Numerical ^ Integration
Deterministic

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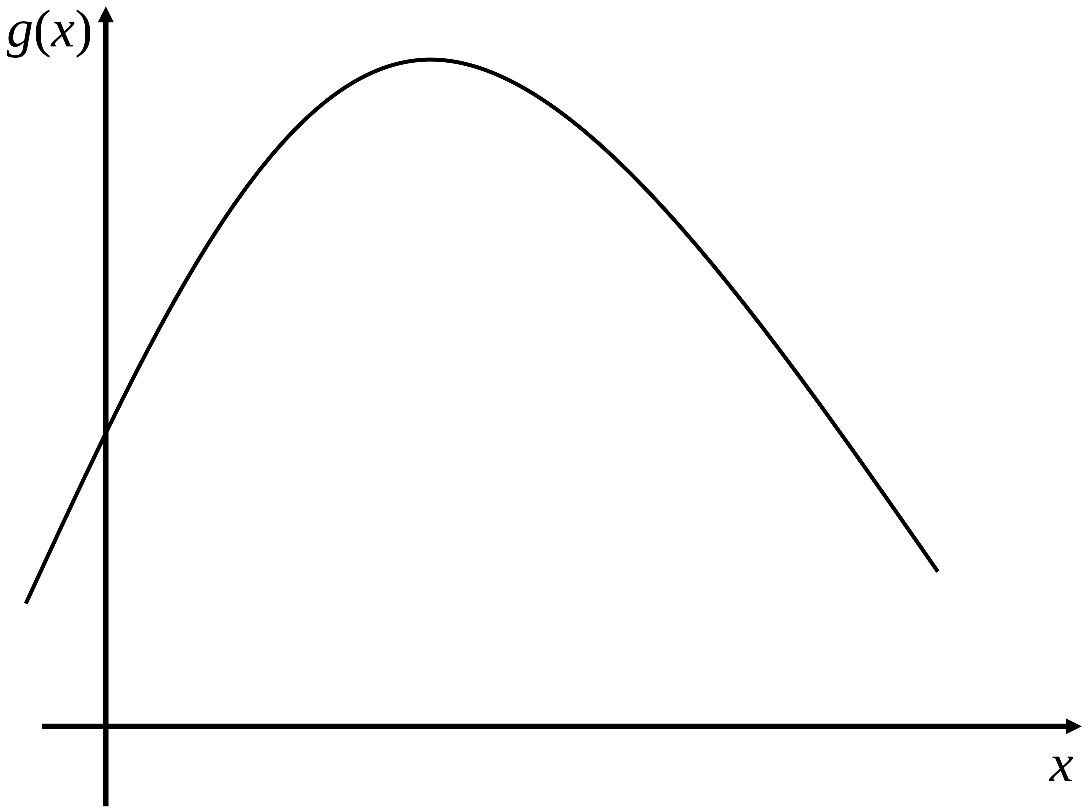
Deterministic Integration

goal

$$E(x) = \int_x g(x)dx$$



Deterministic Integration

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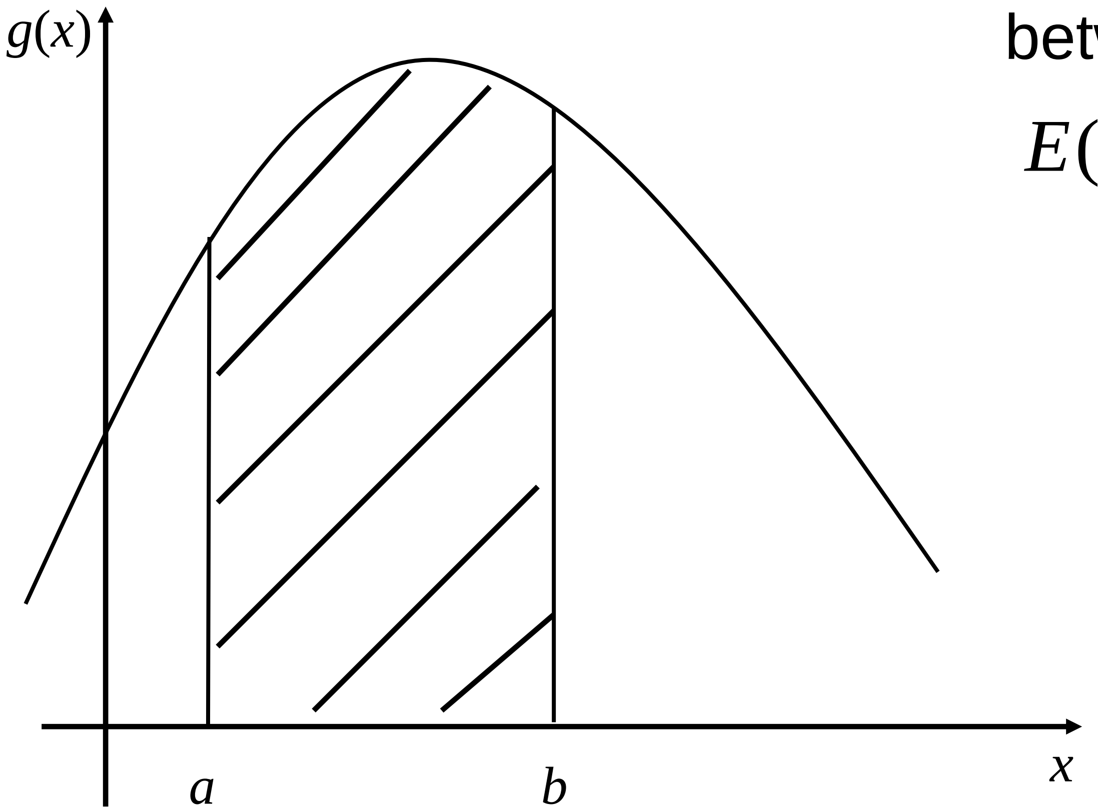


