Chapter 4: Generating Discrete Random Variables

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Agenda

4.1 The Inverse Transform Method

4.2 Generating a Poisson Random Variable

4.3 Generating Binomial Random Variables

4.4 The Acceptance-Rejection Technique ⟵

4.5 The Composition Approach

4.1 The Inverse (CDF) Transform Method

To generate discrete RVs with PMF

$$P\{X = x_j\} = p_j, \quad j = 0,1,\ldots \quad \sum_j p_j = 1$$

generate a random U that is uniformly distributed in (0,1) ,

and set

$$X = \begin{cases} x_0 & \text{If} \quad 0 < U < p_0 \\ x_1 & \text{If} \quad p_0 < U < p_0 + p_1 \\ \vdots & \\ x_j & \text{If} \quad \sum_{i=0}^{j-1} p_i < U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases}$$

then X has the desired PMF.

4.1 The Inverse (CDF) Transform Method

This is called the inverse method because two are the same!

$$X = \begin{cases} x_0 & \text{If} \quad 0 < U < p_0 \\ x_1 & \text{If} \quad p_0 < U < p_0 + p_1 \\ \vdots & \\ x_j & \text{If} \quad \sum_{i=0}^{j-1} p_i < U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases} \quad \longleftrightarrow \quad X = \begin{cases} x_0 & \text{If} \quad 0 < U < F(x_0) \\ x_1 & \text{If} \quad F(x_0) \le U < F(x_1) \\ \vdots & \\ x_j & \text{If} \quad F(x_{j-1}) \le U < F(x_j) \\ \vdots & \end{cases}$$

Recall:

$$F(x_{j-1}) = \sum_{i=0}^{j-1} P(X = x_i) \qquad F(x_j) = \sum_{i=0}^{j} P(X = x_i)$$

4.1 The Inverse (CDF) Transform Method

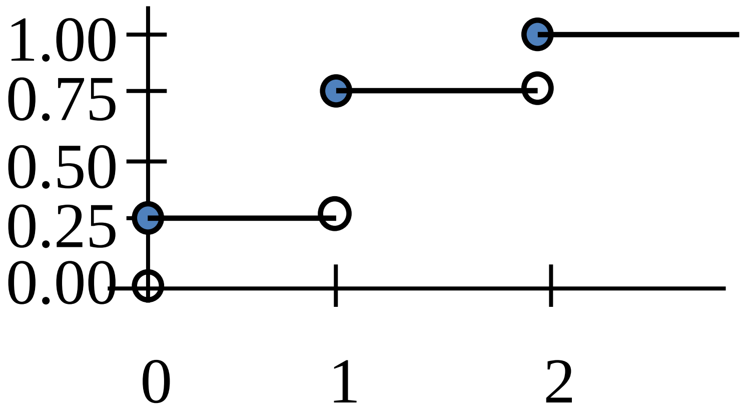
Example: Tossing a coin twice

x	0	1	2
$P(X=x)$	1/4	1/2	1/4

$$F(x) = \begin{cases} 0/4 & -\infty < x < 0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 4/4 & 2 \le x < \infty \end{cases}$$

Generate U, then

$$X = \begin{cases} 0 & \text{If} \quad 0 \le U < 1/4 \\ 1 & \text{If} \quad 1/4 \le U < 3/4 \\ 2 & \text{If} \quad 3/4 \le U < 1 \end{cases} \quad \longleftrightarrow$$



4.1 The Inverse (CDF) Transform Method

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STEP 1: Generate a random U.
STEP 2: If $0 \le U < 0.25$, then $X=0$. Go to Step 1.
STEP 3: If $0.25 \le U < .75$, then $X=1$. Go to Step 1.
STEP 4: If $0.75 \le U < 1.00$, then $X=2$. Go to Step 1.

