

Continuous Probability Functions (Cont.)

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Outline

- **Continuous Gamma Distribution**
PDF, Moments, CDF, Matlab
- **Continuous Exponential Distribution**
PDF, Moments, CDF, Matlab
- **Continuous Chi-square Distribution**
PDF, Moments, CDF, Matlab

Continuous Distributions

Gamma:

A random variable x has a continuous gamma distribution, $x \sim \text{gamma}(\alpha, \beta)$ if

$$f(x | \alpha, \beta) = \begin{cases} 0 & x \leq 0 \\ x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & x > 0 \end{cases},$$

where, $\alpha, \beta > 0$.

$$\Gamma(\alpha) = \int_{t=0}^{\infty} t^{\alpha-1} e^{-t} dt$$

$$\Gamma(\alpha) = (\alpha - 1)!$$

If α a pos integer

Continuous Distributions

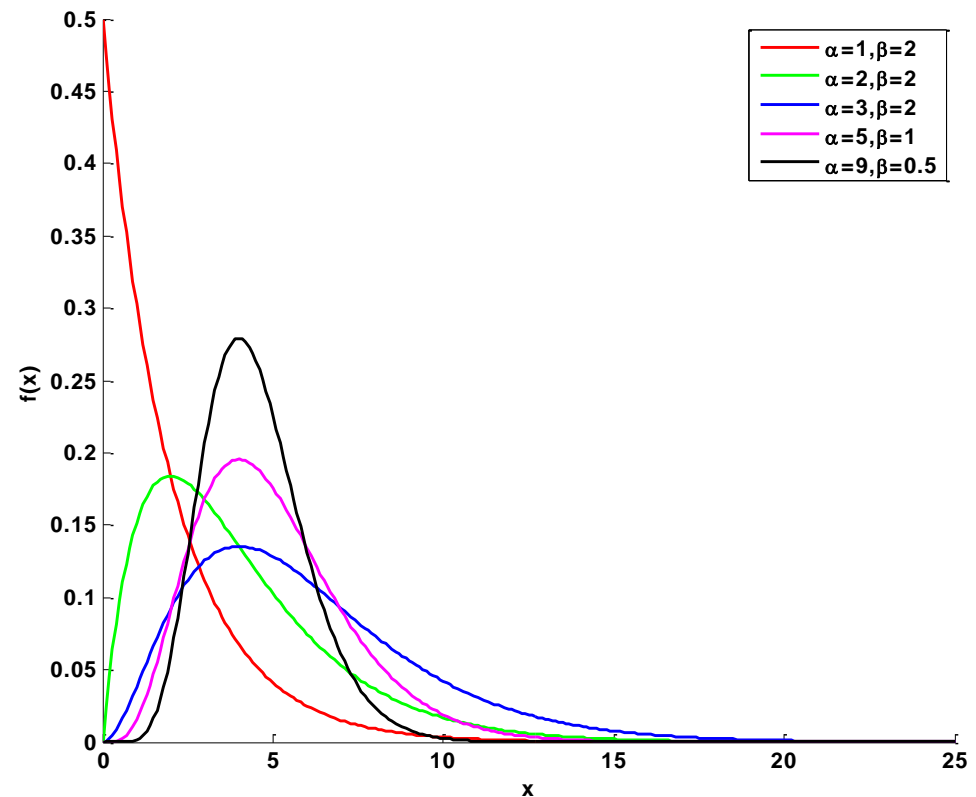
Gamma:

```

x=(0:.1:50)';
alpha=[1,2,3,5,9];, beta=[2,2,2,1,.5];
figure(1)
hold on
for count=1:5
    y = gampdf(x,alpha(count),beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 25])

```

$$f(x | \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}$$



Continuous Distributions

Gamma:

