Bi(Multi)variate Transformation of Variables
(continued)

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Outline

Distributions

Uniforms to Normals

Normals to Chi-Square

Normal and Chi-Square to t

Chi-Squares to F

Multivariate Transformation of Variables

Bivariate Change of Variable

Given two continuous random variables, (x_1, x_2)

with joint probability distribution function $f_{X_1,X_2}(x_1, x_2 \mid \theta)$.

Let $\begin{pmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{pmatrix}$ be a transformation from (x_1, x_2) to (y_1, y_2)

with inverse transformation $\begin{pmatrix} x_1(y_1, y_2) \\ x_2(y_1, y_2) \end{pmatrix}$.

Bivariate Change of Variable

Then, the joint probability distribution function $f_{Y_1,Y_2}(y_1,y_2\,|\,\theta)$

of (y_1,y_2) can be found via

$$f_{Y_1,Y_2}(y_1,y_2\,|\,\theta)=f_{X_1,X_2}\big(x_1(y_1,y_2),x_2(y_1,y_2)\,|\,\theta\big)\times|\,J(x_1,x_2\rightarrow y_1,y_2)\,|$$

where $J(x_1,x_2\rightarrow y_1,y_2)=\begin{vmatrix}\dfrac{dx_1(y_1,y_2)}{dy_1} & \dfrac{dx_1(y_1,y_2)}{dy_2}\\[2ex] \dfrac{dx_2(y_1,y_2)}{dy_1} & \dfrac{dx_2(y_1,y_2)}{dy_2}\end{vmatrix}$.

Bivariate Change of Variable - Normals

Let u_1~uniform(0,1) and u_2~uniform(0,1).
The joint PDF of (u_1,u_2) is

$$f(u_1,u_2)=\begin{cases}1 & \text{if} \quad u_1\in[0,1] \text{ and } u_2\in[0,1] \\ 0 & \text{if} \quad u_1\notin[0,1] \text{ or } u_2\notin[0,1]\end{cases}.$$

If $z_1=z_1(u_1,u_2)$, $z_2=z_2(u_1,u_2)$, the joint distribution of (z_1,z_2) is

$$f_{Z_1,Z_2}(z_1,z_2\,|\,\theta)=f_{U_1,U_2}\left(u_1(z_1,z_2),u_2(z_1,z_2)\,|\,\theta\right)\times|J(u_1,u_2\to z_1,z_2)|$$

$$J(u_1,u_2\to z_1,z_2)=\begin{vmatrix}\dfrac{du_1(z_1,z_2)}{dz_1} & \dfrac{du_1(z_1,z_2)}{dz_2} \\ \dfrac{du_2(z_1,z_2)}{dz_1} & \dfrac{du_2(z_1,z_2)}{dz_2}\end{vmatrix}$$

Bivariate Change of Variable - Normals

Let $z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$ and $z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$

then $u_1(z_{1,}z_2) = e^{-\frac{1}{2}(z_1^2+z_2^2)}$ and $u_2(z_1,z_2) = \frac{1}{2\pi}\text{atan}\left(\frac{z_2}{z_1}\right)$.

$$J(u_1,u_2 \to z_1,z_2) = \begin{vmatrix} \frac{du_1(z_1,z_2)}{dz_1} & \frac{du_1(z_1,z_2)}{dz_2} \\ \frac{du_2(z_1,z_2)}{dz_1} & \frac{du_2(z_1,z_2)}{dz_2} \end{vmatrix} = -\frac{1}{2\pi}e^{-\frac{1}{2}(z_1^2+z_2^2)}$$

Bivariate Change of Variable - Normals

Therefore,

$$f_{Z_1,Z_2}(z_1,z_2 \mid \theta) = f_{U_1,U_2}\left(u_1(z_1,z_2),u_2(z_1,z_2) \mid \theta\right) \times \mid J(u_1,u_2 \to z_1,z_2) \mid$$

which upon insertion yields

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2 \mid \theta) &= \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2} \quad . \end{aligned}$$

Joint PDF factors thus independent

This means $z_1 \sim N(0,1)$, $z_2 \sim N(0,1)$, z_1 and z_2 are independent.

Bivariate Change of Variable - Normals

Generate 10^6 independent uniform(0,1)'s.

The first half of the 10^6 standard uniform random variates were used as u_1's and the second half used as u_2's.

Take each (u_1, u_2) pair to produce a (z_1, z_2) pair.

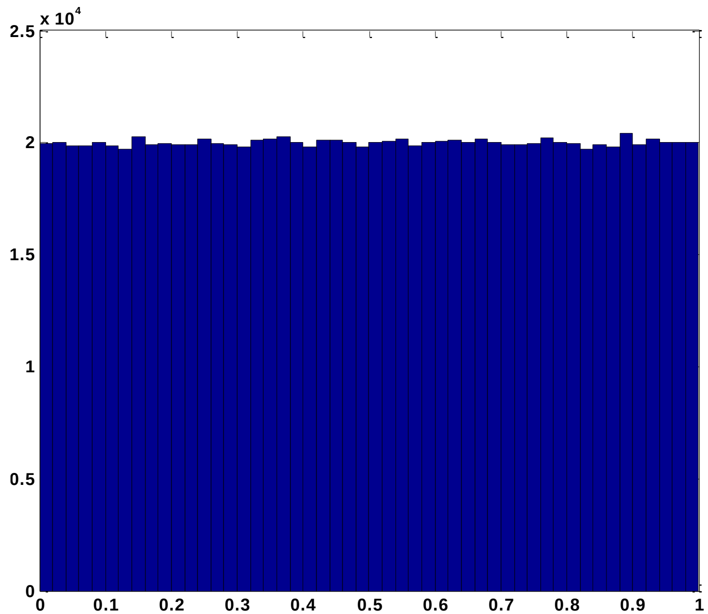
$$z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \qquad z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$$