U	X
0.8147	2
0.9058	2
0.1270	0
0.9134	2
0.6324	1

```
n=5;
U=rand(n,1);, X=zeros(n,1);
for j=1:n
  if (0<=U(j,1))&(U(j,1)<.25)
    X(j,1)=0;
  elseif(.25<=U(j,1))&(U(j,1)<.75)
    X(j,1)=1;
  elseif(.75<=U(j,1))&(U(j,1)<1.0)
    X(j,1)=2;
  end
end
[U,X]
```

4.2 Generating a Poisson Random Variable

To generate discrete RVs with PMF

$$p_i = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0,1,\ldots$$

$$p_{i+1} = \frac{\lambda}{i+1} p_i$$

$$\frac{p_{i+1}}{p_i} = \frac{e^{-\lambda}\frac{\lambda^{i+1}}{(i+1)!}}{e^{-\lambda}\frac{\lambda^i}{i!}}$$

use the identity that

STEP 1: Generate a random U.
STEP 2: i=0, $p=e^{-\lambda}=p_0$, $F=p$.
STEP 3: If $U < F$, set $X=i$ and stop.
STEP 4: $p=\lambda p/(i+1)$, $F=F+p$, $i=i+1$.
STEP 5: Go to Step 3.

then X has the desired PMF

4.3 Generating a Binomial Random Variable

To generate discrete RVs with PMF

$$P\{X=i\}=\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i},\quad i=0,1,\ldots n$$

use the identity that

$$P\{X=i+1\}=\frac{n-i}{i+1}\frac{p}{1-p}P\{X=i\}$$

$$\frac{P\{X=i+1\}}{P\{X=i\}}=\frac{\frac{n!}{(n-i-1)!(i+1)!}p^{i+1}(1-p)^{n-i-1}}{\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i}}$$

4.3 Generating a Binomial Random Variable

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STEP 1: Generate a random U.
STEP 2: $c=p/(1-p)$, $i=0$, pr= $(1-p)^n=p_0$, F=pr.
STEP 3: If $U<F$, set $X=i$ and stop.
STEP 4: pr=$[c(n-i)/(i+1)]$pr, $F=F+$pr, $i=i+1$.
STEP 5: Go to Step 3.

then X has the desired PMF.

4.4 The Acceptance-Rejection Technique

Sometimes it is difficult to generate X from PMF $\{p_j, j\geq 0\}$.

If we have a technique to generate Y from PMF $\{q_j, j\geq 0\}$.

Then we can use Y to generate an X with PMF $\{p_j, j\geq 0\}$.

The PMF q_j is called the instrumental distribution.

4.4 The Acceptance-Rejection Technique

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If we have a technique to generate Y from PMF $\{q_j, j\geq 0\}$.
Then we can use Y to generate an X with PMF $\{p_j, j\geq 0\}$.
The PMF q_j is called the instrumental distribution.

$$p_j < cq_j, \quad c > 1$$

Here cq_j is called the envelope distribution, a function whose "probabilities" are all larger than p_j.

We accept the random $X=Y$ if $U < p_Y/(cq_Y)$
and X has the desired PMF$\{p_j, j\geq 0\}$.

4.4 The Acceptance-Rejection Technique

Rejection Method

STEP 1: Simulate the value of Y, having PMF q_j.

STEP 2: Generate a random U.

STEP 3: If $U < p_Y/(cq_Y)$, set $X=Y$ and stop. Otherwise go to 1.

Theorem

The acceptance-rejection algorithm generates a RV X such that

$$P\{X = j\} = p_j, \quad i = 0,1,\ldots n$$

The number of iterations needed to obtain X is a geometric RV with mean c.

4.4 The Acceptance-Rejection Technique

Example

Use rejection sampling to generate Poisson random variables.

