# Multivariate Distributions

Daniel B. Rowe, Ph.D.

Professor
Department of Mathematical and Statistical Sciences

# Outline

Recall Univariate:
Uniforms to Normals
Normals to Chi-Square
Normal and Chi-Square to t

- Bi(Multi)variate Normal Distribution
- Wishart Distribution (symmetric matrix)
- Bi(Multi)variate Student t
- Matrix Normal and Matrix T

# Bivariate Normal Distribution

A bivariate (2D) PDF $f(x_1, x_2 \mid \theta)$

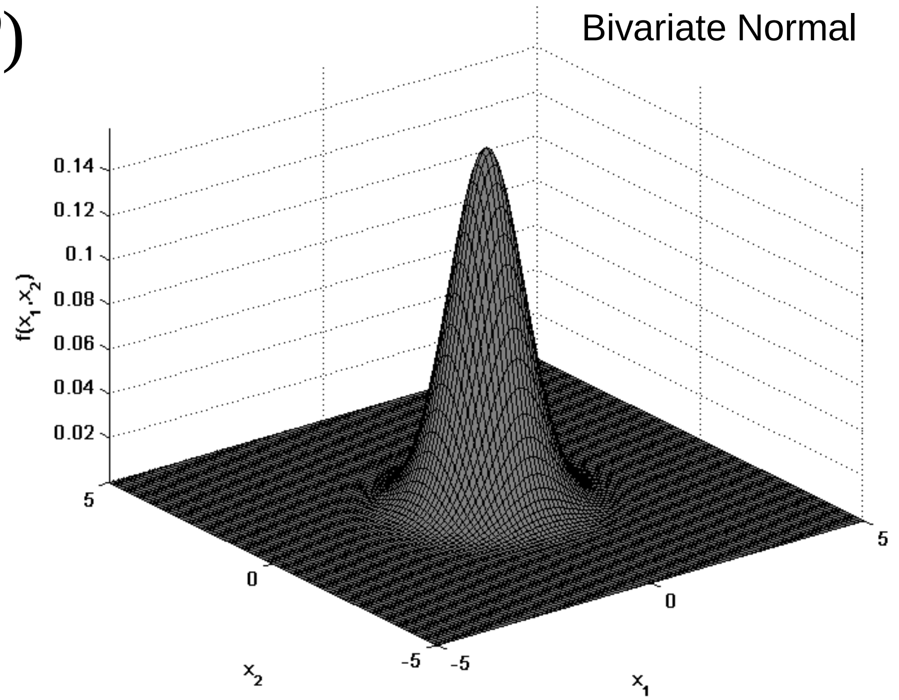
of two continuous random

variables $(x_1, x_2)$ depending

upon parameters $\theta$ satisfies

1) $0 \leq f(x_1, x_2 \mid \theta), \quad \forall (x_1, x_2)$

2) $\iint_{x_1 x_2} f(x_1, x_2 \mid \theta) dx_1 dx_2 = 1$ .



# Bivariate Normal Distribution

Let $\underset{2\times 1}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a 2-dimensional (or $p$-dimensional)

random variable with PDF of $\underset{2\times 1}{x}$ being $f(x|\theta)$, then

Marginal means.

$$E(x|\theta) = \begin{pmatrix} E(x_1|\theta) \\ E(x_2|\theta) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$
$$= \underset{2\times 1}{\mu}$$

Marginal variances.

$$\mathrm{var}(x|\theta) = \begin{pmatrix} \mathrm{var}(x_1|\theta) & \mathrm{cov}(x_1,x_2|\theta) \\ \mathrm{cov}(x_1,x_2|\theta) & \mathrm{var}(x_2|\theta) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$
$$= \underset{2\times 2}{\Sigma}$$

Vectors are lower case, matrices are upper case.

# Bivariate Normal Distribution

Note the way I have written this.

**Recall:** If $\underset{1\times 1}{z} \sim N(0,1)$ , then

mean mean

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}=\left(2\pi\right)^{-1/2}(1)^{-1/2}e^{-\frac{1}{2}(z-0)(1)^{-1}(z-0)}$$

variance variance

We can obtain a random variable $\underset{1\times 1}{x}$ that has a general normal distribution with mean $\underset{1\times 1}{\mu}$ and variance $\underset{1\times 1}{\sigma^2}$ via the transformation $\underset{1\times 1}{x}=\underset{1\times 1}{\sigma}\,\underset{1\times 1}{z}+\underset{1\times 1}{\mu}$.

# Bivariate Normal Distribution

The PDF of $x$ can be obtained by

$$f(x|\mu,\sigma^2)=f(z(x))\times|J(z\to x)| \qquad J(z\to x)=\frac{dz(x)}{dx}$$

where $\underset{1\times 1}{z}=z(x)$ and $J(\cdot)$ is the Jacobian of the transformation.

The PDF of $\underset{1\times 1}{x}$ is

$$f(x|\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

which can be written as

Note the way I have written this.

$$f(x|\mu,\sigma^2)=(2\pi)^{-1/2}\left(\sigma^2\right)^{-1/2}e^{-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)}$$

# Bivariate Normal Distribution

Given two continuous random variables $(z_1, z_2)$, we write

them as a 2-dimensional vector $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, and this vector

has the PDF $f_{\underset{\text{vector}}{Z}}(\overset{\text{vector}}{z} \,|\, \theta)$.

If $z_1$ and $z_2$ are independent, then

$$f_Z(z \,|\, \theta) = f_{Z_1}(z_1 \,|\, \theta_1) f_{Z_2}(z_2 \,|\, \theta_2) \quad .$$

# Bivariate Normal Distribution

Let $\underset{1\times 1}{z_1}$ and $\underset{1\times 1}{z_2}$ be iid $N(0,1)$ random variables. Then, $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

has PDF
$$
\begin{aligned}
f_{Z_1,Z_2}(z_1,z_2) &= f_{Z_1}(z_1)\,f_{Z_2}(z_2) \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}(z_1^2+z_2^2)} \quad .
\end{aligned}
$$

With vector $\underset{2\times 1}{z}$, this can be rewritten as

$$
f_Z(z) = \frac{1}{2\pi}e^{-\frac{1}{2}z'z} \quad .
$$

# Bivariate Normal Distribution

This can also be written as

$$\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$f_Z(z) = \frac{1}{2\pi} e^{-\frac{1}{2}z'z}$$

$$= (2\pi)^{-2/2} \, \underbrace{|I_p|}_{\text{covariance matrix}}^{-1/2} \, e^{-\frac{1}{2}(\overbrace{z-0}^{\text{mean vector}})'\underbrace{(I_p)^{-1}}_{\text{covariance matrix}}(\overbrace{z-0}^{\text{mean vector}})}$$

and we write that $\underset{2\times 1}{z} \sim N(\underbrace{0}_{\text{mean vector}}, \underbrace{I_2}_{\text{covariance matrix}})$ .

$I_2$ is 2×2 Identity matrix

That is, the 2-dimensional random vector $\underset{2\times 1}{z}$ has a

mean vector of zero and identity variance-covariance matrix.