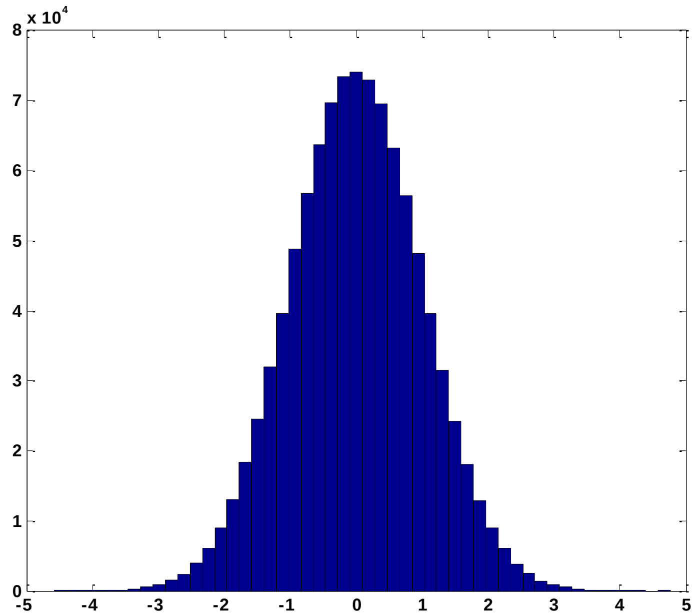
(z_1, z_2) are independent normally distributed.

Bivariate Change of Variable - Normals

```
n=10^6;
u1=rand(n/2,1);        z1=sqrt(-2*log(u1)).*cos(2*pi*u2);
u2=rand(n/2,1);        z2=sqrt(-2*log(u1)).*sin(2*pi*u2);
figure(1)              figure(2)
hist([u1;u2],50)       hist([z1;z2],50)
```

```
[mean(u1),var(u1)]
[mean(u2),var(u2)]
[mean(z1),var(z1)]
[mean(z2),var(z2)]
[corr(u1,u2),corr(z1,z2)]
```





0.5000	0.0832
0.5006	0.0833
0.0000	1.0011
-0.0016	0.9970
0.0025	0.0013

Uncorrelated and since normal are independent

Bivariate Change of Variable - Chi-Square

We discussed how we can obtain a random variable x

that has a general normal distribution with mean μ and

variance σ^2 via the transformation $x = \sigma z + \mu$.

The PDF of x can be obtained by the change of variable

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where, $x, \mu \in \mathbb{R}$, $0 < \sigma$. That is, x~normal(μ,σ^2).

Bivariate Change of Variable - Chi-Square

We also discussed how the change of variable technique can be repeated. If x_i~normal(μ,σ^2) for i=1,…,n, and x_i's are independent, then

$$y=\frac{1}{n}\sum_{i=1}^{n}x_i \sim N\left(\mu,\frac{\sigma^2}{n}\right) \quad .$$

Bivariate Change of Variable - Chi-Square

We also discussed how the change of variable technique can be applied to $y_1 = \left(\frac{x_1 - \mu}{\sigma}\right)^2$. If x_1~normal(μ, σ^2), then the distribution y_1 is $\chi^2(1)$. This process can be duplicated so that if x_2~normal(μ, σ^2), then the distribution of

$$y_2 = \left(\frac{x_2 - \mu}{\sigma}\right)^2$$ is $\chi^2(1)$.

Now what is the distribution of $y_1 + y_2$?

Bivariate Change of Variable - Chi-Square

Let y_1 and y_2 have independent chi-square PDFs

$$f(y_i) = \frac{y_i^{1/2-1} e^{-y_i/2}}{\Gamma(1/2) 2^{1/2}} \,, \quad y_i > 0 \,, \; i = 1,2 \,.$$

We can find the distribution of $w_1 = y_1 + y_2$ (and w_2=y_2)

via the bivariate change of variable technique

$$f_{W_1,W_2}(w_1, w_2 \mid \theta) = f_{Y_1,Y_2}(y_1(w_1,w_2), y_2(w_1,w_2) \mid \theta) \times | J(y_1, y_2 \to w_1, w_2) |$$

with marginalization $f_{W_1}(w_1 \mid \theta) = \int_{w_2} f_{W_1,W_2}(w_1, w_2 \mid \theta) dw_2 \,.$

Bivariate Change of Variable - Chi-Square

It turns out that if $y_1 \sim \chi^2(1)$, $y_2 \sim \chi^2(1)$, and independent, then

$w_1 = y_1 + y_2 \sim \chi^2(2)$. Or more generally, if $y_1 \sim \chi^2(\nu_1)$,

$y_2 \sim \chi^2(\nu_2)$, and independent, then $w_1 = y_1 + y_2 \sim \chi^2(\nu_1 + \nu_2)$.