Area under curve between a and b.

$$E(x)=\int_x g(x)dx$$

think of as

$$g(x)=xf(x)$$

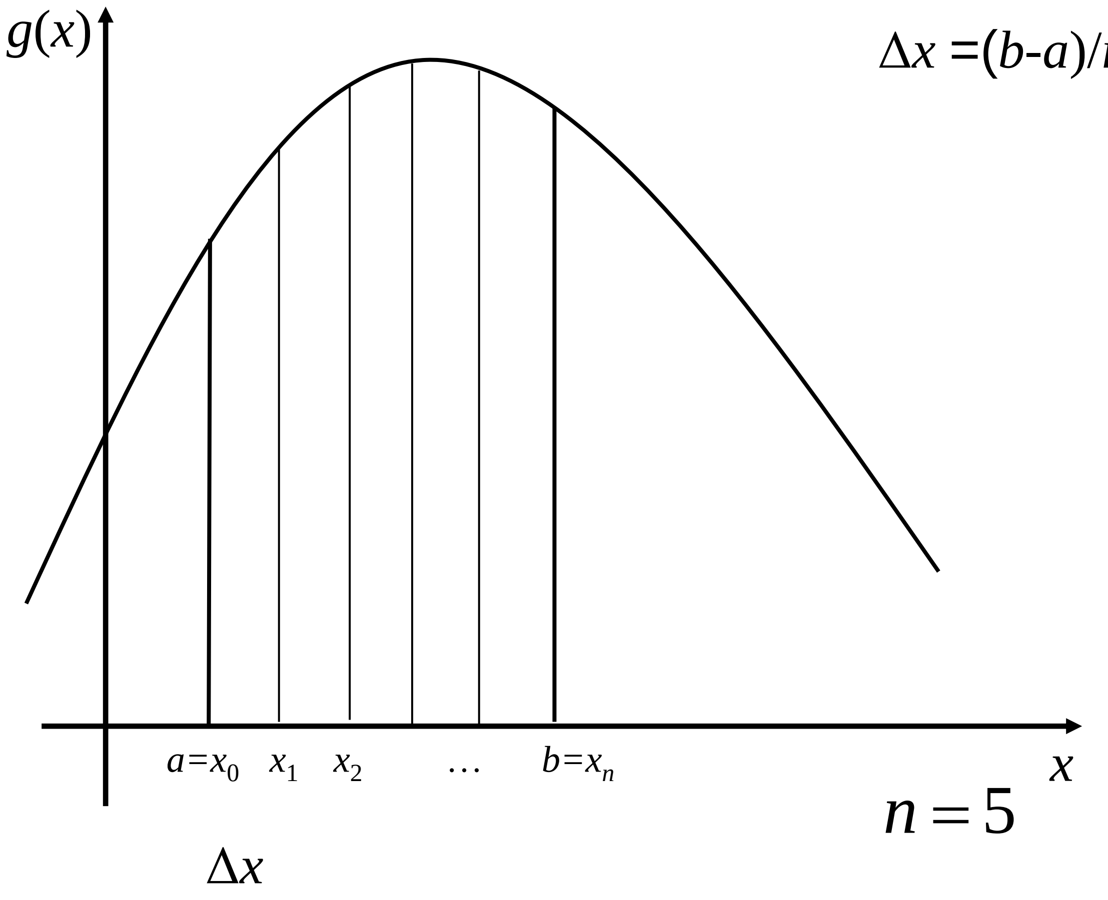
Deterministic Integration

goal

$$E(x)=\int_x g(x)dx$$

Divide into intervals: Δx small

$\Delta x = (b\text{-}a)/n$ $\Delta x = x_i - x_{i-1}$



think of as

$$g(x)=xf(x)$$

Deterministic Integration

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$$E(x)=\int_x g(x)dx$$

Divide into intervals: Δx small

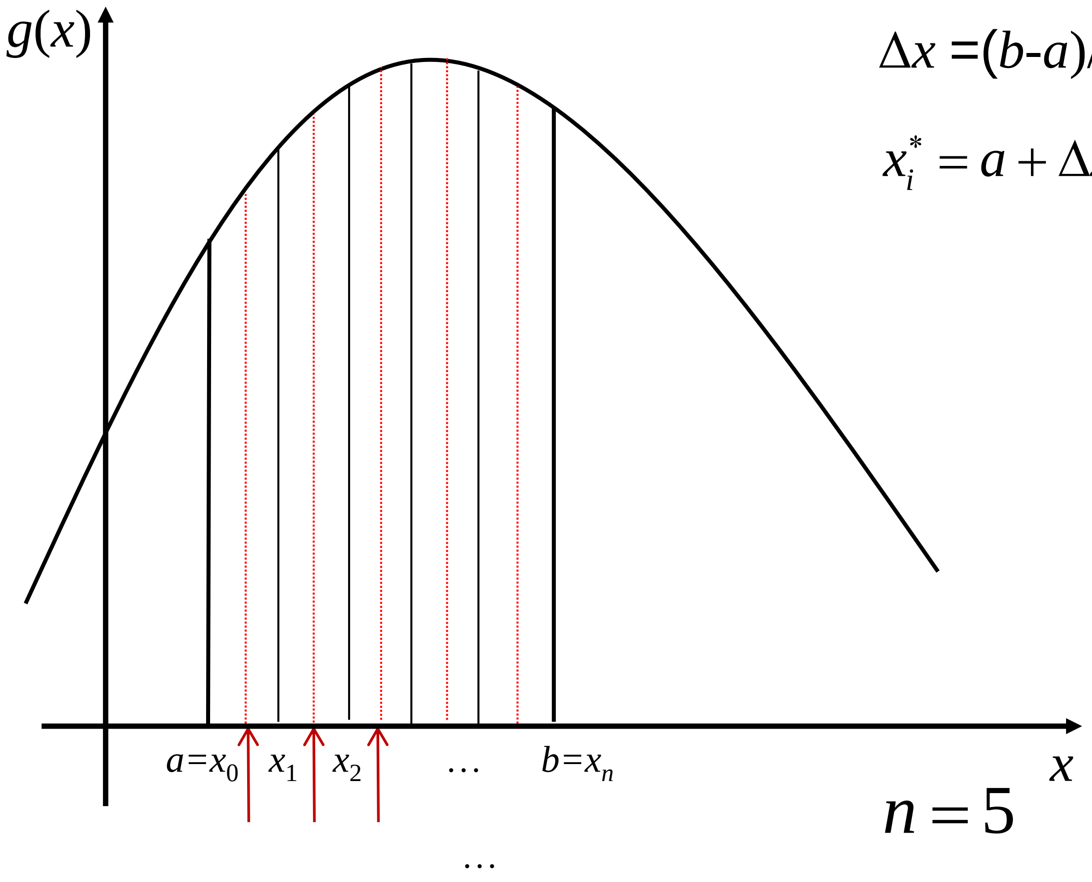
Δx =(b-a)/n $\qquad \Delta x = x_i - x_{i-1}$