# Bivariate Normal Distribution

This means that

$$\underset{2\times1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$f_Z(z) = (2\pi)^{-2/2} \, | I_p |^{-1/2} \, e^{-\frac{1}{2}(z-0)'(I_p)^{-1}(z-0)}$$

$$E(z) = \begin{pmatrix} E(z_1) \\ E(z_2) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \underset{2\times1}{\mu} = 0$$

2×2 Identity matrix

$$\mathrm{var}(z) = \begin{pmatrix} \mathrm{var}(z_1) & \mathrm{cov}(z_1, z_2) \\ \mathrm{cov}(z_1, z_2) & \mathrm{var}(z_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \underset{2\times2}{\Sigma} = I_2$$

# Bivariate Normal Distribution

If $\underset{2\times 1}{z} \sim N(0, I_2)$ , then

$$f(z) = (2\pi)^{-2/2} \, | I_p |^{-1/2} \, e^{-\frac{1}{2}(z-0)'(I_p)^{-1}(z-0)}$$

We can obtain a random variable $\underset{2\times 1}{x}$ that has a general normal distribution with mean vector $\underset{2\times 1}{\mu}$ and variance-covariance matrix $\underset{2\times 2}{\Sigma}$ via the transformation $\underset{2\times 1}{x} = \underset{2\times 2}{A} \; \underset{2\times 1}{z} + \underset{2\times 1}{\mu}$

where $\underset{2\times 2}{\Sigma} = \underset{2\times 2}{A} \underset{2\times 2}{A'}$, is a factorization (i.e. Cholesky or Eigen).

# Bivariate Normal Distribution

If a random variable $\underset{2\times 1}{x}$ has a normal distribution with

mean vector $\underset{2\times 1}{\mu}$ and variance-covariance matrix $\underset{2\times 2}{\Sigma}$, then

mean vector mean vector

$$f(x\,|\,\mu,\Sigma) = (2\pi)^{-p/2}\,|\Sigma|^{-1/2}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

covariance matrix

$x,\mu \in \mathbb{R}^p$
$p = 2$
$\Sigma > 0$

set of pos def matrices

mean vector covariance matrix
covariance matrix

and we write $\underset{2\times 1}{x} \sim N(\underset{2\times 1}{\mu}, \underset{2\times 2}{\Sigma})$ . The covariance matrix $\underset{2\times 2}{\Sigma}$, has to

be of full rank (there is an inverse in PDF).

make sure you know what this means

# Bivariate Normal Distribution

Let's take a closer look at this bivariate transformation.

$$\underset{2\times1}{x} = \underset{2\times2}{A}\ \underset{2\times1}{z} + \underset{2\times1}{\mu}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$x_1 = a_{11}z_1 + a_{12}z_2 + \mu_1$$

$$x_2 = a_{21}z_1 + a_{22}z_2 + \mu_2$$

We can solve for $\underset{1\times1}{z_1}$ and $\underset{1\times1}{z_2}$ in terms of $\underset{1\times1}{x_1}$ and $\underset{1\times1}{x_2}$.

# Bivariate Normal Distribution

This will give us

$$\underset{2\times1}{x} = \underset{2\times2}{A}\ \underset{2\times1}{z} + \underset{2\times1}{\mu}$$

$$\underset{2\times1}{z} = \underset{2\times2}{A^{-1}}\left(x - \underset{2\times1}{\mu}\right)$$

$A$ invertible

$$\longrightarrow$$

$$\underset{2\times2}{A^{-1}} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\underset{2\times2}{B} = \underset{2\times2}{A^{-1}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\underset{2\times1}{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \underset{2\times2}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}\ \underset{2\times1}{\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}}$$

$$z_1 = b_{11}(x_1 - \mu_1) + b_{12}(x_2 - \mu_2)$$

$$z_2 = b_{21}(x_1 - \mu_1) + b_{22}(x_2 - \mu_2)$$

# Bivariate Normal Distribution

$$z_1 = b_{11}(x_1 - \mu_1) + b_{12}(x_2 - \mu_2)$$
$$z_2 = b_{11}(x_1 - \mu_1) + b_{12}(x_2 - \mu_2)$$

Continuing on, this leads to

$$J(z_1, z_2 \to x_1, x_2) = \begin{vmatrix} \dfrac{\partial z_1(x_1, x_2)}{\partial x_1} & \dfrac{\partial z_1(x_1, x_2)}{\partial x_2} \\ \dfrac{\partial z_2(x_1, x_2)}{\partial x_1} & \dfrac{\partial z_2(x_1, x_2)}{\partial x_2} \end{vmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \underset{2\times 2}{B}$$

i.e. with $\underset{2\times 1}{z} = B\left(x - \mu\right)$ , the vector derivative is $J = \dfrac{\partial z}{\partial x} = \underset{2\times 2}{B}$

# Bivariate Normal Distribution

$$z = B\left(x - \mu\right)$$

The distribution of vector variable $\underset{2\times 1}{x}$ (joint of $x_1$ and $x_2$) is

$$f_X(x\,|\,\theta) = f_Z(z(x)) \times |\,J(z \to x)\,|$$

$$f(z) = (2\pi)^{-2/2}\,|\,I_n\,|^{-1/2}\,e^{-\frac{1}{2}(z-0)'(I_n)^{-1}(z-0)}$$

$$J = \frac{\partial z}{\partial x} = \underset{2\times 2}{B}$$

$$f_X(x\,|\,\mu,\Sigma) = (2\pi)^{-2/2}\,|\,I_n\,|^{-1/2}\,e^{-\frac{1}{2}\left(B\left(x-\mu\right)-0\right)'(I_n)^{-1}\left(B\left(x-\mu\right)-0\right)}\left|B\right|$$

$$\Sigma = AA'\ ,\ |\,\Sigma\,| = |\,A\,||\,A'\,| = |\,A\,|^2,\ |\,\Sigma\,|^{1/2} = |\,A\,|\ ,\ B = A^{-1},\ |\,\Sigma\,|^{-1/2} = |\,B\,|$$

$$f_X(x\,|\,\mu,\Sigma) = (2\pi)^{-2/2}\,|\,\Sigma\,|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

$$x,\mu \in \mathbb{R}^p$$

$$\Sigma > 0$$

# Bivariate Normal Distribution

This form may be more familiar

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q}$$

$$Q = \frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$\sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1$$

$$\rho = \sigma_{12} / (\sigma_1\sigma_2)$$

$$\sigma_{12} = \operatorname{cov}(x_1, x_2)$$

# Bivariate Normal Distribution

**Theorem:**

If $x$ is a 2-D (or $p$-D) random variable from $f(x|\mu,\Sigma)$, with

$$E(x \mid \mu,\Sigma) = \underset{p\times 1}{\mu}$$ think of $p$=2

$$\mathrm{var}(x \mid \mu,\Sigma) = \underset{p\times p}{\Sigma}$$

then we form $\underset{r\times 1}{y} = \underset{r\times p}{A}\,\underset{p\times 1}{x} + \underset{r\times 1}{\delta}$ where dimensions match

and $\underset{r\times p}{A}$ full column rank ($A$: $r$×$p$, $r$≤$p$), then

$$E(y \mid \mu,\Sigma,\delta,A) = \underset{r\times p}{A}\,\underset{p\times 1}{\mu} + \underset{r\times 1}{\delta}$$

$$\mathrm{var}(y \mid \mu,\Sigma,A) = \underset{r\times p}{A}\,\underset{p\times p}{\Sigma}\,\underset{r\times p}{A'}\ .$$

# Bivariate Normal Distribution

**Recall:** Let $u_1$~uniform(0,1) and $u_2$~uniform(0,1).

Let $z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$ and $z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$

then $u_1(z_{1,}z_2) = e^{-\frac{1}{2}(z_1^2+z_2^2)}$ and $u_2(z_1,z_2) = \frac{1}{2\pi}\text{atan}\left(\frac{z_2}{z_1}\right)$.