So what this means is that

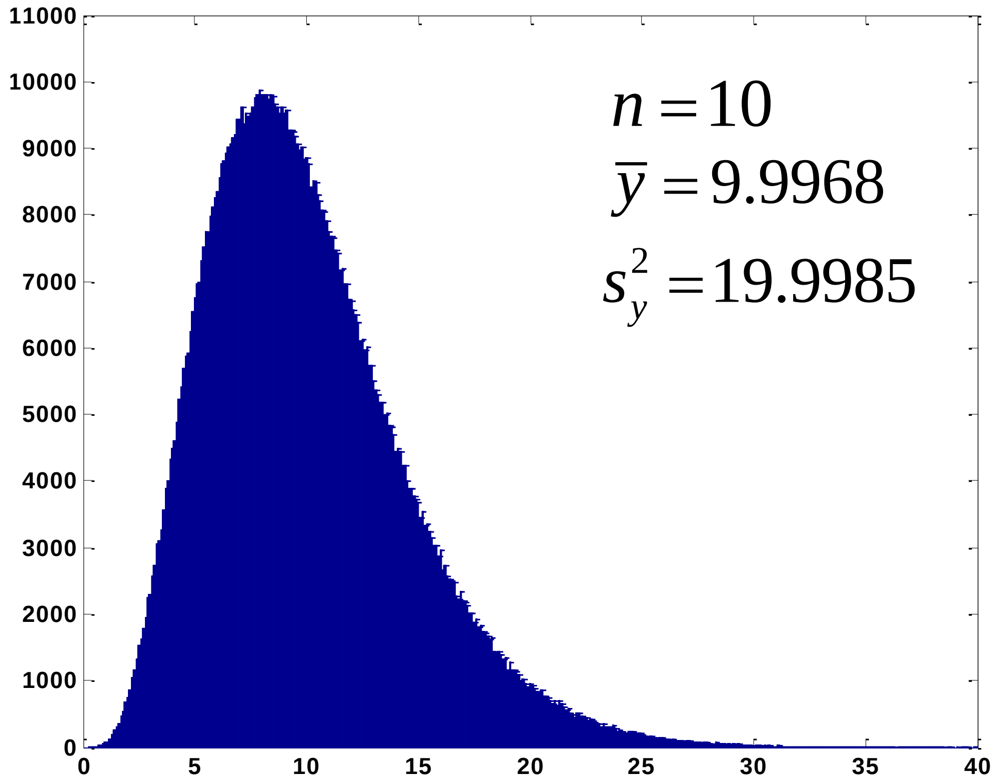
$$y = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n) \; !$$

⟵ Homework problem.

Bivariate Change of Variable - Chi-Square

If $\underset{i=1,\ldots,n}{y_i} = \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2(1)$, then $y = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$.

```
n=10;, mu=5;, sigma=2;
x=sigma*randn(10^6,n)+mu;
y=sum(((x-mu)/sigma).^2,2);
figure(1)
hist(y,(0:.1:40)')
axis([0 40 0 11000])
mean(y)
var(y)
```



Bivariate Change of Variable - Chi-Square

If the mean μ is unknown, then we can estimate it by $\bar{x}$ and lose one degree of freedom!

$$\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}_{\chi^2(n)} = \underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}_{\chi^2(n-1)} + \underbrace{\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2(1)}$$

We just showed

add and subtract $\bar{x}$ in the numerator

Because df add, or by transformation!

Since $\bar{x} \sim N(\mu, \sigma^2/n)$

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

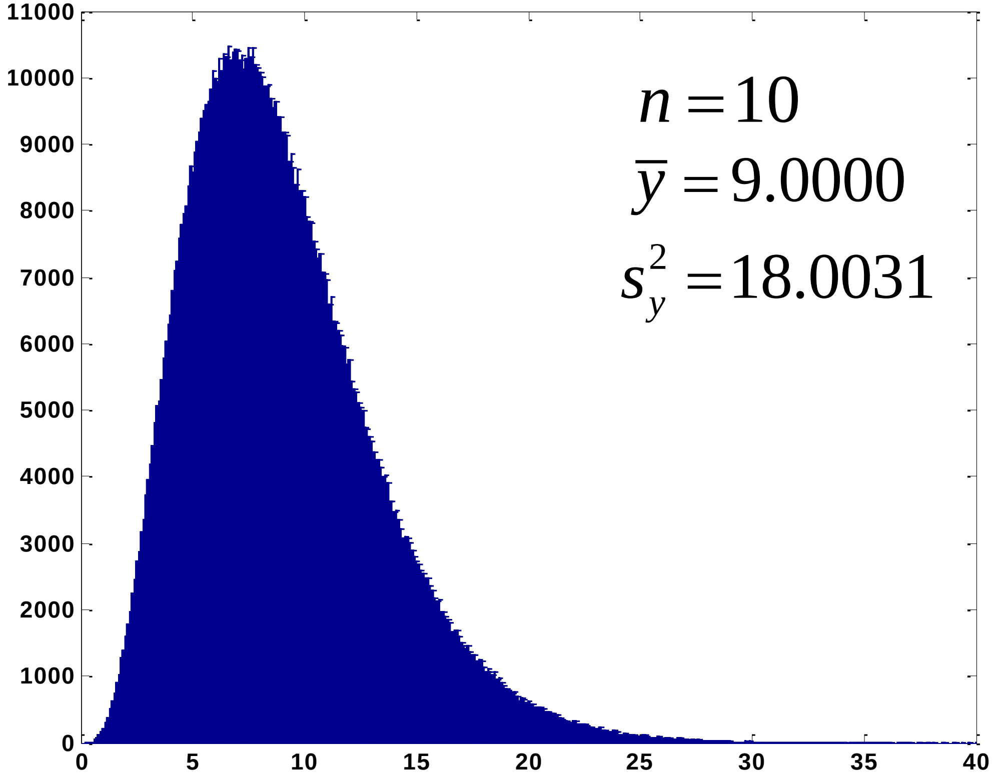
$$\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$$

Bivariate Change of Variable - Chi-Square

$$y_2 = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2(n-1)$$

Already have x's.

```
xbar=mean(x,2);
y2=sum(((x-xbar*ones(1,n))...
    /sigma).^2,2);
figure(2)
hist(y2,(0:.1:40)')
axis([0 40 0 11000])
mean(y2)
var(y2)
```