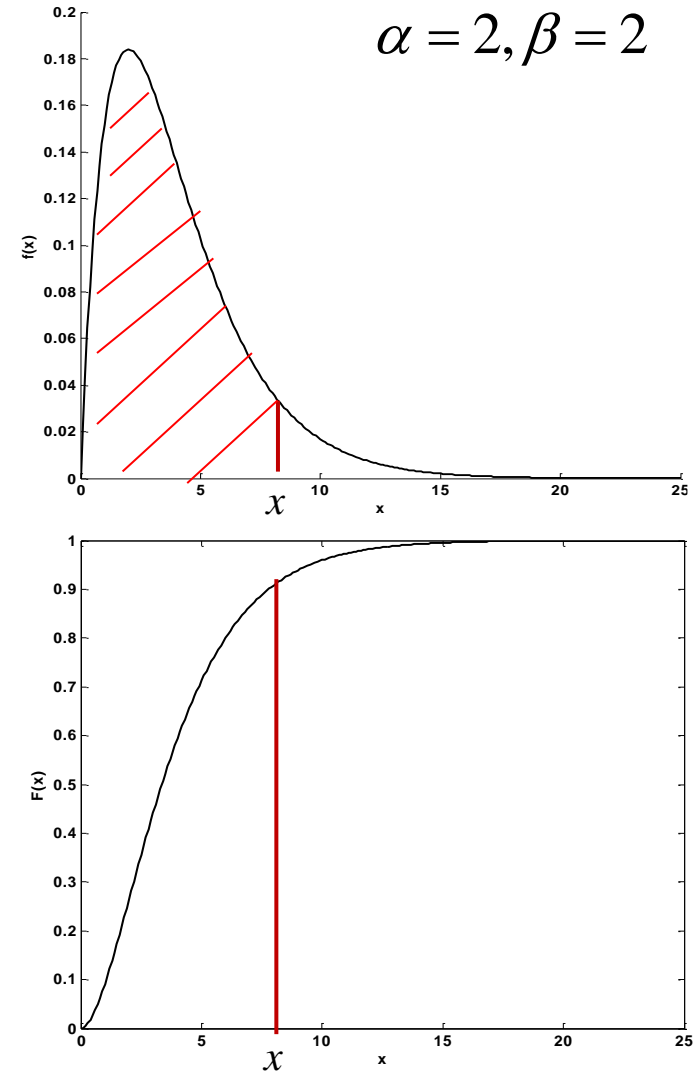
The CDF of the continuous gamma distribution is

$$F(x|\theta) = \int_{t=-\infty}^x f(t|\theta)dt$$

$$F(x|\alpha, \beta) = \int_{t=0}^x t^{\alpha-1} \frac{e^{-t/\beta}}{\beta^\alpha \Gamma(\alpha)} dt$$

$$= \begin{cases} 0 & x \leq 0 \\ \frac{\gamma(\alpha, x/\beta)}{\Gamma(\alpha)} & x > 0 \end{cases}$$

Where $\gamma(\alpha, x/\beta)$ is the lower incomplete gamma function.



Continuous Distributions

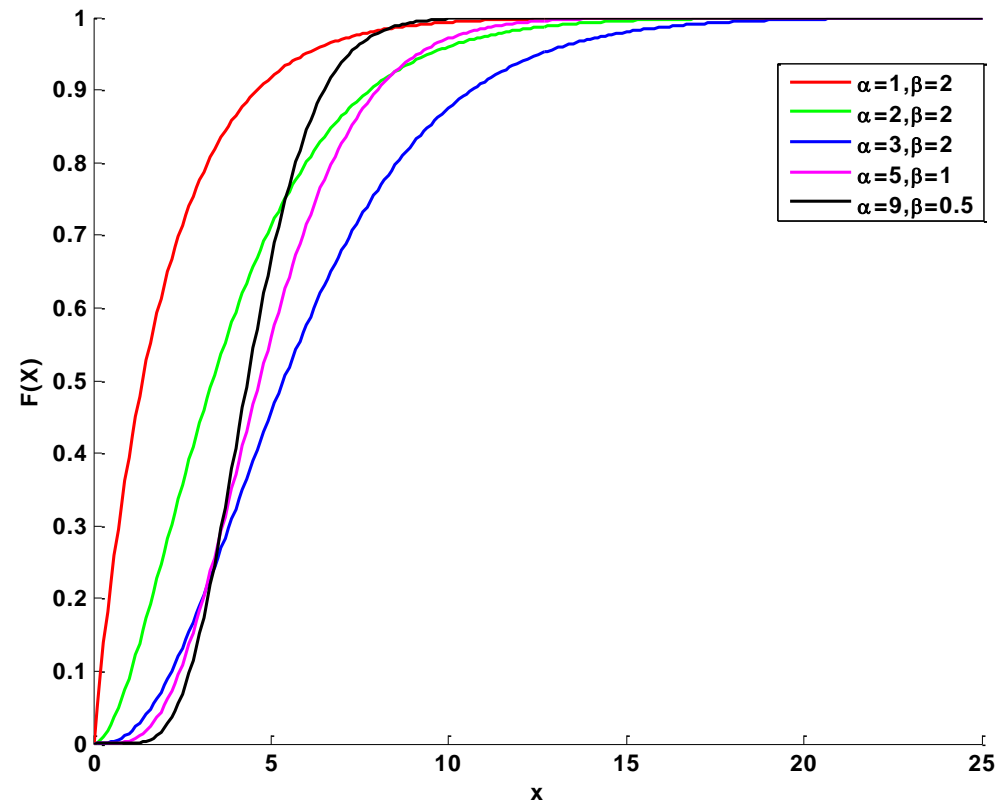
Gamma:

```

x=(0:.1:50)';
alpha=[1,2,3,5,9];, beta=[2,2,2,1,.5];
figure(1)
hold on
for count=1:5
    y = gamcdf(x,alpha(count),beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 25])