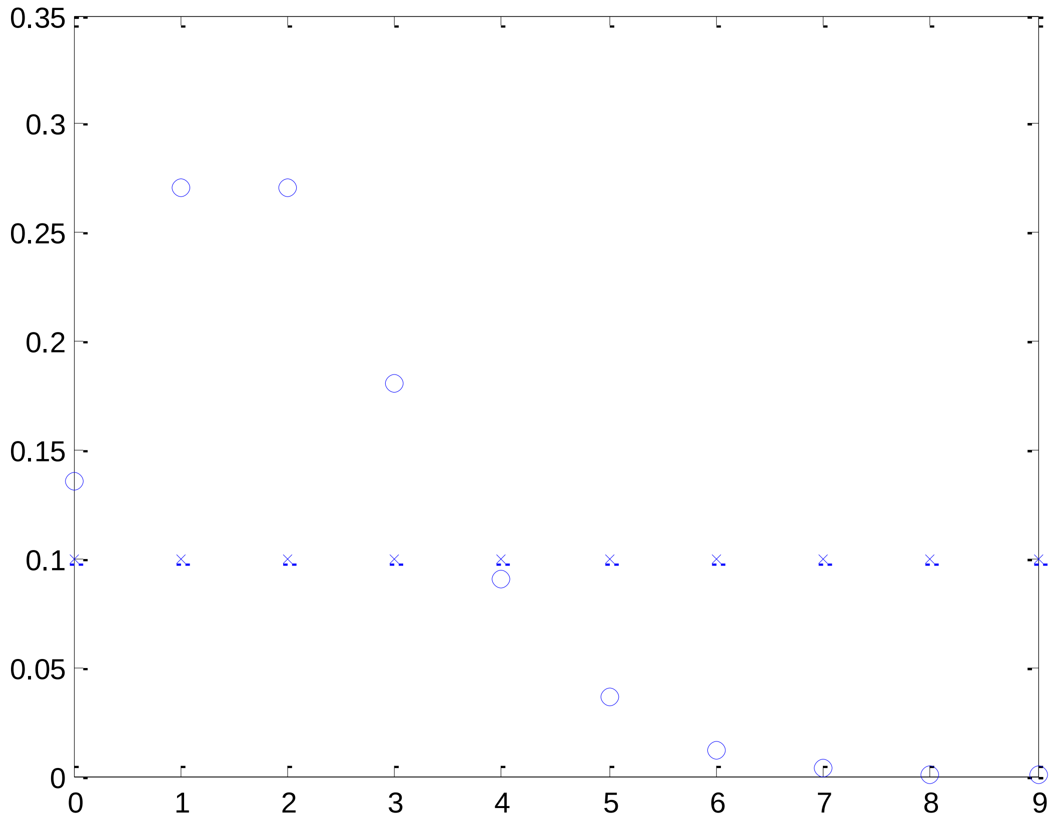
Let λ=2 and use discrete uniform(0,N-1) instrumental distribution.

$$p_j = e^{-\lambda}\frac{\lambda^j}{j!}, \quad j = 0,1,\ldots$$

and

$$q_j = \frac{1}{N}, \;\; j = 0,1,\ldots,N-1$$

$$N-1 = \max(j) \ni p_j > 0$$



j	p_j	q_j
0	.1353	.1
1	.2707	.1
2	.2707	.1
3	.1804	.1
4	.0902	.1
5	.0361	.1
6	.0120	.1
7	.0034	.1
8	.0009	.1
9	.0002	.1