$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

$$i=1,\ldots,n$$

think of as

$$g(x)=xf(x)$$



$$n=5$$

Deterministic Integration

goal

$$E(x)=\int_x g(x)dx$$

Divide into intervals: Δx small

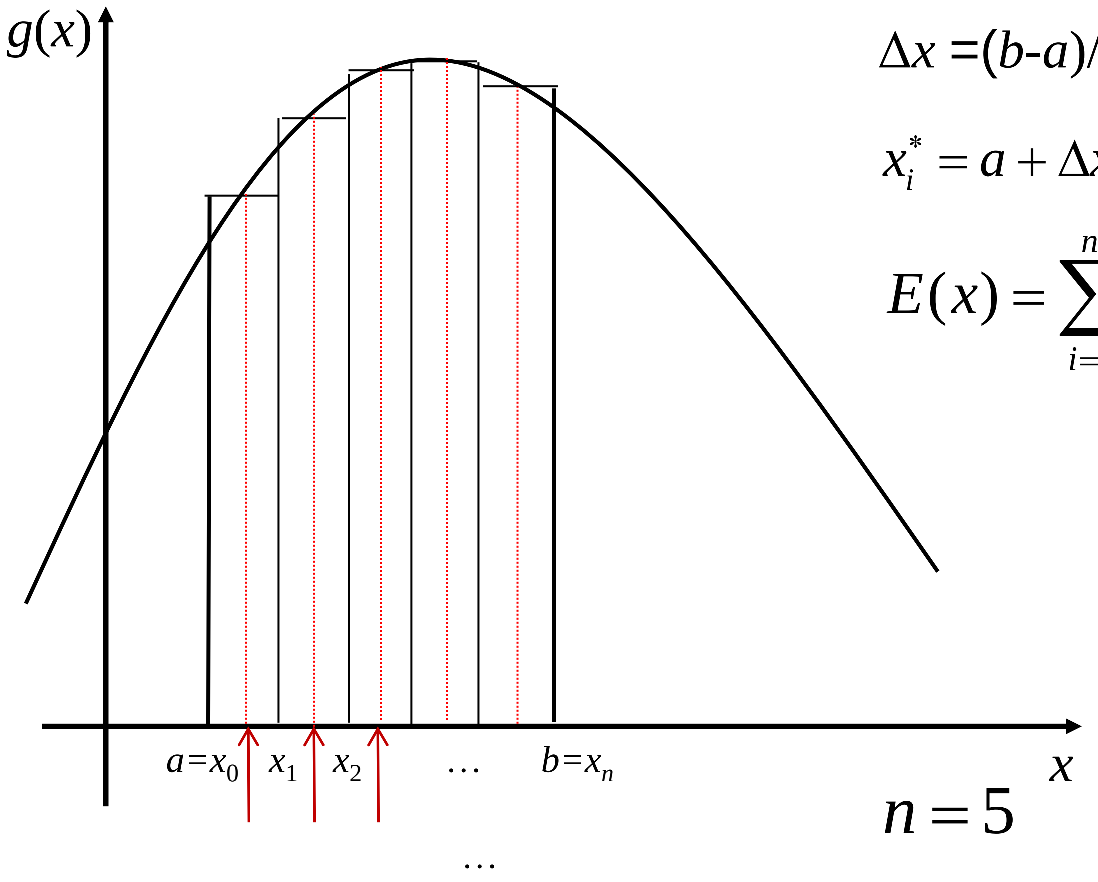
$\Delta x = (b-a)/n$ $\qquad \Delta x = x_i - x_{i-1}$

