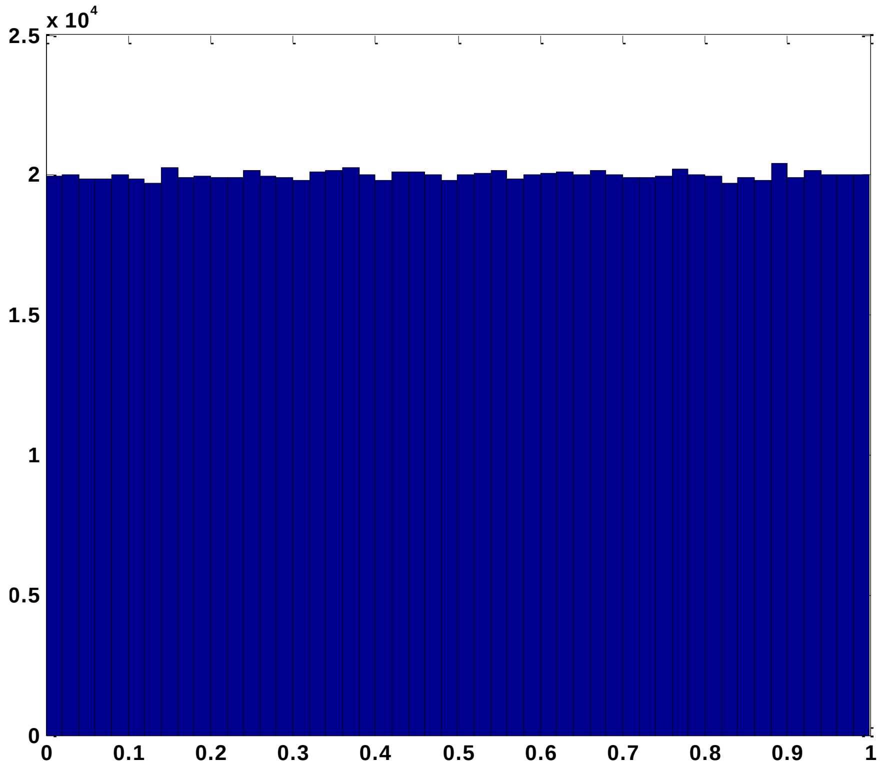
$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

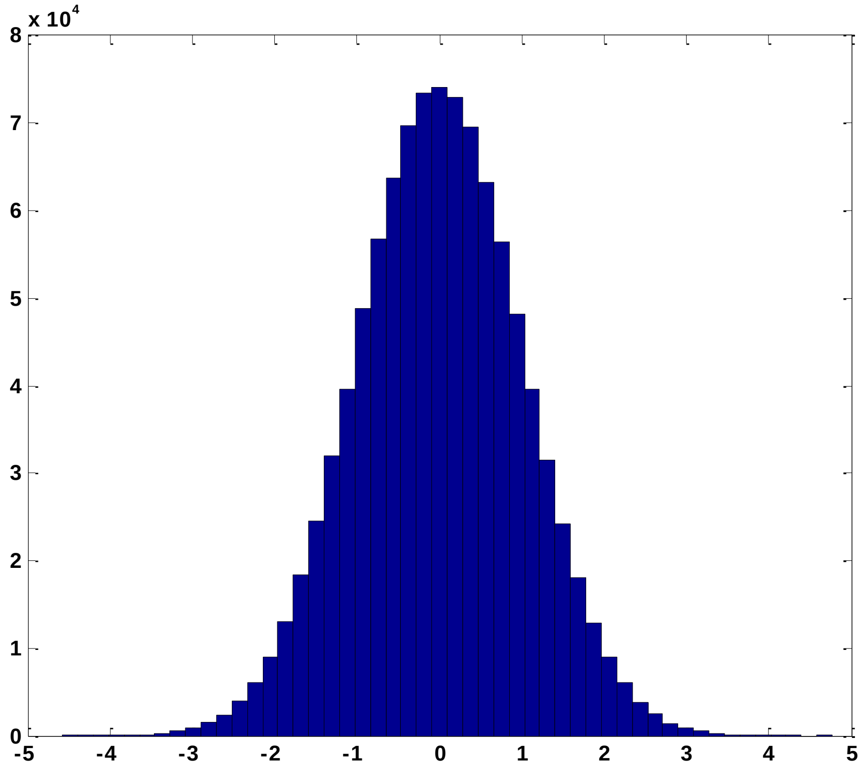
# 5.3 The Polar Method for Generating Normal Random Variables

```
n=10^6;
u1=rand(n/2,1);
u2=rand(n/2,1);
figure(1)
histogram([u1;u2],50)
```

```
x=sqrt(-2*log(u1)).*cos(2*pi*u2);
y=sqrt(-2*log(u1)).*sin(2*pi*u2);
figure(2)
histogram([x;y],50)
```

```
[mean(u1),var(u1)]
[mean(u2),var(u2)]
[mean(x),var(x)]
[mean(y),var(y)]
[corr(u1,u2),corr(x,y)]
```