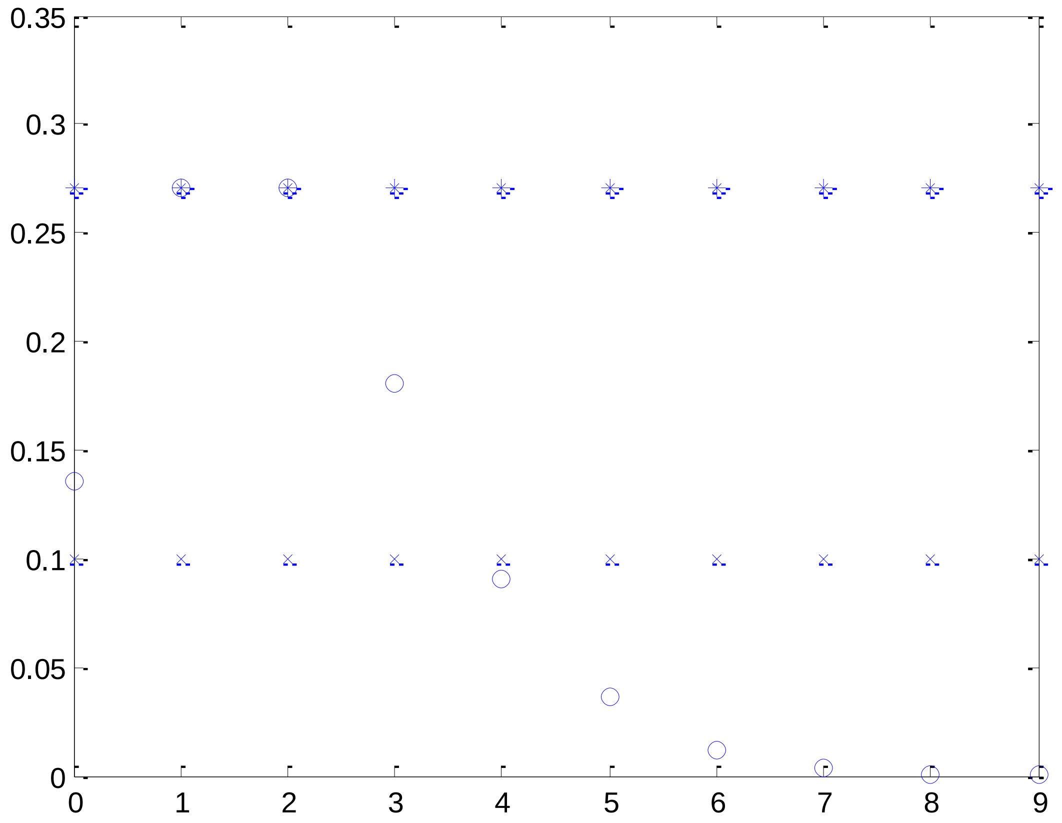
4.4 The Acceptance-Rejection Technique

Example

Use rejection sampling to generate Poisson random variables.

Let λ=2 and use discrete uniform(0,N-1) instrumental distribution.

Get envelope "distribution" cq_j, where c=max(p_j/q_j).
c= (0.2707/0.1)=2.707



j	p_j	q_j	cq_j
0	.1353	.1	0.2707
1	.2707	.1	0.2707
2	.2707	.1	0.2707
3	.1804	.1	0.2707
4	.0902	.1	0.2707
5	.0361	.1	0.2707
6	.0120	.1	0.2707
7	.0034	.1	0.2707
8	.0009	.1	0.2707
9	.0002	.1	0.2707