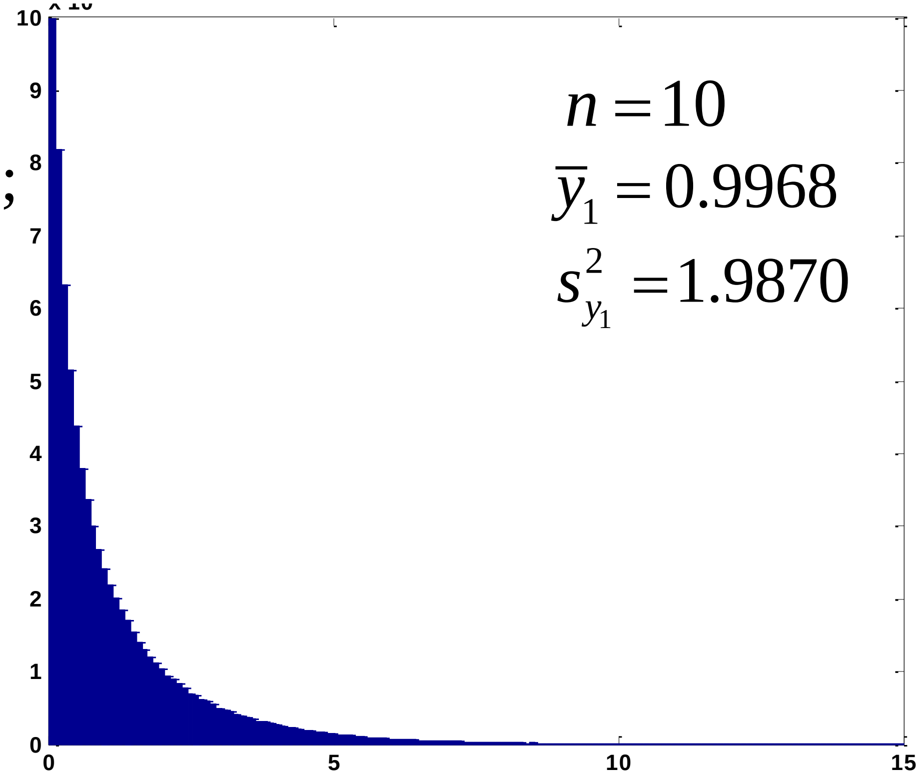


Bivariate Change of Variable - Chi-Square

$$y_1 = \left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2 \sim \chi^2(1)$$

Already have *x*-bars's.

```
y1=((xbar-mu)/(sigma/sqrt(n))).^2;
figure(3)
hist(y1,(0:.1:15)')
axis([0 15 0 10^5])
mean(y1)
var(y1)
```