$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

$$E(x)=\sum_{i=1}^{n} g(x_i^*)\Delta x \qquad i=1,\ldots,n$$

think of as

$$g(x)=xf(x)$$

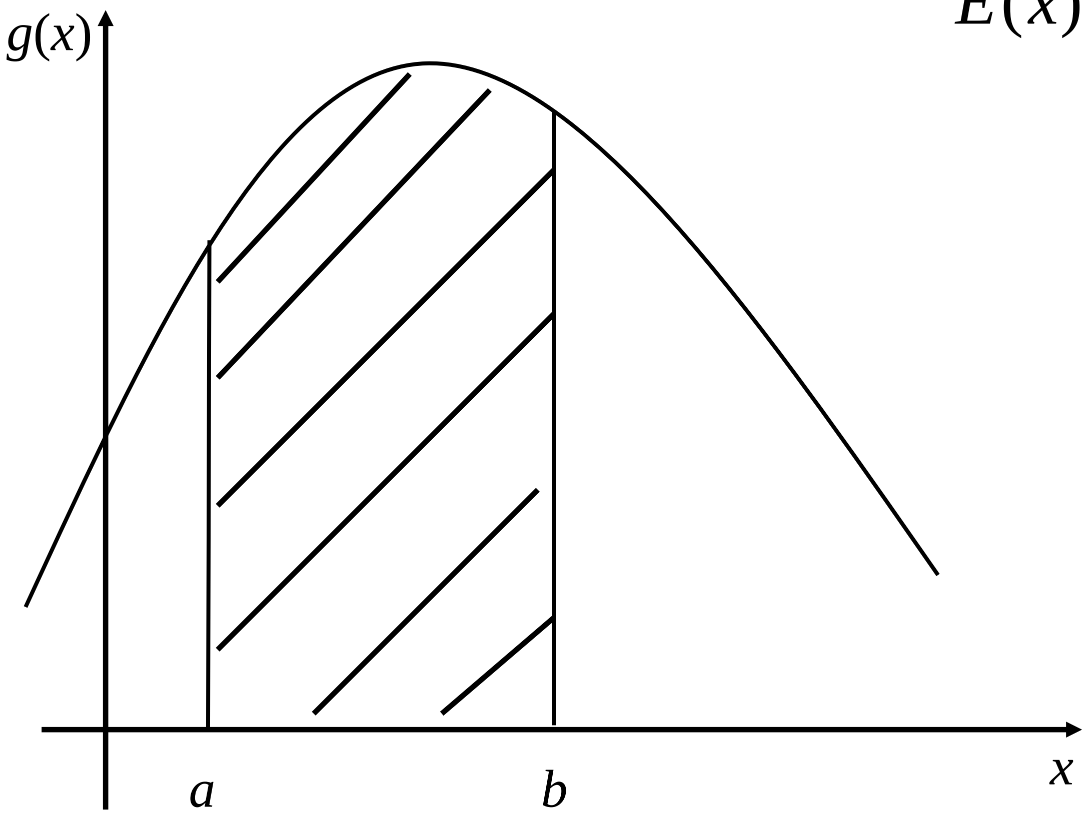


$n=5$

Deterministic Integration

goal

$$E(x) = \int_x g(x)dx$$



$$E(x) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} g(x_i^*)\Delta x$$

$$= \int_{x=a}^{b} g(x)dx$$

think of as

$$g(x) = xf(x)$$

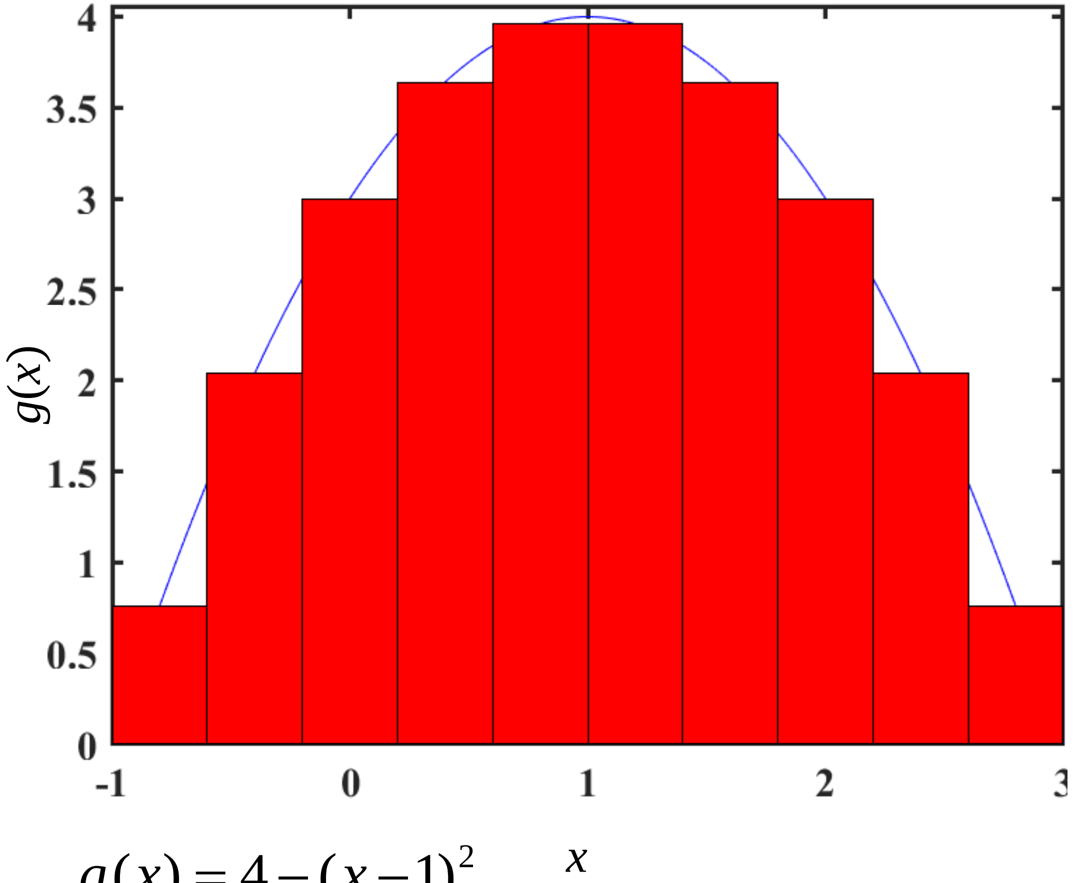
$$\Delta x = (b-a)/n$$

$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

Deterministic Integration

goal

$$E(x)=\int_x g(x)dx$$



$g(x)=4-(x-1)^2$

← numerical

n=10, Δx=0.400

$$\hat{E}(x)=\Delta x\sum_{i=1}^{10} g(x_i^*)=10.7200$$

analytic

$$\int_{x=-1}^{3}[4-(x-1)^2]dx=10.6667$$

```
% Numerical Integral
a=-1; b=3;
n=10;   dx=(b-a)/n;
tpts=(a+dx/2:dx:b)';
gpts=4-(xpts-1).^2;
Exhat=dt*sum(gpts)
```