4.4 The Acceptance-Rejection Technique

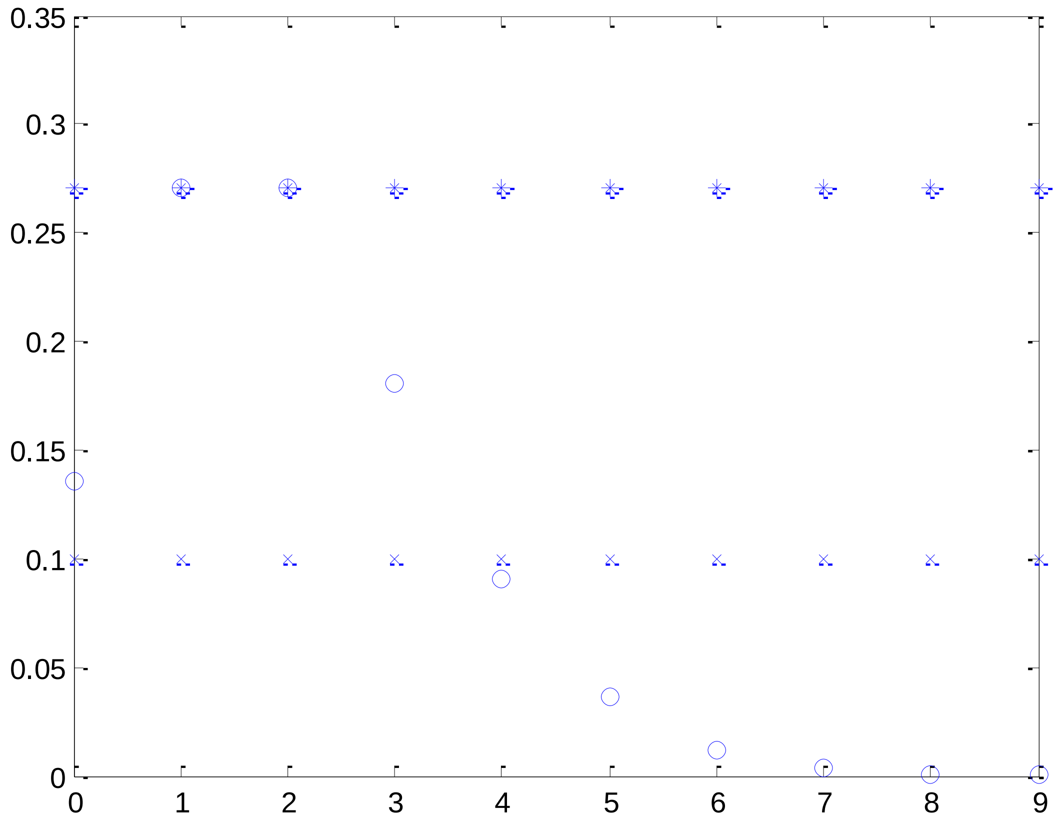
Example

Use rejection sampling to generate Poisson random variables.

Let λ=2 and use discrete uniform(0,N-1) instrumental distribution.

c= (0.2707/0.1)=2.707

STEP 1: Generate Y from q_j.
STEP 2: Generate a U.
STEP 3: If $U < p_Y/(cq_Y)$,
set X=Y and stop.
Otherwise go to 1.



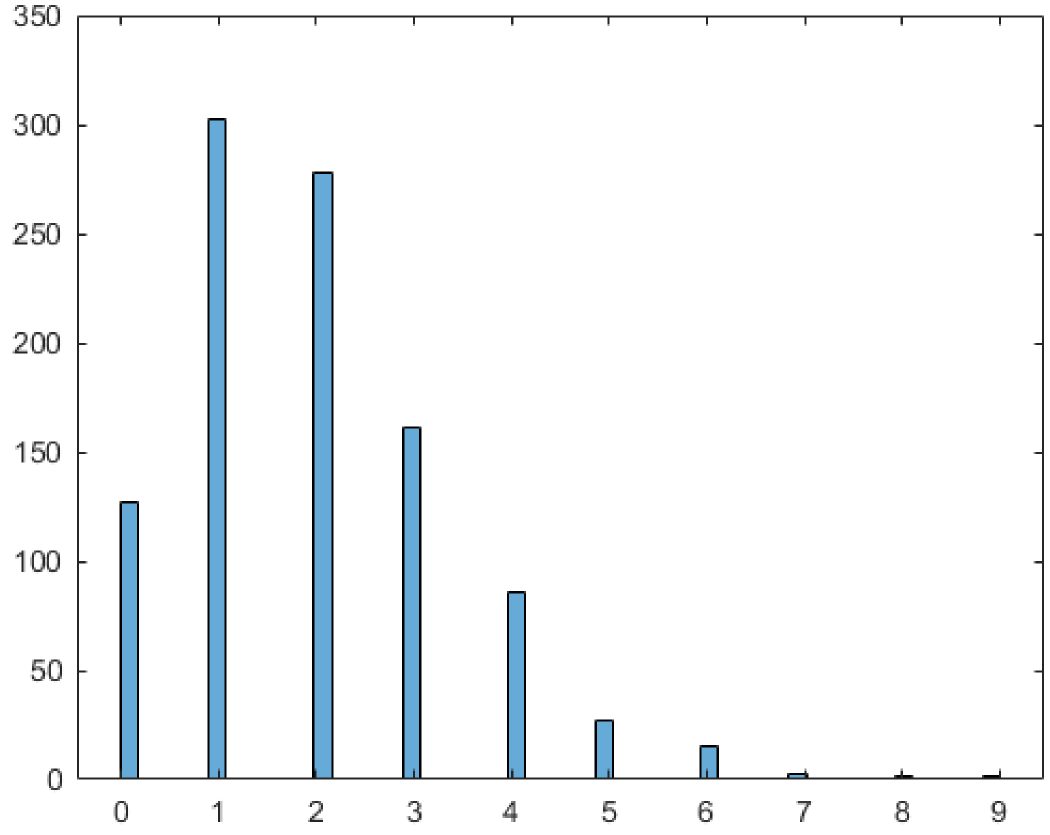
j	p_j	q_j	cq_j	$p_j/(cq_j)$
0	.1353	.1	0.2707	0.5000
1	.2707	.1	0.2707	1.0000
2	.2707	.1	0.2707	1.0000
3	.1804	.1	0.2707	0.6667
4	.0902	.1	0.2707	0.3333
5	.0361	.1	0.2707	0.1333
6	.0120	.1	0.2707	0.0444
7	.0034	.1	0.2707	0.0127
8	.0009	.1	0.2707	0.0032
9	.0002	.1	0.2707	0.0007