```

$$f(x | \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}$$

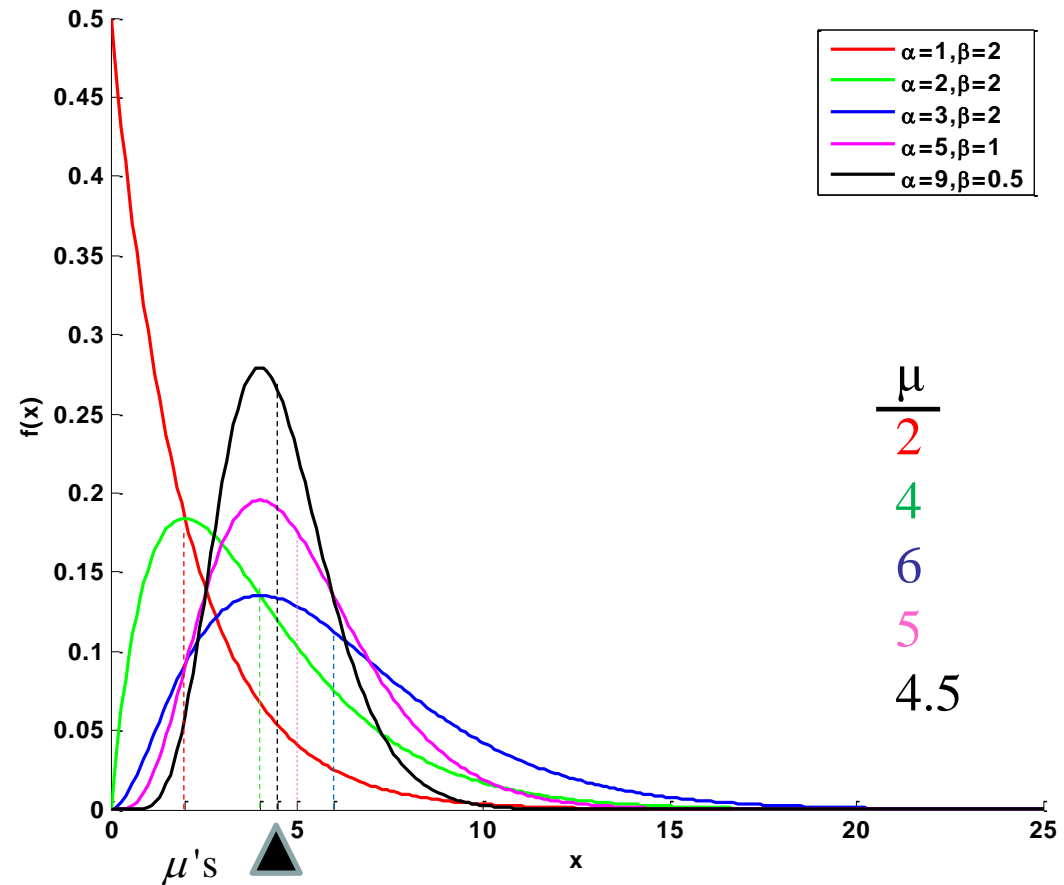


Continuous Distributions

Gamma:

It can be shown that

$$\begin{aligned}\mu &= \int_0^{\infty} xf(x|\theta)dx \\ &= \int_0^{\infty} x \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)} dx \\ &= \alpha\beta\end{aligned}$$



Continuous Distributions

Gamma:

It can be shown that

median

$$\int_{x=0}^{\tilde{x}} f(x | \theta) dx = \frac{1}{2}$$

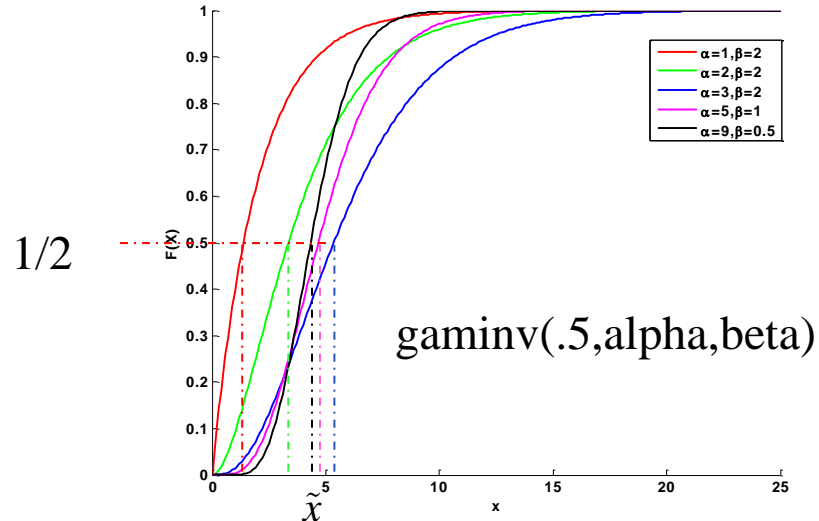
\tilde{x} has no closed form solution

mode

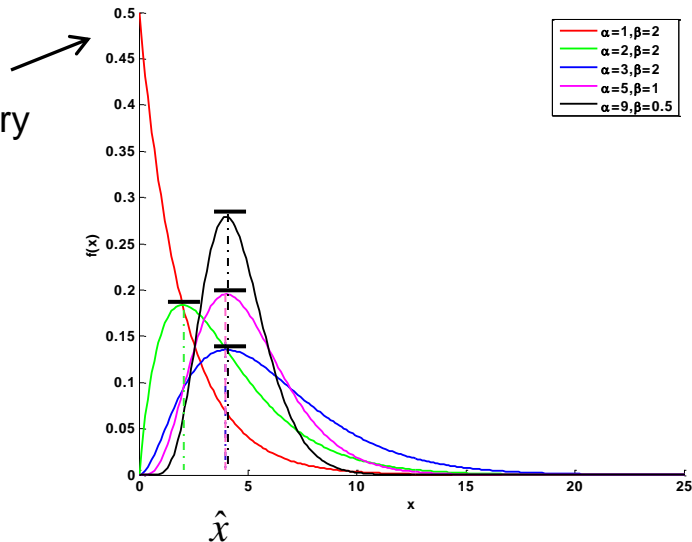
$$\left. \frac{\partial}{\partial x} f(x | \theta) \right|_{\hat{x}} = 0$$

$$\hat{x} = (\alpha - 1)\beta \quad \alpha \geq 1$$

Check boundary points for max.