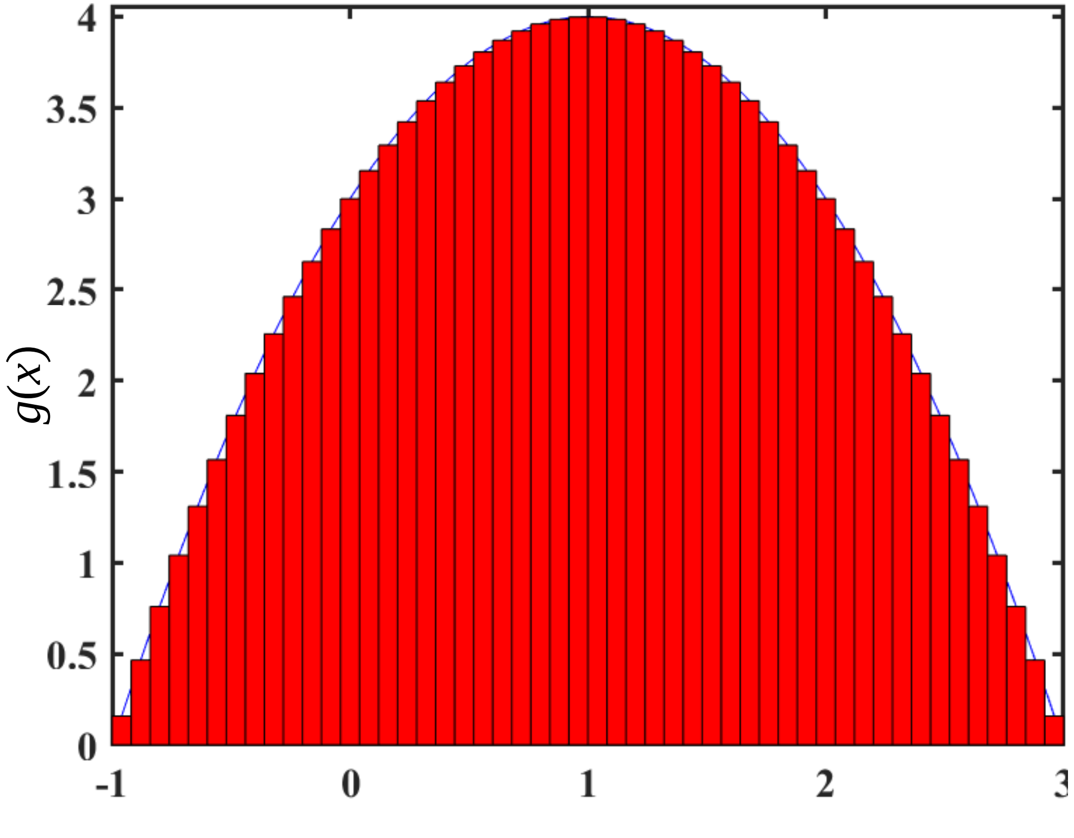
think of as

$$g(x)=xf(x)$$

Deterministic Integration

goal

$$E(x)=\int_x g(x)dx$$



$g(x)=4-(x-1)^2$

← numerical

n=50, Δx=0.080

$$\hat{E}(x)=\Delta x\sum_{i=1}^{50} g(x_i^*)=10.6688$$

analytic

$$\int_{x=-1}^{3}[4-(x-1)^2]dx=10.6667$$

```
% Numerical Integral
a=-1; b=3;
n=50;   dx=(b-a)/n;
tpts=(a+dx/2:dx:b)';
gpts=4-(xpts-1).^2;
Exhat=dt*sum(gpts)
```