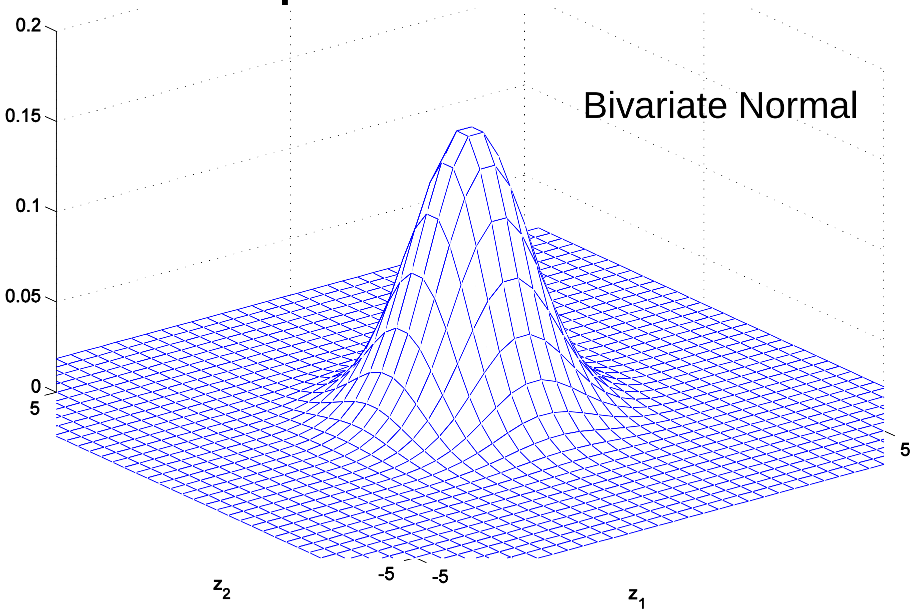
$$f_{Z_1,Z_2}(z_1,z_2|\theta) = f_{U_1,U_2}\left(u_1(z_1,z_2),u_2(z_1,z_2)|\theta\right)\times|J(u_1,u_2\to z_1,z_2)|$$

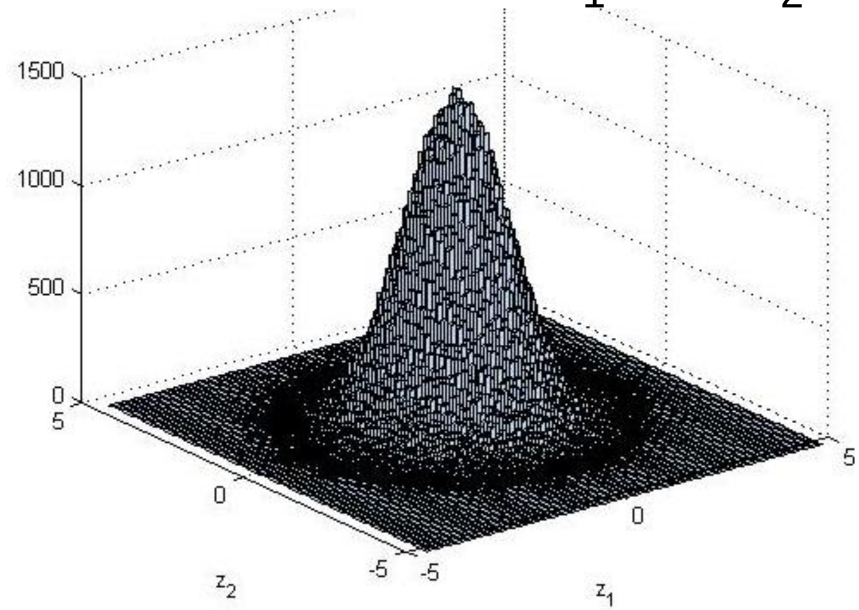
$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_2^2}$$

This means $z_1 \sim N(0,1)$, $z_2 \sim N(0,1)$, $z_1$ and $z_2$ are independent.

# Bivariate Normal Distribution

Obtain 2-D standard normal variates $\underset{2\times 1}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by transforming two independent standard uniform random variates $u_1$ and $u_2$.




First half of $10^6$ standard normal variates as $z_1$'s and second half as $z_2$'s. Produce $5\times10^5$ statistically independent $z$'s.

# Bivariate Normal Distribution

$$\underset{2\times1}{\mu}=\begin{pmatrix}3\\5\end{pmatrix}\quad \underset{2\times2}{\Sigma}=\begin{pmatrix}3 & 2\\2 & 4\end{pmatrix}$$