\tilde{x}
1.3863
3.3567
5.3481
4.6709
4.3345



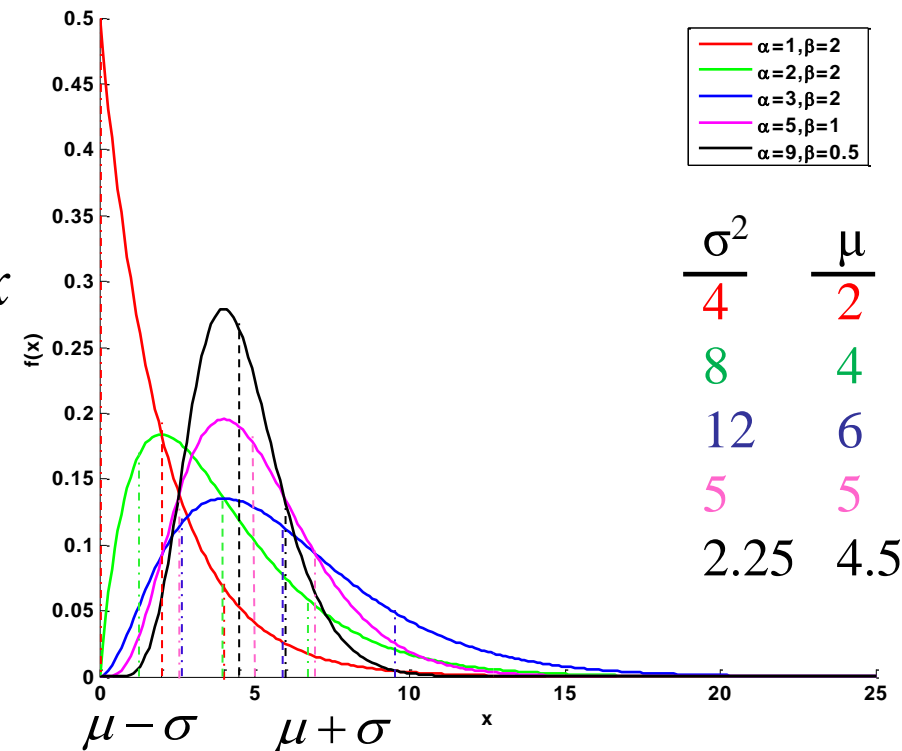
\hat{x}
0
2
4
4
4

Continuous Distributions

Gamma:

that

$$\begin{aligned}\sigma^2 &= \int_x (x - \mu)^2 f(x | \theta) dx \\ &= \int_{x=0}^{\infty} (x - \mu)^2 \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx \\ &= \alpha \beta^2\end{aligned}$$



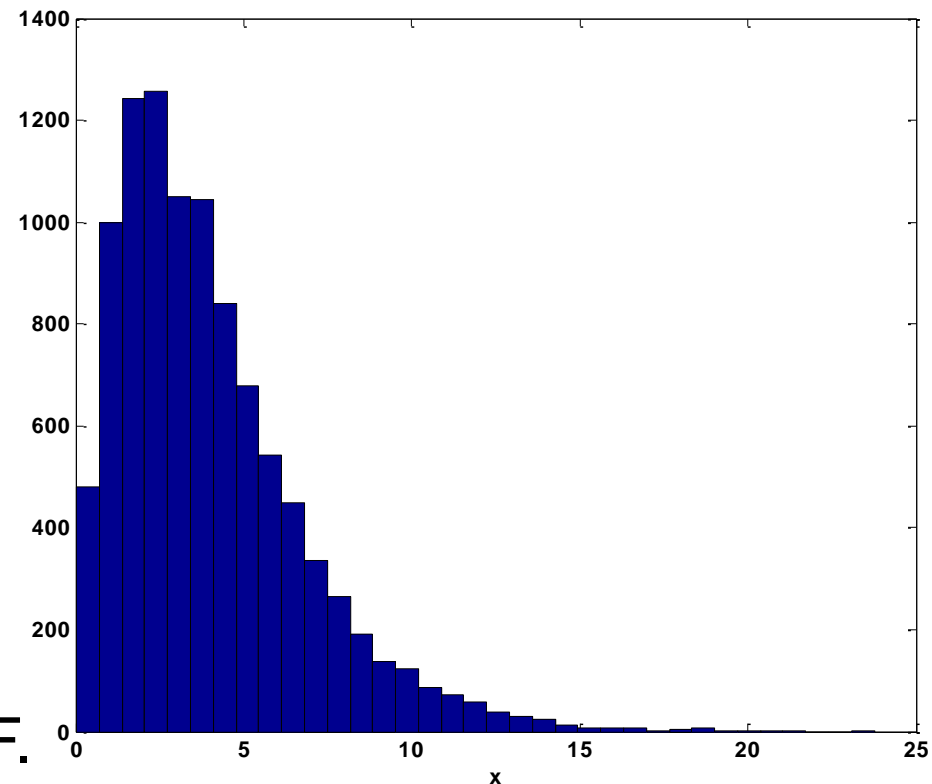
Continuous Distributions

Gamma: $f(x | \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$

```
alpha=2; beta=2; num=10^4;
x=gamrnd(alpha,beta,num,1);
mean(x)
var(x)
hist(x,35)
```