| | |
|---|---|
| 0.5000 | 0.0832 |
| 0.5006 | 0.0833 |
| 0.0000 | 1.0011 |
| -0.0016 | 0.9970 |
| 0.0025 | 0.0013 |

Uncorrelated and
since normal
are independent

# 5.4 Generating A Poisson Process

Suppose we want to generate a homogeneous Poisson Process with rate $\lambda$ observed from time 0 to time $T$.

We know that time between successive arrivals is exponentially distributed with rate $\lambda$. So all we need to do is generate exponential RVs and accumulate them as arrival times.

Generate random numbers $U_1$, $U_2$,…, $U_n$ and set $X_i = -\log U_i$.
Cumulatively sum $X_i$'s until $t > T$.

# 5.4 Generating A Poisson Process

## The algorithm is:

STEP 1: $t$=0, $I$=0.
STEP 2: Generate a random number $U$.
STEP 3: $t=t-\ \log U$. If $t>T$, stop.
STEP 4: $I=I+1$, S($I$)=$t$.
STEP 5: Go to Step 2.

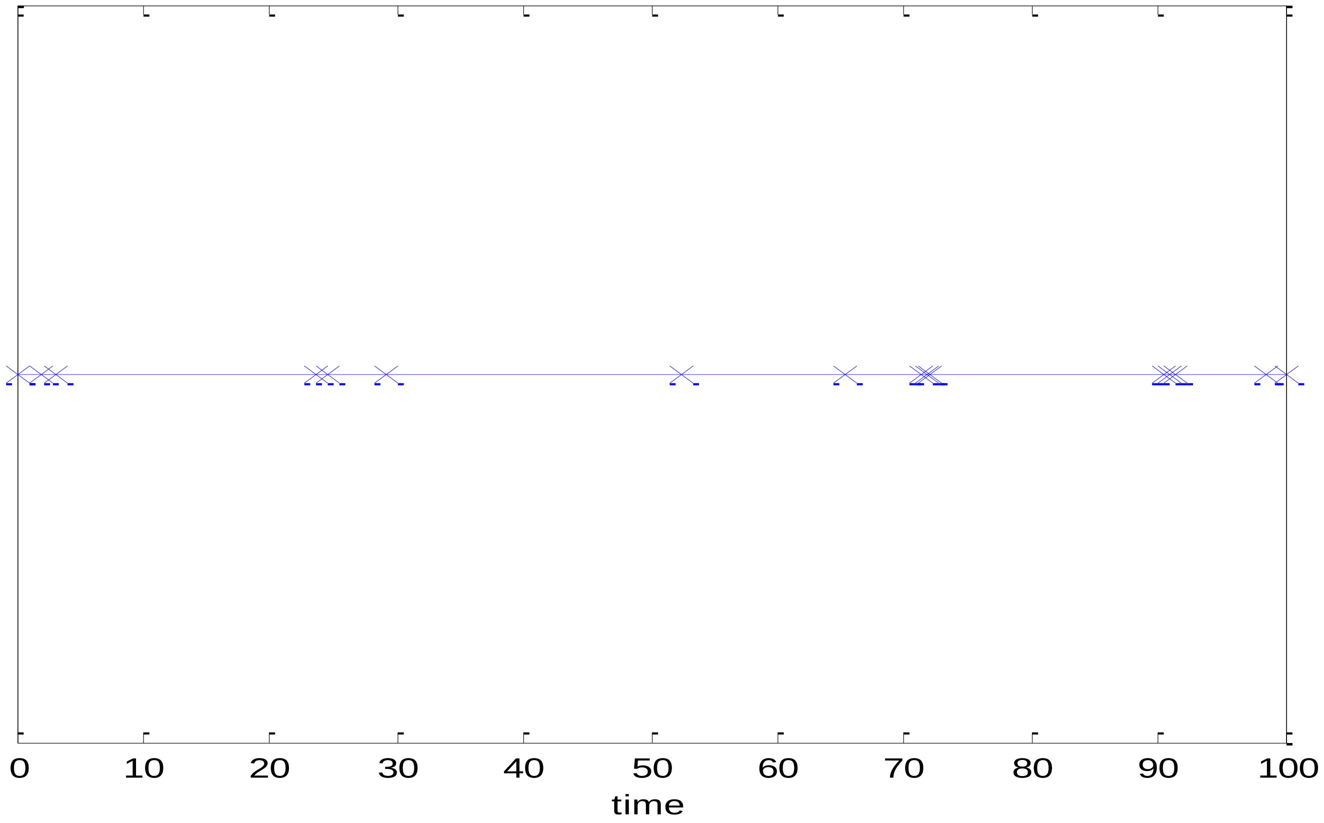
## Example: $\lambda$=.1, $T$=100.

```
clear all
close all

lam=.1;, T=100;, n=15;
U=rand(n,1);
X=-log(U)/lam;
t=cumsum(X);
nits=max(find(t<=T));a
U(nits+1:n)=[];,X(nits+1:n)=[];,t(nits+1:n)=[];