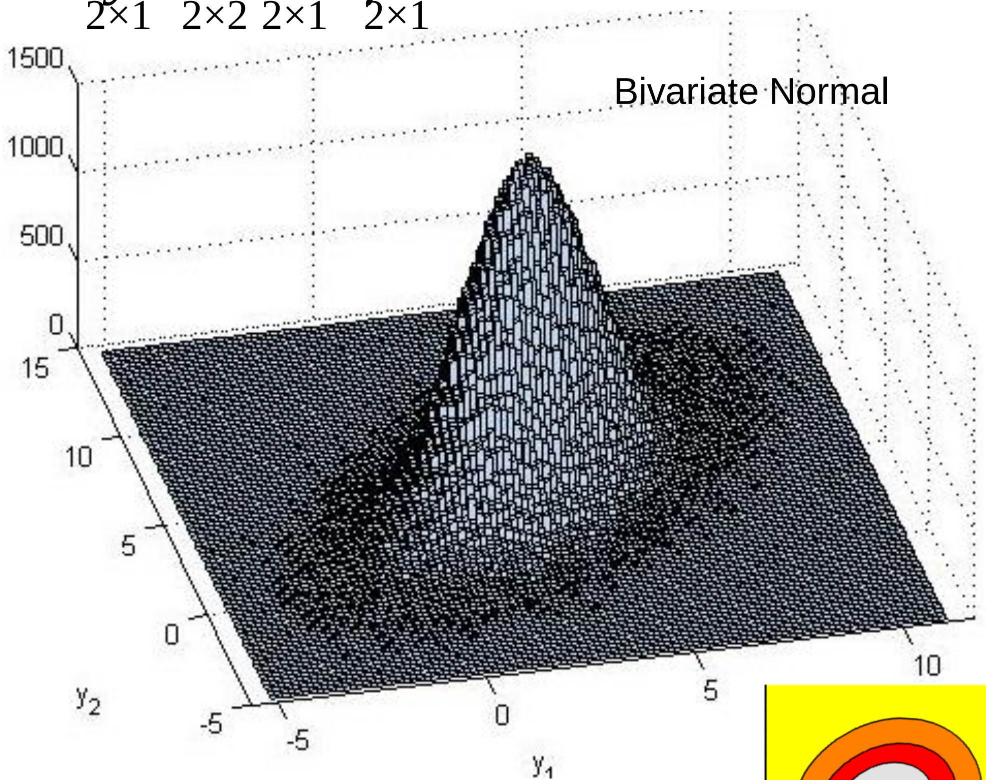
$$\rho=0.58$$

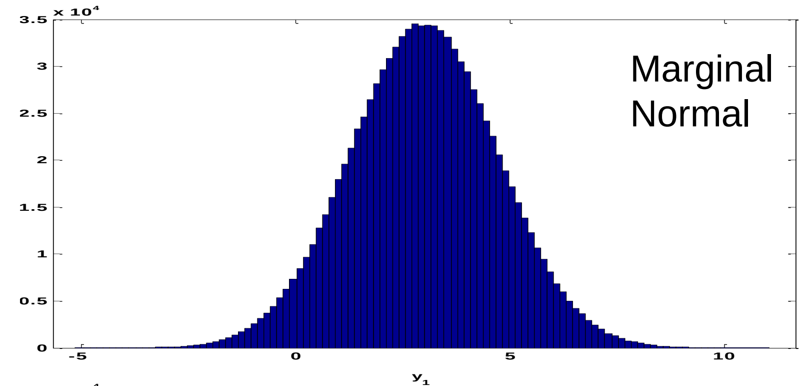
Multiplied $5\times10^5$ simulated $z$'s by $\underset{2\times2}{A}$ and added $\underset{2\times1}{\mu}$.
The $y$'s are now $N(\mu,\Sigma=AA')$.

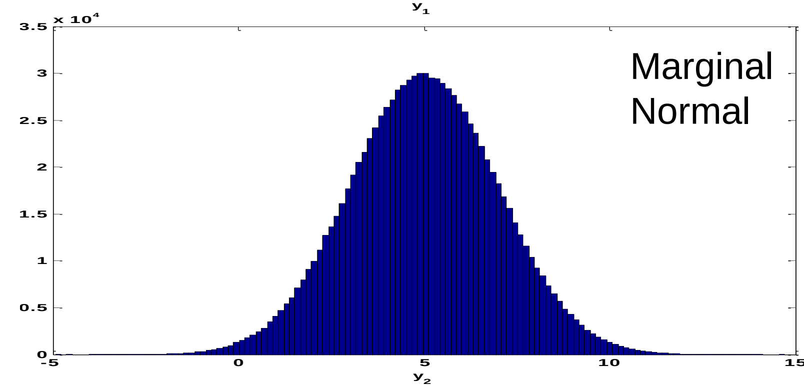
$$\underset{2\times1}{y}=\underset{2\times2}{A}\,\underset{2\times1}{z}+\underset{2\times1}{\mu}$$

Cholesky

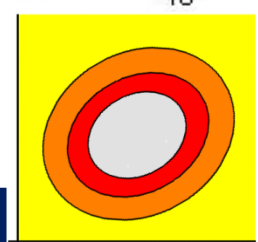
$$\underset{2\times2}{A}=\begin{pmatrix}1.7321 & 0\\1.1547 & 1.6330\end{pmatrix}$$




cross section



$A$=chol(Sigma)’

# Bivariate Normal Distribution

**Theorem:**

If $x_1$ and $x_2$ are independent 2-D (or $p$-D) RVs with

$$\underset{p\times 1}{E(x_1}\,|\,\mu_1,\Sigma_1)=\underset{p\times 1}{\mu_1} \qquad \underset{p\times 1}{E(x_2}\,|\,\mu_2,\Sigma_2)=\underset{p\times 1}{\mu_2}$$

think of $p$=2

$$\underset{p\times p}{\mathrm{var}(x_1}\,|\,\mu_1,\Sigma_1)=\underset{p\times p}{\Sigma_1} \qquad \underset{p\times p}{\mathrm{var}(x_2}\,|\,\mu_2,\Sigma_2)=\underset{p\times p}{\Sigma_2}$$

then if we let $\underset{p\times 1}{y}=(x_1\underset{p\times 1}{+}x_2)/2$,

$$\underset{p\times 1}{E(y}\,|\,\mu_1,\Sigma_1,\mu_2,\Sigma_2)=(\mu_1\underset{p\times 1}{+}\mu_2)/2$$

$$\underset{p\times p}{\mathrm{var}(y}\,|\,\mu_1,\Sigma_1,\mu_2,\Sigma_2)=(\Sigma_1\underset{p\times p}{+}\Sigma_2)/4$$

Recall in 1D if we let
$\mu=\mu_1=\mu_2$ and
$\sigma^2=\sigma_1^2=\sigma_2^2$ then
$E(y\,|\,\mu)=\mu$ and
$\mathrm{var}(y\,|\,\mu,\sigma^2)=\sigma^2/2$ .

# Bivariate Normal Distribution

**Theorem:**

If $x_1,\ldots,x_n$ are independent 2-D (or $p$-D) RVs with

$$E(x_i \mid \mu,\Sigma) = \underset{p\times 1}{\mu} \qquad i=1,\ldots,n$$

think of $p$=2

$$\text{var}(x_i \mid \mu,\Sigma) = \underset{p\times p}{\Sigma}$$

then if we let $\underset{p\times 1}{\bar{x}} = (x_1+\cdots+x_n)/n$ ,

$$E(\bar{x} \mid \mu,\Sigma) = \underset{p\times 1}{\mu}$$

$$\text{var}(\bar{x} \mid \mu,\Sigma) = \underset{p\times p}{\Sigma}/n \quad .$$

If $x$'s are normal, then

$$\underset{p\times 1}{\bar{x}} \sim N(\underset{p\times 1}{\mu},\underset{p\times p}{\Sigma}/n)$$

Otherwise CLT type result?

# Bivariate Normal Distribution

If $\underset{p\times 1}{x_1},\ldots,\underset{p\times 1}{x_n}$ are IID $N(\mu,\Sigma)$ 2-D (or $p$-D) RVs and

$$\underset{p\times 1}{\bar{x}} = (x_1 + \cdots + x_n)/n \sim N(\underset{p\times 1}{\mu},\underset{p\times p}{\Sigma}/n)$$

then

$$f_{\bar{X}}(\bar{x}\,|\,\mu,\Sigma) = (2\pi)^{-2/2}\,|\Sigma/n|^{-1/2}\,e^{-\frac{1}{2}(\bar{x}-\mu)'(\Sigma/n)^{-1}(\bar{x}-\mu)}\,.$$

Also note that

Multivariate version of

$$\frac{(\bar{x}-\mu)}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$n^{1/2}\Sigma^{-1/2}(\bar{x}-\mu) \sim N(0,I_2)\,.$$

$$\underset{p\times p}{\Sigma}^{-1/2} = \underset{p\times p}{A}^{-1} = \underset{p\times p}{B}$$

Estimate $\Sigma$ by its MLE $\underset{p\times p}{\hat{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})\,.$

# Wishart Distribution

Multivariate version of gamma distribution.