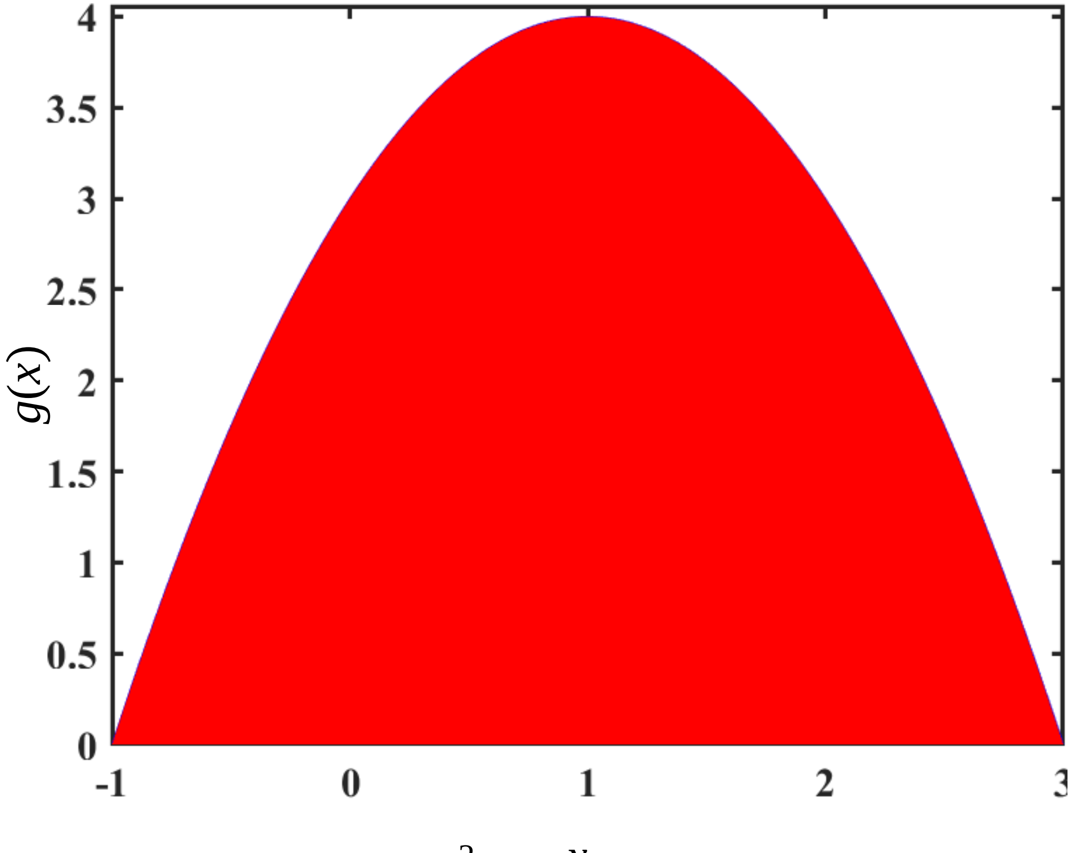
think of as

$$g(x)=xf(x)$$

Deterministic Integration

goal

$$E(x) = \int_x g(x) dx$$



$g(x) = 4-(x-1)^2$

← numerical

n=1000, Δx=0.004

$$\hat{E}(x) = \Delta x \sum_{i=1}^{1000} g(x_i^*) = 10.6667$$

analytic

$$\int_{x=-1}^{3} [4-(x-1)^2] dx = 10.6667$$

```
% Numerical Integral
a=-1; b=3;
n=1000; dx=(b-a)/n;
tpts=(a+dx/2:dx:b)';
gpts=4-(xpts-1).^2;
Exhat=dt*sum(gpts)
```

think of as

$$g(x) = xf(x)$$

Chapter 3: Random Numbers

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Agenda

3.1 Pseudorandom Number Generation

3.2 Using Random numbers to Evaluate Integrals

3.1 Pseudorandom Number Generation - “Random” (0,1) numbers

Multiplicative Congruential Method

(1) Start with seed x_0

(2) Compute $x_n = a x_{n-1}$ modulo m , n=0,1,2,…
where a and m are given positive integers

choose $m = 2^{31}-1$ and $a=7^5$ for 32 bit machines

3.1 Pseudorandom Number Generation - “Random” (0,1) numbers

Mixed Congruential Method

(1) Start with seed x_0

(2) Compute $x_n=(ax_{n\text{-}1}+c)$ modulo m , n=0,1,2,…
where a, c, and m are given positive integers

choose m = to be the computer’s word length

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

Let $g(x)$ be a function and suppose we want

$$\theta = \int_0^1 g(x)dx$$

If U is uniformly distributed over (0,1), then

$$\theta = E(g(U))$$

Think of $f(u) = \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$

and

$$\theta = \int_0^1 g(u) f(u) du$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

If we generate $U_1,\ldots,U_k$ independent uniforms

then $g(U_1),\ldots, g(U_k)$ will be IID with mean $\theta = \int_0^1 g(x)dx$.

$$\theta = \int_0^1 g(u) f(u) du$$

$$\sum_{i=1}^{k} \frac{g(U_i)}{k} \to E[g(U)] = \theta \quad \text{as } k \to \infty$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

If we want $\theta = \int_a^b g(x)dx$, then we can transform x to y as y =(x -a)/(b-a), with dy =dx/(b-a)

$$\theta = \int_a^b g(x)dx \qquad dx = (b-a)dy \qquad \text{limits } x = a \rightarrow y = 0$$

$$x = a + y[b-a] \qquad x = b \rightarrow y = 1$$

$$\theta = \int_0^1 g(a+[b-a]y)(b-a)dy$$

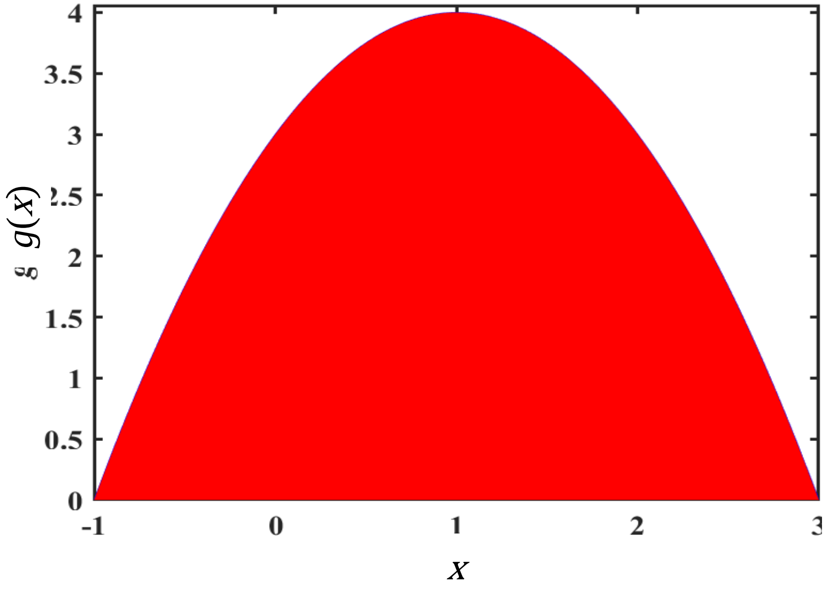
$$\theta = \int_0^1 h(y)dy \qquad h(y) = g(a+[b-a]y)(b-a)$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