	True	Simulated
μ	4	3.9841
σ^2	8	7.7597

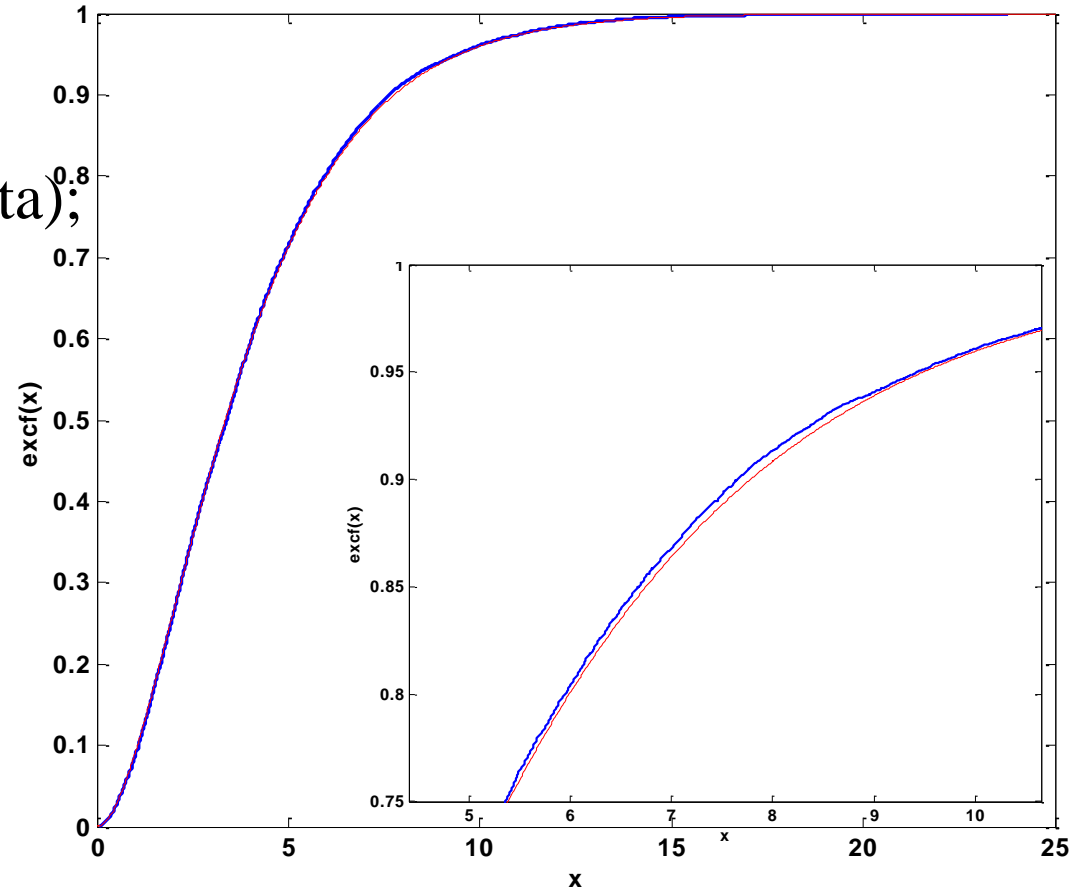
Can also find and plot ECDF.