A random $p \times p$ matrix variate $\underset{p\times p}{G}$ follows the Wishart

distribution with scale matrix $\underset{p\times p}{\Sigma}$ and $\nu$ *df* denoted $\underset{p\times p}{G} \sim W(\underset{p\times p}{\Sigma}, \nu)$

$G, \Sigma > 0$

iff $f(G \mid \Sigma, \nu) = k_W \mid \Sigma \mid^{-\frac{\nu}{2}} \mid G \mid^{\frac{\nu - p - 1}{2}} e^{-\frac{1}{2} tr(\Sigma^{-1} G)}$

$\nu > p + 1$

$\nu \in \mathbb{N}$

where $k_W^{-1} = 2^{\frac{\nu p}{2}} \pi^{-\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(\frac{\nu + 1 - j}{2}\right)$

If $p$=1, $f(g \mid \sigma^2, \nu) = \dfrac{g^{\frac{\nu}{2}-1} e^{-\frac{g}{2\sigma^2}}}{\Gamma(\nu/2)(2\sigma^2)^{\nu/2}}$

Gamma dist by $\alpha = \nu / 2$ , Chi-square by $y = g / \sigma^2$ .

# Wishart Distribution

$$f(G|\Sigma,\nu) = k_W \, |\Sigma|^{-\frac{\nu}{2}} |G|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\Sigma^{-1}G)}$$

$$k_W^{-1} = 2^{\frac{\nu p}{2}} \pi^{-\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(\frac{\nu+1-j}{2}\right)$$

The mean, variance, and covariance of elements are

$$E(G|\Sigma,\nu) = \nu \underset{p\times p}{\Sigma}$$

$$\mathrm{var}(G_{ij}|\Sigma,\nu) = \nu(\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj})$$

$$\mathrm{cov}(G_{ij}G_{kl}|\Sigma,\nu) = \nu(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk})$$

# Wishart Distribution

We obtain a (singular) $W(\Sigma,1)$ variate $G_1$ by transforming a centered normal vector variate $z_1=(x_1-\mu)$ via $G_1=z_1 z_1'$ and a $W(\Sigma,n)$ variate by $G=\sum_{i=1}^{n} z_i z_i'$ . If the mean vector is not known, then we can estimate it and lose one *df*.

$$\underbrace{\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)}_{W(\Sigma,n),n\geq p} = \underbrace{\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})}_{W(\Sigma,n-1),n-1>p} + \underbrace{n(\bar{x}-\mu)'(\bar{x}-\mu)}_{W(\Sigma,1),\ \text{singular}}$$

$G$ $G_2$ $G_1$

add and subtract $\bar{x}$ in parentheses

Let $p$=1 and get $\sigma^2$ times $\chi^2$ result.

# Wishart Distribution

**Theorem:**

If $G$ is a $p \times p$ random matrix variable from $f(G|\Sigma,\nu)$, with

$$f(G|\Sigma,\nu) = k_W |\Sigma|^{-\frac{\nu}{2}} |G|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\Sigma^{-1}G)}$$

then if we form $Q = AGA'$ where dimensions match

and $A$ full row rank ($A$: $r \times p$, $r \leq p$), then $Q \sim W(\Delta = A\Sigma A', \nu)$

$$E(Q|\Delta,\nu) = \nu\Delta$$

$$\mathrm{var}(Q_{ij}|\Delta,\nu) = \nu(\Delta_{ij}^2 + \Delta_{ii}\Delta_{jj})$$

$$\mathrm{cov}(Q_{ij}Q_{kl}|\Delta,\nu) = \nu(\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk}) \quad .$$

# Wishart Distribution

$$\underset{2\times 1}{\mu} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \underset{2\times 2}{\Sigma} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

Took 5×10$^4$ sets of $n$=10 variates $x$, subtracted mean $\underset{2\times 1}{\mu} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

from each set, transpose multiplied each value, added the

10 values in set to form each $G$. The $G$'s are now $W(\Sigma, \nu = n)$ .

$$E(G \mid \Sigma, \nu) = \nu\Sigma$$

$$\nu = 10 \quad \Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\operatorname{var}(G_{ij} \mid \Sigma, \nu) = \nu(\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj})$$

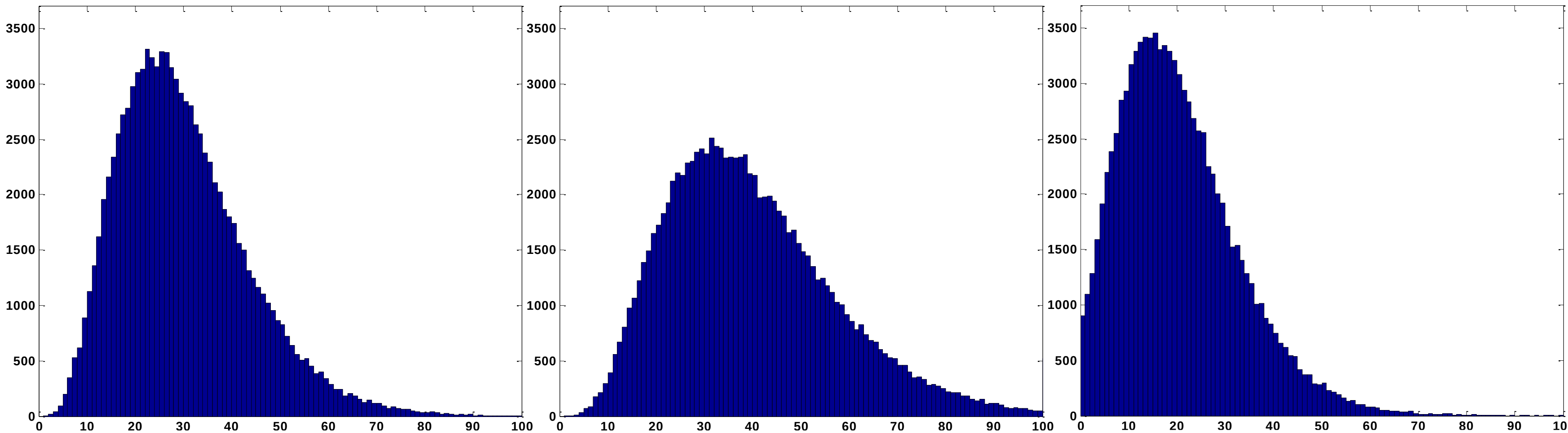
$$G = \sum_{i=1}^{n} (x_i - \mu)'(x_i - \mu)$$

$$\operatorname{cov}(G_{ij} G_{kl} \mid \Sigma, \nu) = \nu(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk})$$

# Wishart Distribution

$$\underset{2\times 1}{\mu} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \underset{2\times 2}{\Sigma} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