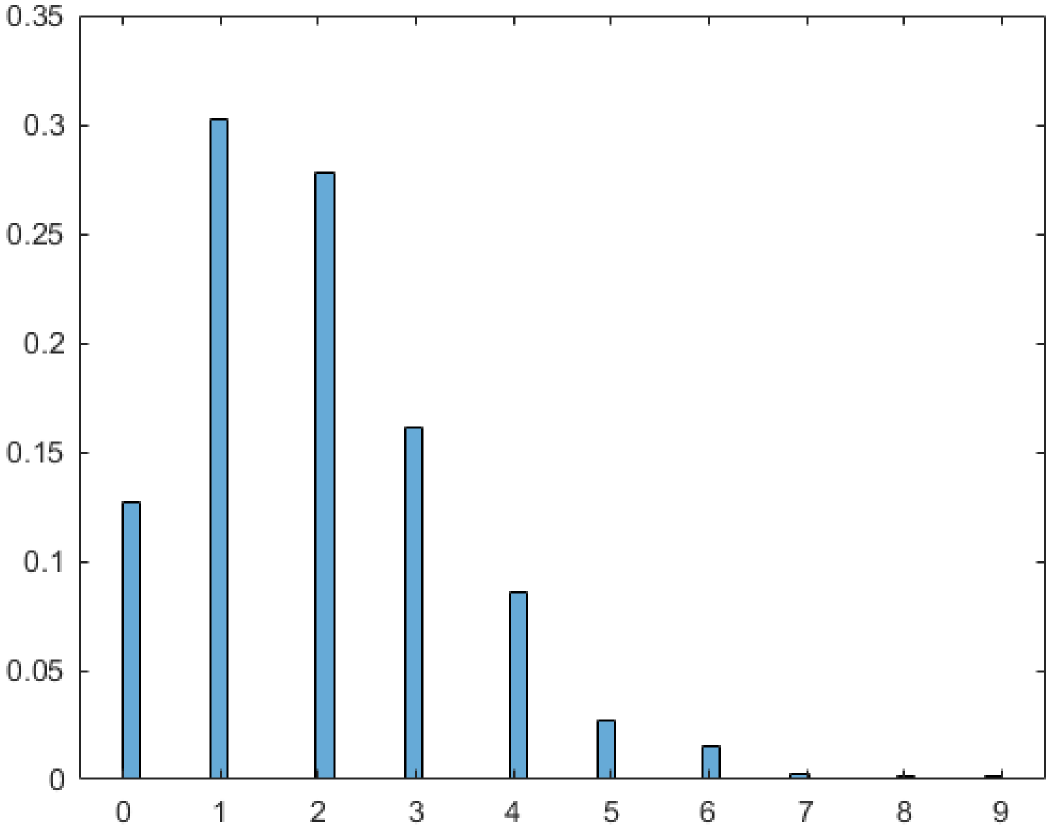
4.4 The Acceptance-Rejection Technique

Example

Use rejection sampling to generate Poisson random variables.