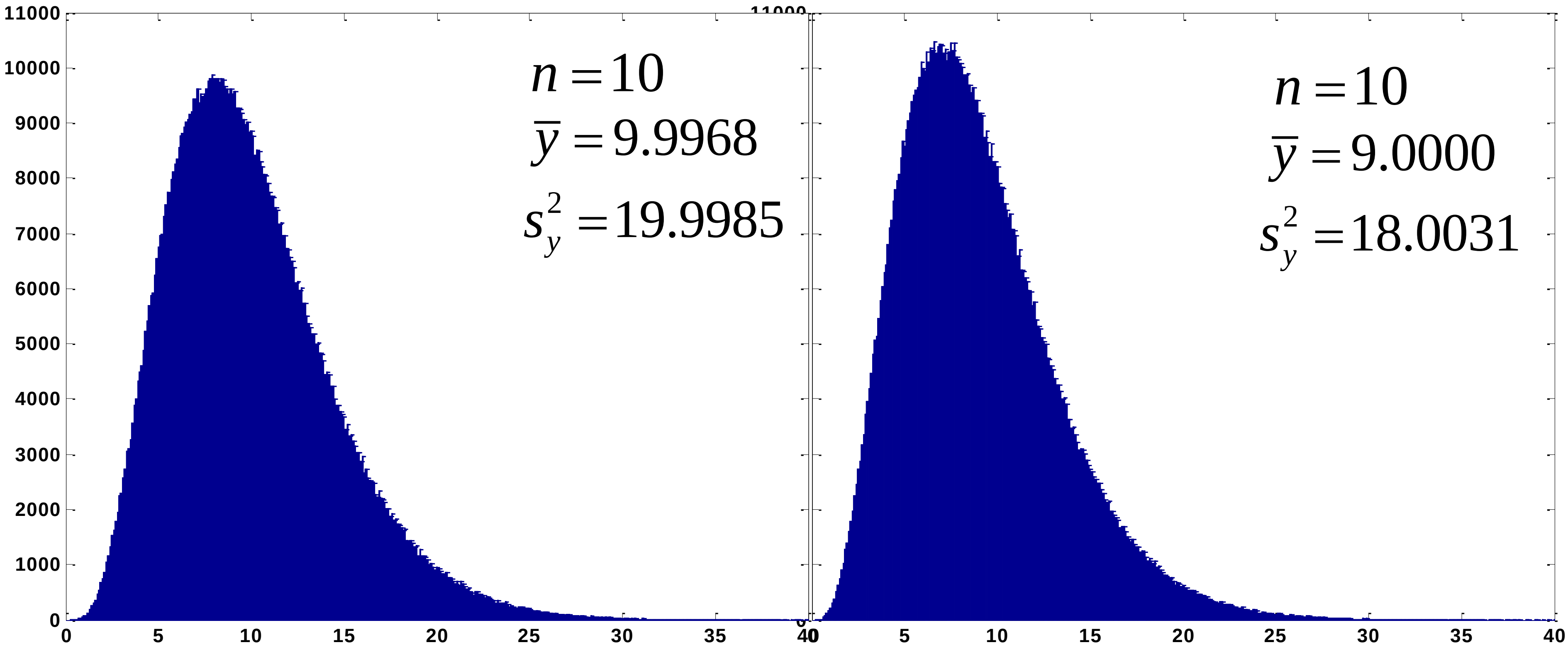
$n = 10$

$\bar{y}_1 = 0.9968$

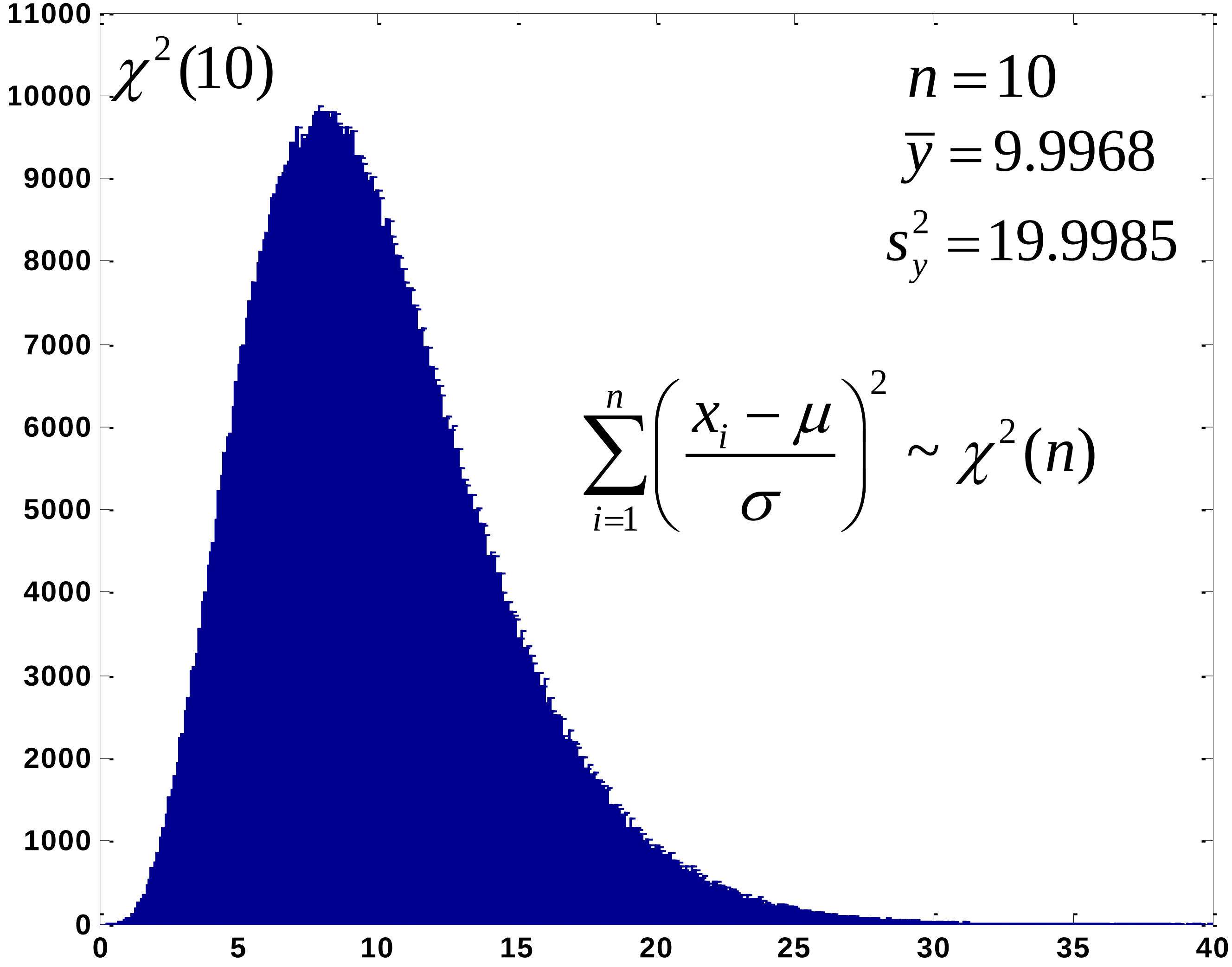
$s^2_{y_1} = 1.9870$

Bivariate Change of Variable - Chi-Square

$$y = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 \sim \chi^2(n) \quad \text{and} \quad y_2 = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2(n-1)$$