The $G$'s, $G = \sum_{i=1}^{n} (x_i - \mu)'(x_i - \mu)$ are now $W(\Sigma, \nu)$ . $\nu = 10$




$G(1,1)$'s $G(2,2)$'s $G(1,2)$'s

$$E(G\,|\,\Sigma,\nu) = \nu\Sigma = \begin{pmatrix} 30 & 20 \\ 20 & 40 \end{pmatrix} \qquad \mathrm{var}(G_{ij}\,|\,\Sigma,\nu) = \nu(\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj}) = \begin{pmatrix} 180 & 160 \\ 160 & 320 \end{pmatrix}$$

$$\mathrm{cov}(G_{ij}G_{kl}\,|\,\Sigma,\nu) = \nu(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) = \underset{11,22}{80}\,,\underset{11,12}{120},\underset{22,12}{160} \leftarrow ij,kl$$

# Wishart Distribution

Took $5\times10^4$ sets of $n$=10 variates $x$, subtracted mean $\bar{x}$

from each set, transpose multiplied each value, added the

10 values to form each $G_2$. The $G_2$'s are now $W(\Sigma, \nu = n-1)$.

$$E(G_2 \mid \Sigma, \nu) = \nu\Sigma \qquad \nu = 9 \quad \Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\operatorname{var}(G_{2ij} \mid \Sigma, \nu) = \nu(\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj})$$

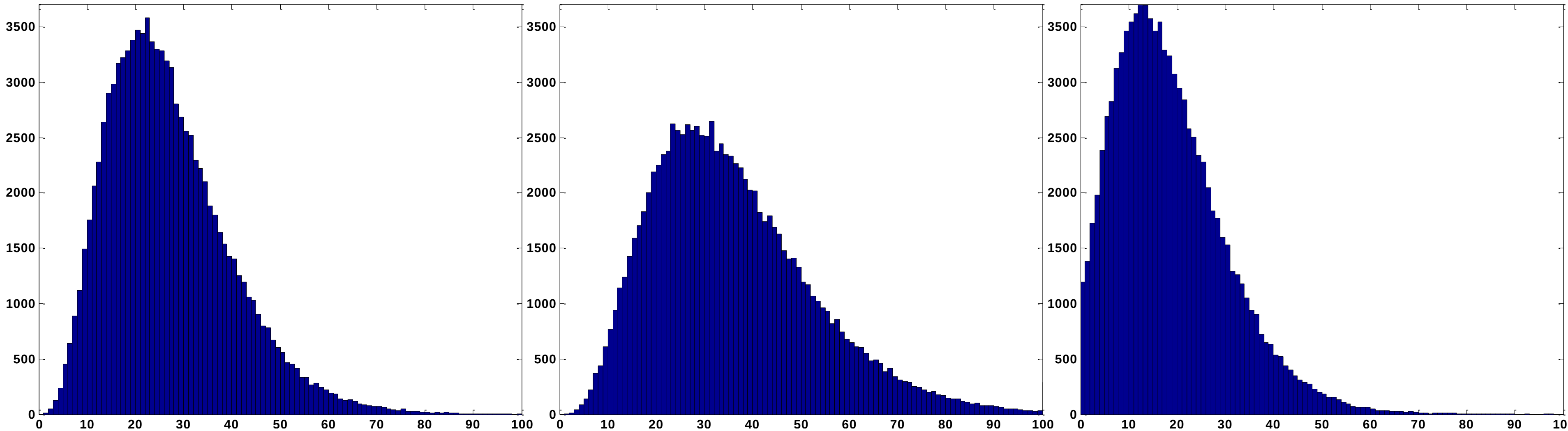
$$\operatorname{cov}(G_{2ij} G_{2kl} \mid \Sigma, \nu) = \nu(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk})$$

# Wishart Distribution

$$\underset{2\times1}{\mu} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \underset{2\times2}{\Sigma} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

Toggle with Next slide

The $G_2$'s, $G_2 = \sum_{i=1}^{n} (x_i - \bar{x})'(x_i - \bar{x})$ are now $W(\Sigma, \nu)$. $\nu = 9$




$G_2\,(1,1)$'s $\quad G_2\,(2,2)$'s $\quad G_2\,(1,2)$'s

$$E(G_2 \mid \Sigma, \nu) = \nu\Sigma = \begin{pmatrix} 27 & 18 \\ 18 & 36 \end{pmatrix} \quad \mathrm{var}(G_{2ij} \mid \Sigma, \nu) = \nu(\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj}) = \begin{pmatrix} 162 & 144 \\ 144 & 288 \end{pmatrix}$$

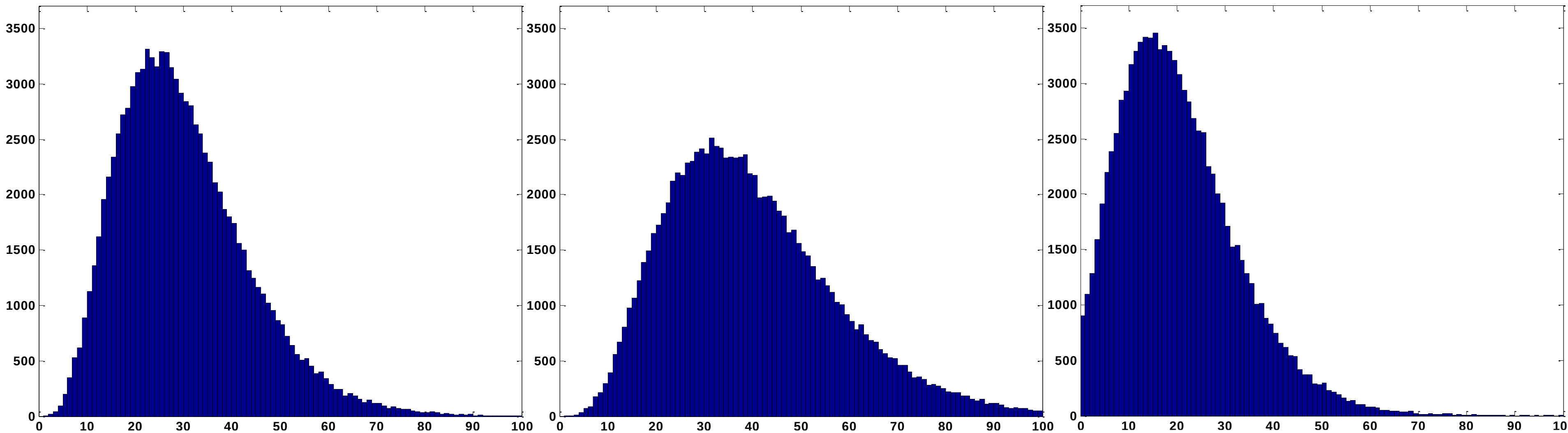
$$\mathrm{cov}(G_{2ij} G_{2kl} \mid \Sigma, \nu) = \nu(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) = \underset{11,22}{72}, \underset{11,12}{108}, \underset{22,12}{144} \leftarrow ij,kl$$