Let's use this idea to evaluate the same integral.

$$(b-a)=4$$

Deterministic



$$g(x)=4-(x-1)^2$$

Deterministic

$$E(x\mid\cdot)=\int_x g(x)dx$$

$$\hat{E}_n(x\mid\cdot)=\Delta x\sum_{i=1}^{n} g(x_i^*)$$

$$\hat{E}_{10}(x\mid\cdot)=10.7200$$

Stochastic

$$h(u)=g\big(a+(b-a)u\big)(b-a)$$

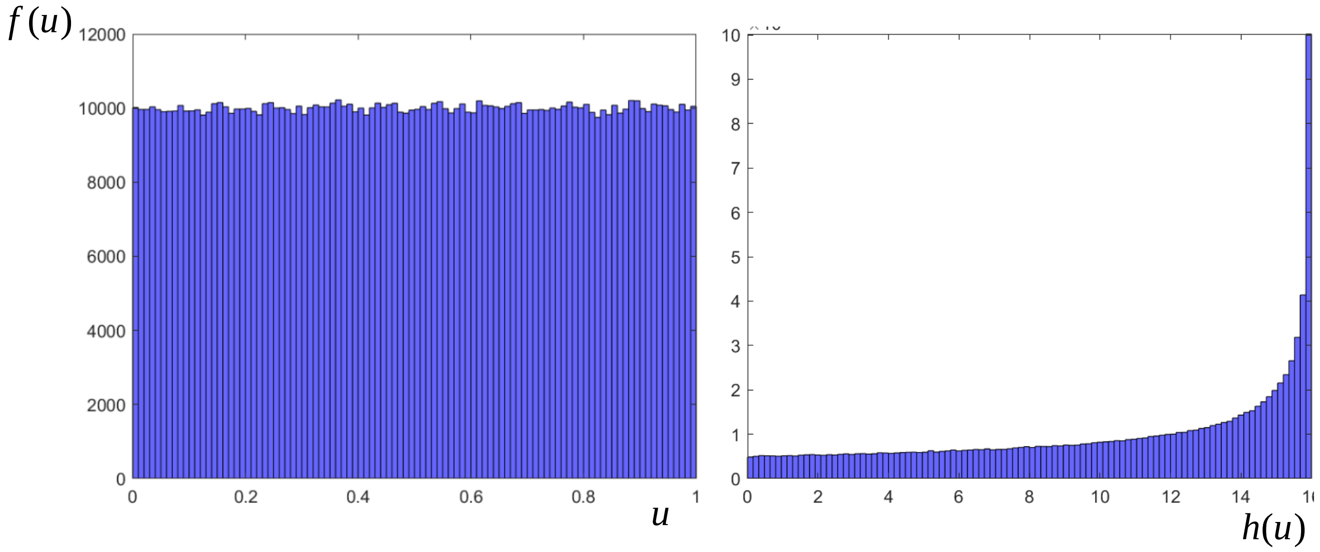
$$\frac{1}{n}\sum_{i=1}^{n} h(u_i)=10.6718$$

```
% stochastic integral
rng('default')
a=-1; b=3; n=10^6;
u=rand(n,1);
t=(a+(b-a)*u);
hu=(4-(t-1).^2)*(b-a);
Exhat=sum(hu)/n
```

Theoretical

$$\int_{x=-1}^{3}[4-(x-1)^2]dx=10.6667$$

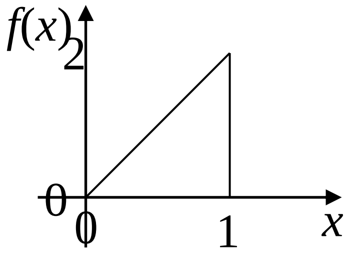
Stochastic



3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

Integrate $\theta = \int_0^1 2x\, dx$ using $U(0,1)$ random numbers.

Analytically, we'd get $\theta = \int_0^1 2x\, dx = 2\frac{x^2}{2}\Big|_0^1 = 1$.



Or we can generate uniform numbers $u_1,\ldots,u_n$ and

Calculate $\theta \approx \frac{1}{n}\sum_{i=1}^{n}(2u_i)$.

Matlab Code:

```
rng('default')
n=10^6;
u=rand(n,1);
theta=sum(2*u)/n
```

Matlab Output:

theta = 1.0006

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

If we want $\theta = \int_0^\infty g(x)dx$, then we can transform x to y as y =1/(x +1), with dy =-dx/(x+1)^2=-y^2dx

$$\theta = \int_0^\infty g(x)dx \qquad \begin{array}{c} dx = -dy / y^2 \\ x = 1/y - 1 \end{array}$$

limits $x = 0 \rightarrow y = 1$

$x = \infty \rightarrow y = 0$

$$\theta = \int_0^1 g(1/y - 1)dy / y^2$$

$$\theta = \int_0^1 h(y)dy \qquad h(y) = g(1/y - 1) / y^2$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

If we want $\theta = \int_{-\infty}^{+\infty} g(x)dx$, then we can transform x to y as $y = e^x / (1+e^x)$, with $dy = dxe^x / (1+e^x)^2 = ydx / (1-y)$

$$\theta = \int_{-\infty}^{+\infty} g(x)dx$$

$$dx = ydy / (1-y)$$

$$x = \ln\left(\frac{y}{1-y}\right)$$

limits $x = -\infty \rightarrow y = 0$

$x = +\infty \rightarrow y = 1$

$$y = e^x / (1+e^x)$$

$$\theta = \int_0^1 g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{dy}{y(1-y)}$$

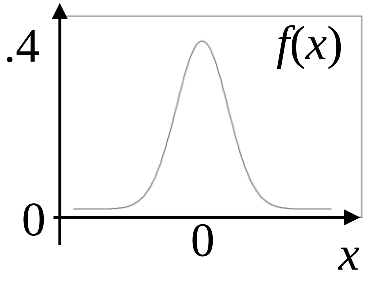
$$\theta = \int_0^1 h(y)dy$$

$$h(y) = g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{1}{y(1-y)}$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

Integrate $\theta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$ using $U(0,1)$ random numbers.