[U,X,t]

figure;
plot([0;t;100],zeros(nits+2,1),'-x','MarkerSize',10)
set(gca,'YTick',[ ]), xlabel('time')
```

# 5.4 Generating A Poisson Process

The results are:

Example: $\lambda$=.1, $T$=100.




| $U$ | $X$ | $t$ | $I$ |
|---|---|---|---|
| 0.8147 | 2.0491 | 2.0491 | 1 |
| 0.9058 | 0.9895 | 3.0385 | 2 |
| 0.1270 | 20.6367 | 23.6752 | 3 |
| 0.9134 | 0.9061 | 24.5813 | 4 |
| 0.6324 | 4.5830 | 29.1643 | 5 |
| 0.0975 | 23.2749 | 52.4392 | 6 |
| 0.2785 | 12.7834 | 65.2226 | 7 |
| 0.5469 | 6.0352 | 71.2578 | 8 |
| 0.9575 | 0.4342 | 71.6921 | 8 |
| 0.9649 | 0.3574 | 72.0495 | 9 |
| 0.1576 | 18.4761 | 90.5256 | 11 |
| 0.9706 | 0.2985 | 90.8241 | 12 |
| 0.9572 | 0.4378 | 91.2619 | 13 |
| 0.4854 | 7.2283 | 98.4902 | 14 |

# Homework 3

Chapter 5: # 1,3,13,15,23,25,29.

*If $u$ has a uniform$(0,1)$ PDF, derive the PDF of $x$=–log(1-$u$) using a transformation of variable.

**Generate $10^4$ uniform random numbers $u$.

Take –log(1-$u$) of each to get $10^4$ random $x$'s.

Make a histogram and calculate mean and variance.

Compare to theoretical values.