# Wishart Distribution

$$\underset{2\times 1}{\mu}=\begin{pmatrix}3\\5\end{pmatrix}\quad \underset{2\times 2}{\Sigma}=\begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}$$

Toggle with Previous slide

The $G$'s, $G=\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)$ are now $W(\Sigma,\nu)$ . $\nu=10$




$G(1,1)$'s $G(2,2)$'s $G(1,2)$'s

$$E(G\,|\,\Sigma,\nu)=\nu\Sigma=\begin{pmatrix}30 & 20\\ 20 & 40\end{pmatrix}\qquad \mathrm{var}(G_{ij}\,|\,\Sigma,\nu)=\nu(\Sigma_{ij}^2+\Sigma_{ii}\Sigma_{jj})=\begin{pmatrix}180 & 160\\ 160 & 320\end{pmatrix}$$

$$\mathrm{cov}(G_{ij}G_{kl}\,|\,\Sigma,\nu)=\nu(\Sigma_{ik}\Sigma_{jl}+\Sigma_{il}\Sigma_{jk})=\underset{11,22}{80},\underset{11,12}{120},\underset{22,12}{160}\leftarrow ij,kl$$

# Multivariate Student-t Distribution

A 2-D ($p$=2) random vector variate $\underset{p\times 1}{t}$ follows a standard Student $t$ distribution with $\nu$ $df$ denoted $\underset{p\times 1}{t} \sim t(\nu)$ iff

$$f(t\,|\,\nu) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{(\nu\pi)^{\frac{p}{2}}\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\left[1+\frac{1}{\nu}t't\right]^{\frac{\nu+p}{2}}}$$

, where $t \in \mathbb{R}^p$ , $\nu \in \mathbb{N}$ .

The mean and variance of $t$ are

$$E(t\,|\,\nu) = \underset{p\times 1}{0}$$

$$\nu > 1$$

$$\mathrm{var}(t\,|\,\nu) = \frac{\nu}{\nu-2}\underset{p\times p}{I_p}$$

$$\nu > 2$$

# Multivariate Student-t Distribution

We obtain a standard $t(\nu=n\text{-}1)$ variate $t$ by transforming a normal 2-D ($p$-D) random variate $\underset{p\times 1}{\bar{x}}$ and a Wishart random matrix variate $G$ that are independent

$$\begin{pmatrix} t \\ V \end{pmatrix} = \begin{pmatrix} n^{1/2}\nu^{1/2}G_2^{-1/2}(\bar{x}-\mu) \\ G_2 \end{pmatrix} \qquad \underset{p\times 1}{\bar{x}} \sim N(\mu, \Sigma / n) \quad \underset{p\times p}{G_2} \sim W(\Sigma, \nu) \; .$$

The original variables in terms of the new variables

$$\begin{pmatrix} \bar{x}(t,v) \\ G_2(t,V) \end{pmatrix} = \begin{pmatrix} n^{-1/2}\nu^{-1/2}V^{1/2}t + \mu) \\ V \end{pmatrix}$$

# Multivariate Student-t Distribution

The joint distribution of $(t,V)$ can be obtained as

$$f(t,V\,|\,\mu,\Sigma,\nu) = f(\bar{x}(t,V),G_2(t,V))\times|\,J(\bar{x},G_2\to t,V)\,|$$

The Jacobian of the transformation is

$$|\,J(\bar{x},G_2\to t,V)\,| = n^{-1/2}\nu^{-1/2}V^{1/2}$$

The joint PDF of $(t,V)$ is

$$p(t,V\,|\,\Sigma,\nu) = (2\pi\nu)^{-p/2}k_W\,|\,\Sigma\,|^{-\frac{\nu+1}{2}}|\,V\,|^{\frac{(\nu+1)-p-1}{2}}\;e^{-\frac{1}{2}tr\Sigma^{-1}V(I_p+\frac{1}{\nu}tt')}$$

The distribution of $t$ can be found by integrating out $V$

$$f(t\,|\,\nu) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{(\nu\pi)^{\frac{p}{2}}\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\left[1+\frac{1}{\nu}t't\right]}\;.$$

After integration use

$$|\,I_p+\tfrac{1}{\nu}tt'\,| = |\,1+\tfrac{1}{\nu}t't\,|$$

# Multivariate Student-t Distribution

Recall $\bar{x} \sim N(\mu, \Sigma / n)$

so $n^{1/2}\Sigma^{-1/2}(\bar{x}-\mu) \sim N(0, I)$

also $\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x}) \sim W(\Sigma, n-1)$

thus $(n-1)\Sigma^{-1/2}\left[\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})\right](\Sigma^{-1/2})' \sim W(I, n-1)$

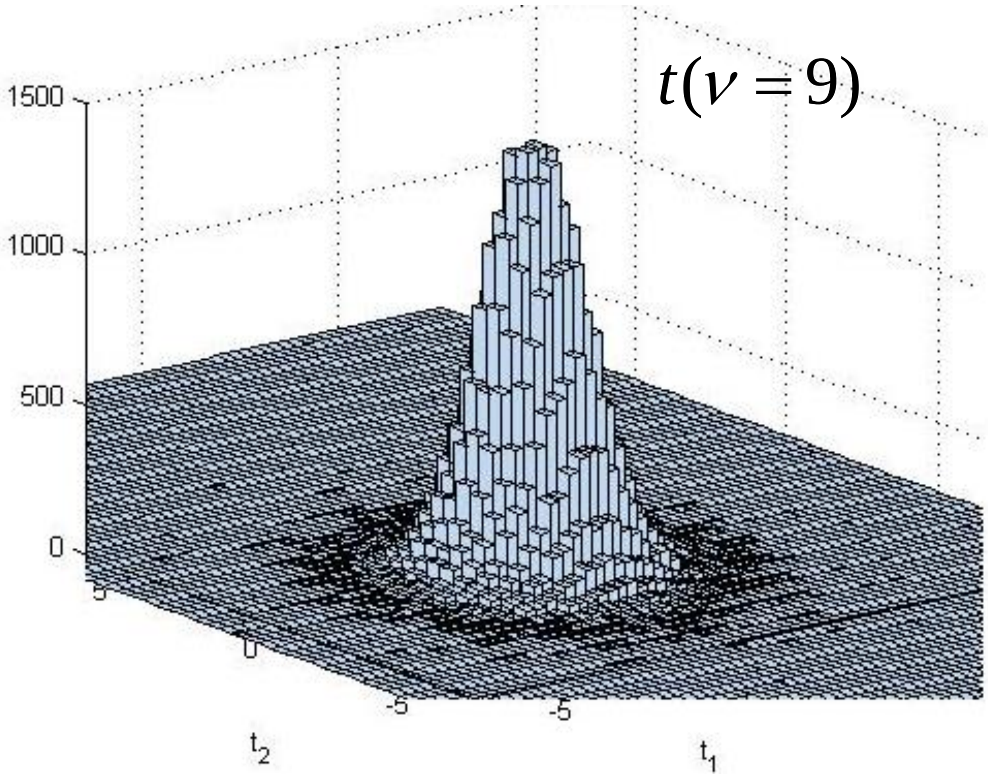
but $\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x}) = S$