Bivariate Change of Variable - Chi-Square



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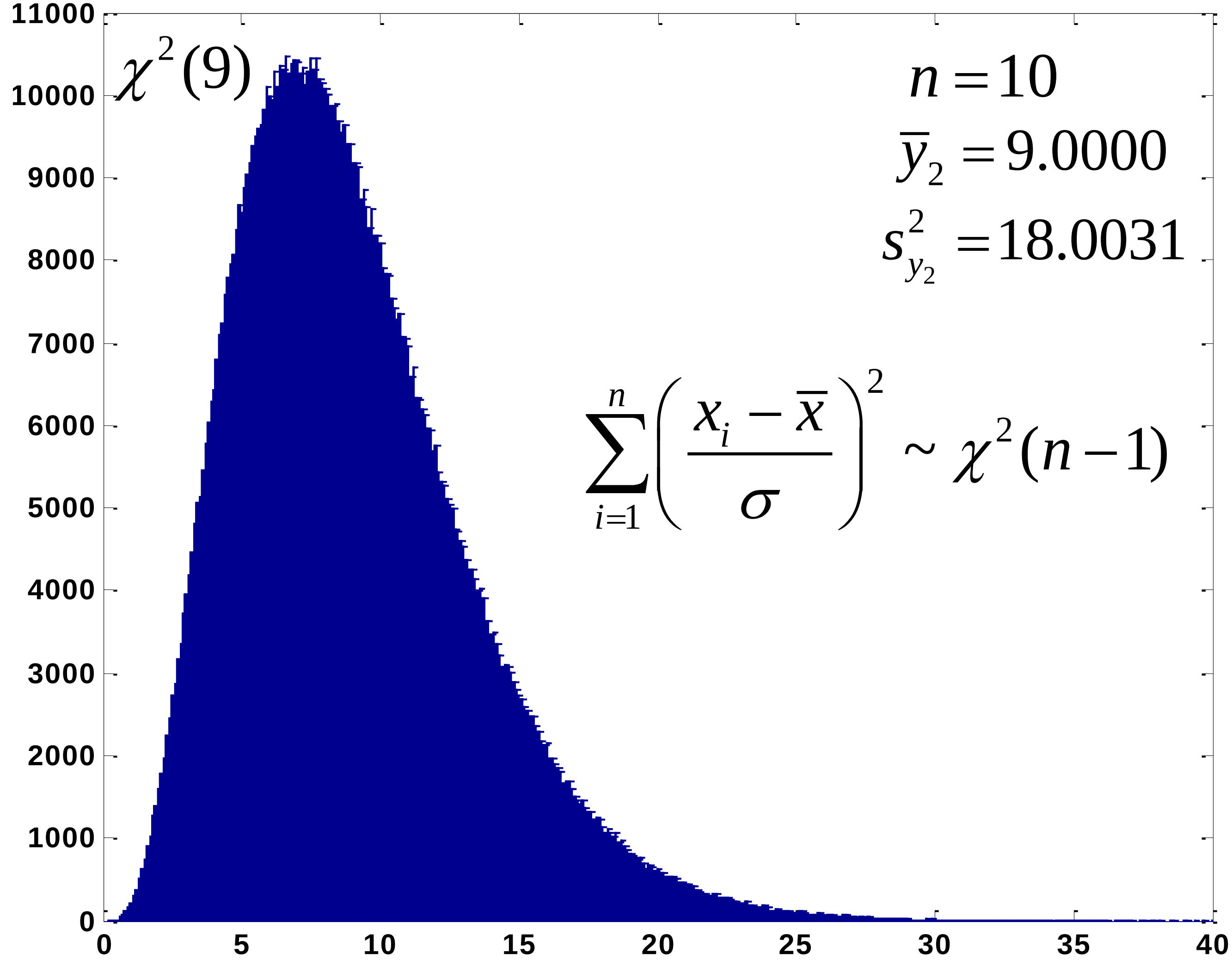
Bivariate Change of Variable - Chi-Square



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Bivariate Change of Variable - Student-t

We showed that if x_i~normal(μ,σ^2) for i=1,…,n, then

the distribution of $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

and that the distribution of $y_2 = \sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma}\right)^2 \sim \chi^2(n-1)$.

Note that $y_2 = \frac{(n-1)s^2}{\sigma^2}$.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

It turns out that z and $\frac{(n-1)s^2}{\sigma^2}$ are statistically independent!

Bivariate Change of Variable - Student-t

So $z=\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$ and $y_2=\dfrac{\nu s^2}{\sigma^2}\sim\chi^2(\nu)$, $\nu=n-1$.

Let $t=\dfrac{z}{\sqrt{y_2/\nu}}$ and $s=y_2$.

Then $z=\dfrac{t\sqrt{s}}{\sqrt{\nu}}$ and $y_2=s$, the Jacobian of the transformation is

$$J(z,y\to t,s)=\begin{vmatrix}\dfrac{dz(t,s)}{dt} & \dfrac{dz(t,s)}{ds}\\[2ex] \dfrac{dy_2(t,s)}{dt} & \dfrac{dy_2(t,s)}{ds}\end{vmatrix}=\frac{\sqrt{s}}{\sqrt{\nu}}$$

Bivariate Change of Variable - Student-t

Here we use the assumption that z and y are independent!

The joint distribution of (t,s) is

$$f_{T,S}(t,s\,|\,\theta)=f_{y_2,z}(y_2(t,s),z(t,s)\,|\,\theta)\times|\,J(y_2,z\to t,s)\,|$$

$$f_{T,S}(t,s\,|\,\theta)=\frac{s^{\frac{\nu}{2}-1}e^{-\frac{\nu}{2}\left(1+\frac{1}{\nu}t^2\right)}}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}\sqrt{2\pi}}\times\left|\frac{\sqrt{s}}{\sqrt{\nu}}\right|$$

and by integrating out s the distribution of t is

$$f_T(t\,|\,\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{1}{\nu}t^2\right)^{-\frac{\nu+1}{2}}.$$

$$z=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

$$y_2=\frac{(n-1)s^2}{\sigma^2}$$

The distribution of $t=\dfrac{z}{\sqrt{y_2/(n-1)}}\sim t(n-1)$!

Bivariate Change of Variable - F

Recall that $$\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}_{\chi^2(n)} = \underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}_{\chi^2(n-1)} + \underbrace{\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2(1)} ,$$

It turns out that $y_1=\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$ and $y_2=\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2$

are statistically independent.

But of interest to us (hypothesis testing) is the distribution of

$f=\dfrac{y_1/\nu_1}{y_2/\nu_2}$, where $y_1\sim\chi^2(\nu_1)$ and $y_2\sim\chi^2(\nu_2)$.