Let λ=2 and use discrete uniform(0,N-1) instrumental distribution.





j	p_j	q_j	cq_j	$p_j/(cq_j)$
0	.1353	.1	0.2707	0.5000
1	.2707	.1	0.2707	1.0000
2	.2707	.1	0.2707	1.0000
3	.1804	.1	0.2707	0.6667
4	.0902	.1	0.2707	0.3333
5	.0361	.1	0.2707	0.1333
6	.0120	.1	0.2707	0.0444
7	.0034	.1	0.2707	0.0127
8	.0009	.1	0.2707	0.0032
9	.0002	.1	0.2707	0.0007

4.4 The Acceptance-Rejection Technique

```
clear all
close all

% set seed to same default value
rng('default')

x = (0:9)';,lam=2;
p = poisspdf(x,lam);

figure;
plot(x,p,'o')
N=10;
q=1/N*ones(10,1);
hold on
plot(x,q,'x')
c=max(p./q)
hold on
plot(x,c*q,’*’)

[x,p,q]
```

```
n=1000;, nn=1000000;
X=zeros(n,1);
count=0;, Y=zeros(n,1);
R=p./(c*q)
for j=1:nn
  Y(j,1)=randi([0,9]);
  U=rand(1,1);
  if   (U<=p(Y(j,1)+1,1)/(c*q(Y(j,1)+1,1)))
    count=count+1;
    X(count,1)=Y(j,1);
  end
  if (count==n)
    return;
    end
 end
[x,p,q]

figure;
histogram(X,50 ,'Normalization’,’count')
figure;
histogram(X,50,'Normalization','probability')
```

4.5 The Composition Approach

Suppose we have a way to simulate random variables From both PMFs $\{p_j, j\geq 0\}$ and $\{p_j, j\geq 0\}$, and want to simulate from

$$P\{X=j\}=\alpha p_j + (1-\alpha)p_j \, , \, j \geq 0$$

$P\{X=j\}$ is called a mixture PMF.

where $0<\alpha<1$. One way is to simulate X such that

$$X = \begin{cases} X_1 \text{ with probability } \alpha \\ X_2 \text{ with probability } 1-\alpha \end{cases}$$

To do this generate U, then

if $U\leq\alpha$, generate an X_1,

elseif $U>\alpha$, generate an X_2.

4.5 The Composition Approach

Example

I am going to flip a coin with $P(H)=\alpha$.

if heads, generate an X_1, by flipping the coin, $\{0,1\}$

elseif tails, generate an X_2 by rolling the die, $\{1,2,3,4,5,6\}$.

$$P\{X=j\}=\alpha p_j + (1-\alpha)p_j \qquad (1)$$

$$X = \begin{cases} X_1 & \text{with probability } \alpha \\ X_2 & \text{with probability } 1-\alpha \end{cases} \qquad (2)$$

4.6 Generating Random Vectors

A random vector $x=(x_1, \ldots, x_n)'$ can be simulated by

1) Generating a random x_1 from $P\{X_1=j\}$
2) Generating a random vector from $P\{X_2=j_2|X_1=j_1\}$
3) …
4) Generating a random vector from $P\{X_n=j_n\,|X_{n\text{-}1}=j_{n\text{-}1},\ldots,X_1=j_1\}$

If $x_1, \ldots, x_n$ are IID, then we can simply fill the elements of x with random random RVs from $P\{X_i=j_i\}$.

Homework 2

Chapter 4: # 1*, 3,12,15,17.

*Repeat using Matlab’s binornd() command.

Compare results to Matlab’s binornd()