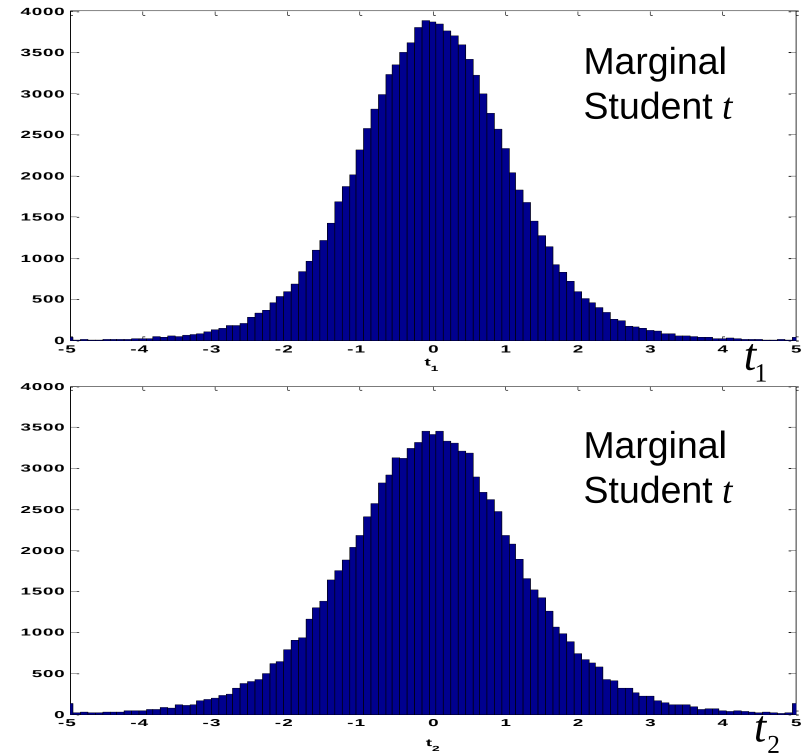
finally $[(n-1)S]^{-1/2} n^{1/2}(\bar{x}-\mu) \sim t(n-1)$

# Multivariate Student-t Distribution

$$\mu = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

Take sample mean $\bar{x}$ from each $5\times10^4$ sets of 10 variates, subtracted $\mu$ from each set, premultiplied the $5\times10^4$ centered normal variates by the $5\times10^4$ $\left[(n-1)S\right]^{-1/2} n^{1/2}(\bar{x}-\mu)$



# Multivariate Student-t Distribution

A 2-D ($p$=2) random vector variate $\underset{p\times 1}{s}$ follows a general

Student $t$ distribution with location $\underset{p\times 1}{\mu}$ and scale $\underset{p\times p}{\Sigma}$ and $df$ $\nu$

$$s \sim t(\nu, \mu, \Sigma)$$

$$f(s \mid \nu, \mu, \Sigma) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{(\nu\pi)^{\frac{p}{2}}\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\left[1 + \frac{1}{\nu}(s-\mu)'\Sigma^{-1}(s-\mu)\right]^{\frac{\nu+p}{2}}}$$

where $s, \mu \in \mathbb{R}^p$, $\nu \in \mathbb{N}$, $\Sigma > 0$ .

The mean and variance of $s$ are

$$E(t \mid \nu, \mu, \Sigma) = \underset{p\times 1}{\mu} \qquad \nu > 1 \qquad\qquad \mathrm{var}(s \mid \nu, \mu, \Sigma) = \frac{\nu}{\nu - 2} \underset{p\times p}{\Sigma} \qquad \nu > 2$$

# Multivariate Distributions

We can also perform hypothesis tests.

Single mean $H_o : \underset{p\times 1}{\mu} = \underset{p\times 1}{\mu_0}$ vs $H_1 : \underset{p\times 1}{\mu} \neq \underset{p\times 1}{\mu_0}$

Two Means $H_o : \underset{p\times 1}{\mu_1} = \underset{p\times 1}{\mu_2}$ vs $H_1 : \underset{p\times 1}{\mu_1} \neq \underset{p\times 1}{\mu_2}$

MANOVA $H_o : \underset{p\times 1}{\mu_1} = \underset{p\times 1}{\mu_2} = \underset{p\times 1}{\mu_3}$ vs $H_1 : \underset{p\times 1}{\mu}$'s not all equal

Multivariate Regression $\underset{n\times p}{Y} = \underset{n\times(q+1)}{X} \underset{(q+1)\times p}{B} + \underset{n\times p}{E}$ $\underset{(q+1)\times p}{\hat{B}} = (X'X)^{-1}X'Y$

$\underset{p\times p}{S} = \frac{1}{n-q-1}(Y - X\hat{B})'(Y - X\hat{B})$

$H_o : \underset{p\times p}{\Sigma} = \underset{p\times p}{diagonal}$ vs $H_1 : \underset{p\times p}{\Sigma} \neq \underset{p\times p}{diagonal}$

# Matrix Distributions

Matrix Normal and Matrix T distributions also exist.

$$\underset{p\times n}{f}(X|M,\Sigma,\Phi)=(2\pi)^{-np/2}|\Phi|^{-p/2}|\Sigma|^{-n/2}\,e^{-\frac{1}{2}tr\Phi^{-1}(X-M)'\Sigma^{-1}(X-M)}$$

$$\underset{p\times n}{f}(T|\nu,T_0,\Sigma,\Phi)=k_T\,\frac{|\Phi|^{\frac{\nu}{2}}|\Sigma|^{-\frac{n}{2}}}{\left|\Phi+\frac{1}{\nu}(T-T_0)'\Sigma^{-1}(T-T_0)\right|^{\frac{\nu+p}{2}}}$$

$$k_T=\frac{\prod_{j=1}^{n}\Gamma\left(\frac{\nu+p+1-j}{2}\right)}{(\nu\pi)^{\frac{np}{2}}\prod_{j=1}^{n}\Gamma\left(\frac{\nu+1-j}{2}\right)}$$

$n=1\rightarrow$multivariate
$n,p=1\rightarrow$univariate

# Homework 12:

1) Repeat the simulation processes performed in this lecture.
Generate two uniforms then transform to two standard normals then to two normals $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $\mu = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$ using the Cholesky factorization. $A$=chol(Sigma)';
Repeat 5 times to get a sample of size $n$=5.
Repeat $10^4$ times to get $10^4$ bivariate samples of $n$ =5.
a) Compute bivariate mean $\bar{x}$ for each sample.
b) Compute bivariate covariance matrix $S$ for each sample.
c) For each sample compute $\left[(n-1)S\right]^{-1/2} n^{1/2}(\bar{x}-\mu)$.
d) Make histograms of items (elements) in a), b), and c).

# Homework 12:

2) Repeat simulation process with eigenvalue-eigenvector decomposition $\Sigma=UD^{1/2}(D^{1/2}U)'$ instead of Cholesky $\Sigma=AA'$.

   [v,d]=eig(Sigma); A=v*sqrt(d);

3) Comment of similarities and/or differences between 1) and 2).