Bivariate Change of Variable - F

Let y_1 and y_2 have independent χ^2 PDFs with ν_1 and ν_2 df

$$f(y_i \mid \nu_i) = \frac{y_i^{\nu_i/2-1} e^{-y_i/2}}{\Gamma(\nu_i/2) 2^{\nu_i/2}} \;, \quad y_i > 0 \;, \; i = 1,2 \;.$$

We can find the distribution of $f = \dfrac{y_1/\nu_1}{y_2/\nu_2}$ (and g=y_2)

via the bivariate change of variable technique

$$f_{F,G}(f,g \mid \theta) = f_{Y_1,Y_2}(y_1(f,g), y_2(f,g) \mid \theta) \times | J(y_1, y_2 \to f,g) |$$

and marginalization $f_F(f \mid \theta) = \int_g f_{F,G}(f,g \mid \theta) dg$.

Bivariate Change of Variable - F

The joint distribution of (f,g) is

$$f_{F,G}(f,g\,|\,\theta)=f_{Y_1,Y_2}(y_1(f,g),y_2(f,g)\,|\,\theta)\times|J(y_1,y_2\rightarrow f,g)|$$

the original variables in terms of the new variables are

$y_1=\dfrac{\nu_1}{\nu_2}gf$ and $y_2=g$ with Jacobian

$$J(y_1,y_2\rightarrow f,g)=\begin{vmatrix}\dfrac{dy_1(f,g)}{df} & \dfrac{dy_1(f,g)}{dg}\\ \dfrac{dy_2(f,g)}{df} & \dfrac{dy_2(f,g)}{dg}\end{vmatrix}=\frac{\nu_1}{\nu_2}g \quad .$$

Bivariate Change of Variable - F

$$y_1 = \frac{\nu_1}{\nu_2} gf \qquad y_2 = g$$

The joint distribution of (f,g) is

$$f_{F,G}(f,g\,|\,\theta) = f_{Y_1,Y_2}(y_1(f,g), y_2(f,g)\,|\,\theta) \times |J(y_1,y_2 \to f,g)|$$

$$f_{F,G}(f,g\,|\,\theta) = \frac{\left(\frac{\nu_1}{\nu_2} gf\right)^{\nu_1/2-1} e^{-\left(\frac{\nu_1}{\nu_2} gf\right)/2}}{\Gamma(\nu_1/2)2^{\nu_1/2}} \frac{g^{\nu_2/2-1} e^{-g/2}}{\Gamma(\nu_2/2)2^{\nu_2/2}} \times \left|\frac{\nu_1}{\nu_2} g\right|$$

$$f_F(f\,|\,\theta) = \int_g f_{F,G}(f,g\,|\,\theta)\,dg$$

$$f_F(f\,|\,\nu_1,\nu_2) = \frac{\Gamma((\nu_1+\nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_1/2} \left(1 - \frac{\nu_1 f}{\nu_1 f + \nu_2}\right)^{\nu_2/2}$$

Bivariate Change of Variable - F

The joint distribution of $f=\dfrac{y_1/\nu_1}{y_2/\nu_2}$ is

F distributed with ν_1 numerator df and ν_2 denominator df

$$f_F(f\,|\,\nu_1,\nu_2)=\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_1 f}{\nu_1 f+\nu_2}\right)^{\nu_1/2}\left(1-\frac{\nu_1 f}{\nu_1 f+\nu_2}\right)^{\nu_2/2}$$

where $\nu_1,\nu_2=1,2,\ldots$

$$\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}_{\chi(n)}=\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}_{\chi(n-1)}+\underbrace{\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi(1)}$$

Therefore,

$$f=\left[\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2\Big/1\right]\Bigg/\left[\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2\Big/(n-1)\right]\sim F(1,n-1)$$

Bivariate Change of Variable - F/Student-t

We just showed that

$$f = \frac{y_1/\nu_1}{y_2/\nu_2} \sim F(1, n-1) \text{ where } y_1 = \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 \text{ and } y_2 = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2$$

Recall that we showed that

$$t = \frac{z}{\sqrt{y_2/(n-1)}} \sim t(n-1) \text{ where } z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \text{ and } y_2 = \frac{(n-1)s^2}{\sigma^2} \text{ ?}$$

What this means is, when $\nu_1 = 1$, $f = t^2$!

$$t^2 = f = \left[\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 \Big/ 1\right] \Big/ \left[\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 \Big/ (n-1)\right]$$

$$\left(\frac{\bar{y}-\mu_0}{s/\sqrt{n}}\right)^2$$

Bivariate Change of Variable - normal, χ^2, t, F

Recap: u_1 and u_2~uniform(0,1) and independent

$$z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \qquad z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$$

$z_1 \sim N(0,1)$, $z_2 \sim N(0,1)$, z_1 and z_2 are independent

$$x_i = \sigma z_i + \mu \sim N(\mu,\sigma^2), \quad \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right), \quad z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$y_1 = \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1), \quad y_2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

y_1 and y are independent

$$t = \frac{z}{\sqrt{y_2/(n-1)}} \sim t(n-1), \qquad f = \frac{y_1/1}{y_2/(n-1)} \sim F(1,n-1).$$

Outline

- **Multivariate Transformation of Variables**

Univariate Change of Variable

Given a continuous RV x, let $y=y(x)$ be a one-to-one transformation with inverse transformation $x=x(y)$.

Then, if $f_X(x|\theta)$ is the PDF of x, the PDF of y is found as

$$f_Y(y|\theta) = f_X(x(y)|\theta) \times |J(x \to y)|$$

where $J(x \to y) = \dfrac{dx(y)}{dy}$.