We know that $\theta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$



Or we can generate uniform numbers $u_1,\ldots,u_n$ and

Calculate $\theta \approx \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\ln\left(\frac{u_i}{1-u_i}\right)\right)^2} \frac{1}{u_i(1-u_i)} \approx 1$

$$\theta = \int_0^1 g\left(\ln\left(\frac{y}{1-y}\right)\right)\frac{dy}{y(1-y)}$$

3.2 Using Random numbers to Evaluate Integrals – Monte Carlo Integration

Most useful in evaluating multiple integrals.

$$\theta = \int_0^1 \int_0^1 \ldots \int_0^1 g(x_1,\ldots,x_n)dx_1\ldots dx_n$$

The key is to use $\theta = E[g(U_1,\ldots,U_n)]$, where $U_1,\ldots,U_n$ are independent $U(0,1)$'s.

If we generate $U_1^{(1)},\ldots,U_n^{(1)}$ then $\theta = E[g(U_1,\ldots,U_n)] \approx \sum_{i=1}^{k} \frac{g(U_1^i,\ldots,U_n^i)}{k}$

$$U_1^{(2)},\ldots,U_n^{(2)}$$

$$\vdots$$

$$U_1^{(k)},\ldots,U_n^{(k)}$$

Homework 1

Problem A: $f(x) = \exp(\exp(x)), \quad x \in [0,1]$

Integrate numerically with Matlab from a=0 to b=1.

Do by pencil and paper with n=4 intervals.

$\Delta x = 0.25$ $\left(x_1^*, x_2^*, x_3^*, x_4^*\right) = \left(0.125, 0.375, 0.625, 0.875\right)$

Write a Matlab program to repeat with n=4 intervals.

Change to n=100. Compare results.

Homework 1

Chapter 3: # 1*, 3#, 7, 9, 11.

*Repeat generating 10^4 of these.

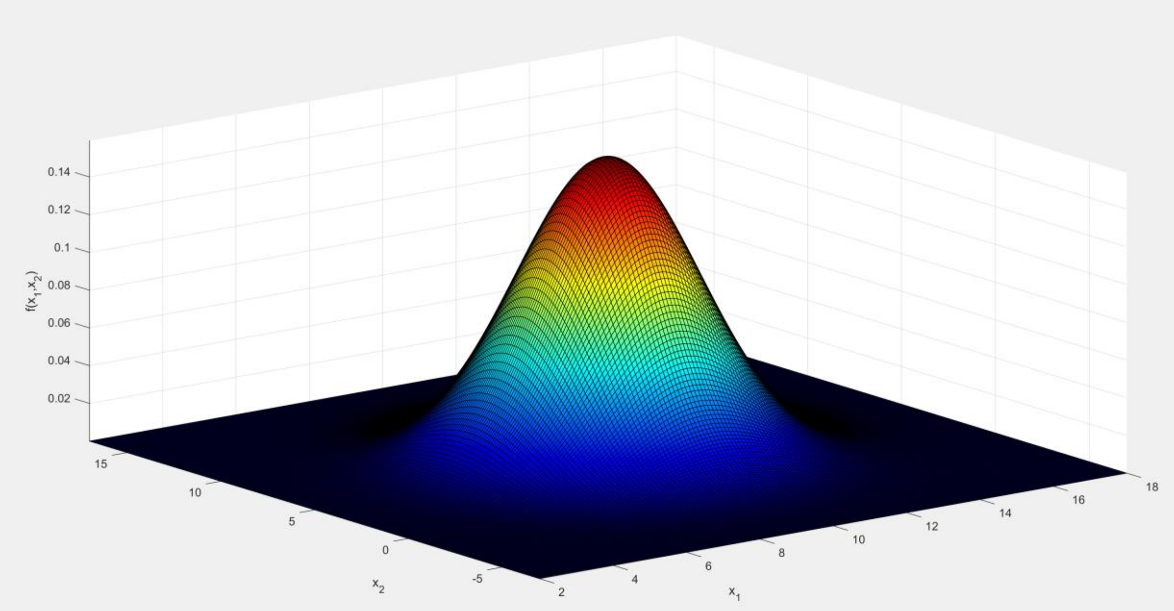
Compute mean, variance, and make a histogram.

Repeat using Matlab's rand() command.

Compare results from to Matlab's rand().

#Compare results to numerical integration in Problem A with n=100.

Homework 1



Problem B:

Evaluate the double integral

$\iint x_1 f(x_1,x_2) dx_1 dx_2$ where $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$

for PDF $f(x_1,x_2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2}$.

$\mu_1 = 10$

$\mu_2 = 5$

$\sigma_1^2 = 4$

$\sigma_2^2 = 9$

a. Use numerical integration by breaking up (x_1,x_2) into rectangles. Use $\Delta x_1 \Delta x_2 \sum_{x_2} \sum_{x_1} x_1 f(x_1,x_2)$.

b. Use stochastic integration $\sum_{i=1}^{k} \frac{g(U_1^i,U_2^i)}{k}$ after transforming (x_1,x_2) to (u_1,u_2).

c. Pencil and paper integration?