Bivariate Change of Variable

Given two continuous random variables, (x_1, x_2)

with joint probability distribution function $f_{X_1,X_2}(x_1, x_2 \mid \theta)$.

Let $\begin{matrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{matrix}$ be a transformation from (x_1, x_2) to (y_1, y_2)

with inverse transformation $\begin{matrix} x_1(y_1, y_2) \\ x_2(y_1, y_2) \end{matrix}$.

Bivariate Change of Variable

Then, the joint probability distribution function $f_{Y_1,Y_2}(y_1,y_2\,|\,\theta)$

of (y_1,y_2) can be found via

$$f_{Y_1,Y_2}(y_1,y_2\,|\,\theta)=f_{X_1,X_2}\left(x_1(y_1,y_2),x_2(y_1,y_2)\,|\,\theta\right)\times|\,J(x_1,x_2\rightarrow y_1,y_2)\,|$$

where $J(x_1,x_2\rightarrow y_1,y_2)=\begin{vmatrix}\dfrac{dx_1(y_1,y_2)}{dy_1} & \dfrac{dx_1(y_1,y_2)}{dy_2}\\ \dfrac{dx_2(y_1,y_2)}{dy_1} & \dfrac{dx_2(y_1,y_2)}{dy_2}\end{vmatrix}$.

Multivariate Change of Variable

Given n continuous random variables, $(x_1,...,x_n)$

with joint probability distribution function $f_X(x_1,...,x_n\,|\,\theta)$.

Let
$$y_1 = y_1(x_1,\ldots,x_n)$$
$$y_2 = y_2(x_1,\ldots,x_n)$$
$$\vdots$$
$$y_n = y_n(x_1,\ldots,x_n)$$
be an n-dimensional transformation from $(x_1,\ldots,x_n)$ to $(y_1,\ldots,y_n)$

with inverse transformation
$$x_1 = x_1(y_1,\ldots,y_n)$$
$$x_2 = x_2(y_1,\ldots,y_n)$$
$$\vdots$$
$$x_n = x_n(y_1,\ldots,y_n)$$
.

Multivariate Change of Variable

Then, the joint probability distribution function

$f_Y(y_1,\ldots,y_n \mid \theta)$ of $(y_1,\ldots,y_n)$ can be found via

$$
\begin{aligned}
f_Y(y_1,\ldots,y_n \mid \theta) &= f_X\left(x_1(y_1,\ldots,y_n),\ldots,x_n(y_1,\ldots,y_n) \mid \theta\right) \\
&\quad \times | J(x_1,\ldots,x_n \to y_1,\ldots,y_n) |
\end{aligned}
$$

where $J(x_1,\ldots,x_n \to y_1,\ldots,y_n) = \begin{vmatrix} \dfrac{dx_1(y_1,\ldots,y_n)}{dy_1} & \cdots & \dfrac{dx_1(y_1,\ldots,y_n)}{dy_n} \\ \vdots & \ddots & \vdots \\ \dfrac{dx_n(y_1,\ldots,y_n)}{dy_1} & \cdots & \dfrac{dx_n(y_1,\ldots,y_n)}{dy_n} \end{vmatrix}$.

Multivariate Change of Variable

The important moral to learn from our study of transformation of variables is:

Measurements have statistical variation and a statistical distribution associated with them and every time we do something with a measurement (i.e. math operation on it) we change its statistical properties and its distribution!

Homework 9:

1) Show analytically that if $y_1 \sim \chi^2(\nu_1)$, $y_2 \sim \chi^2(\nu_2)$, indep. then

$$w_1 = y_1 + y_2 \sim \chi^2(\nu_1 + \nu_2)$$.

2) Generate 10^6 $y_1 \sim \chi^2(5)$ and 10^6 $y_2 \sim \chi^2(7)$ random variates.
   a) Make a histogram of the y_1's. Compute mean and variance.
   b) Make a histogram of the y_2's. Compute mean and variance.
   c) Add y_1 to y_2 to obtain $y=y_1+y_2$ random variates.
   d) Make a histogram of the y's. Compute mean and variance.
   e) Try two things not above.
   f) Comments?

Homework 9:

3) Generate $10*10^6$ independent N(μ=5,σ^2=4) random variates.

a) Compute the sample mean and variance for each group of 10.

b) Make a histogram of the 10^6 z's.

c) Compute mean and variance of z's.

d) Make a histogram of the 10^6 y's.

e) Compute mean and variance of y's.

f) Compute the correlation between z's and y's.

g) Form 10^6 t's. Histogram, mean variance.

h) Square each of the 10^6 t's to get f's. Histogram, mean variance.

i) Try two things not above.

j) Comments.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$y = \frac{(n-1)s^2}{\sigma^2}$$

$$t = \frac{z}{\sqrt{y/(n-1)}}$$

$$f = t^2$$

Homework 9:

$$u = \frac{703w}{h^2}$$

4*) Heights h of Marquette undergraduate students are normal with μ_h=67in and σ_h=2in while their weights w are normal with a mean of μ_w=150lbs and standard deviation σ_w=4lbs.

a) Transform from (h,w) to (u,v) where u is BMI and v is an auxiliary variable that you specify. Present a PDF $f(u,v)$.

b) Make a surface plot of $f(u,v)$ using Matlab.

c) Can you pencil/paper integrate out v to get $f(u)$ and $E(u)$?

d) Can you numerically integrate $uf(u,v)$ and get $E(u)$?

$((u_{min},u_{max})$ and (v_{min},v_{max}) bounds, divide into Δu and $\Delta v)$

e) Generate 10^6 (h,w) pairs. Insert each pair in $u=703w/h^2$.

f) Make a histogram of the u's and compute sample mean.

g) Comment. What if h and w are not independent?

* Show off question