Continuous Distributions

Gamma:
$$f(x | \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

```
alpha=2;beta=2;
y=gamcdf((0:.01:25),alpha,beta);
plot((0:.01:25),y, 'r')
hold on
[F,xx]=ecdf(x);
stairs(xx,F,'LineWidth',2)
```



Continuous Distributions

Exponential:

A random variable x has a continuous exponential distribution, $x \sim \text{exponential}(\lambda)$ if

$$f(x | \lambda) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}, \quad \text{where, } \lambda > 0.$$

Exponential distribution is a special case of the gamma distribution with $\alpha=1$ and $\lambda \equiv 1/\beta$.

$$f(x | \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

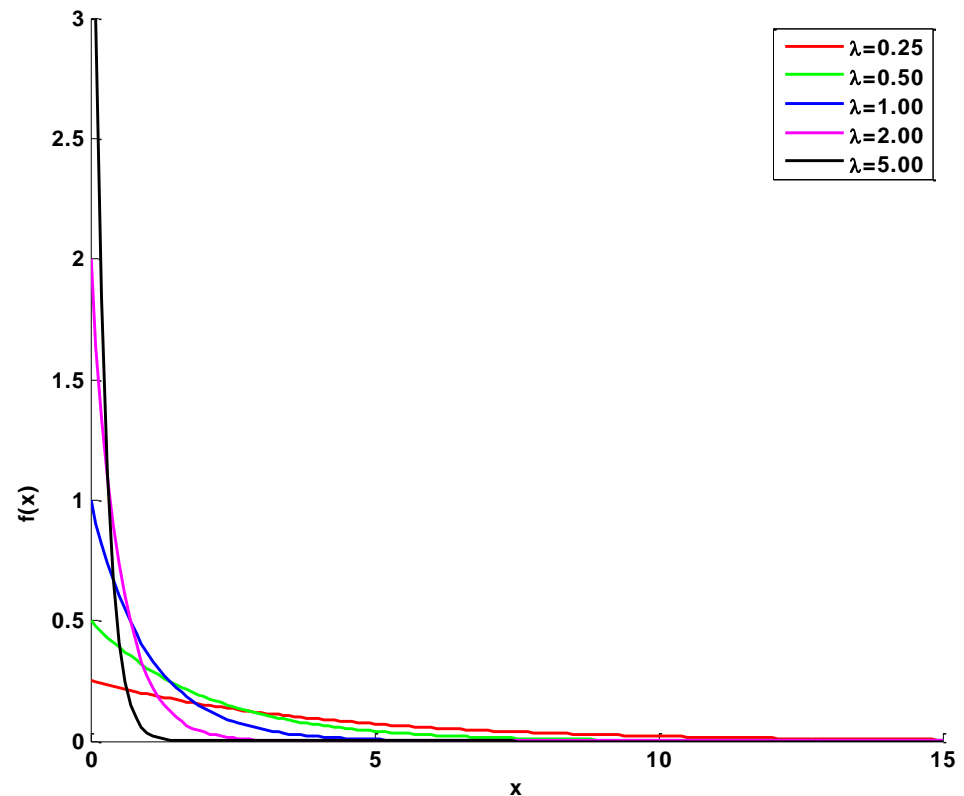
Continuous Distributions

Exponential:

← Matlab uses gamma dist def

```
x=(0:.1:15)';
lambda=[.25,.5,1,2,5];, beta=1./lambda;
figure(1)
hold on
for count=1:length(lambda)
    y = exppdf(x,beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
ylim([0 3]), xlim([0 15])
```

$$f(x | \lambda) = \lambda e^{-\lambda x}$$



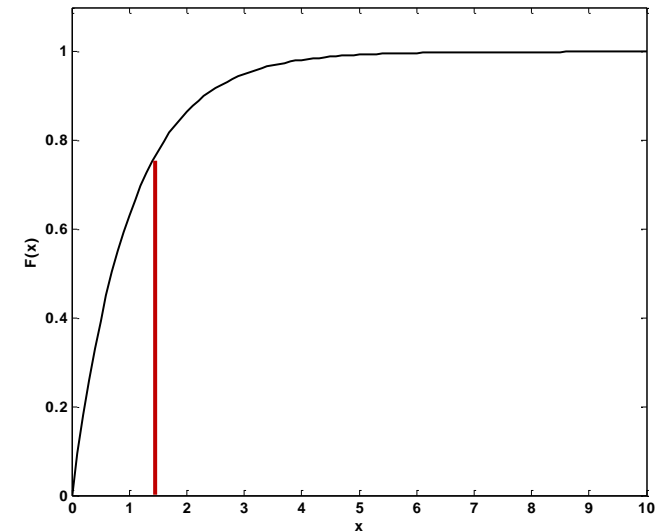
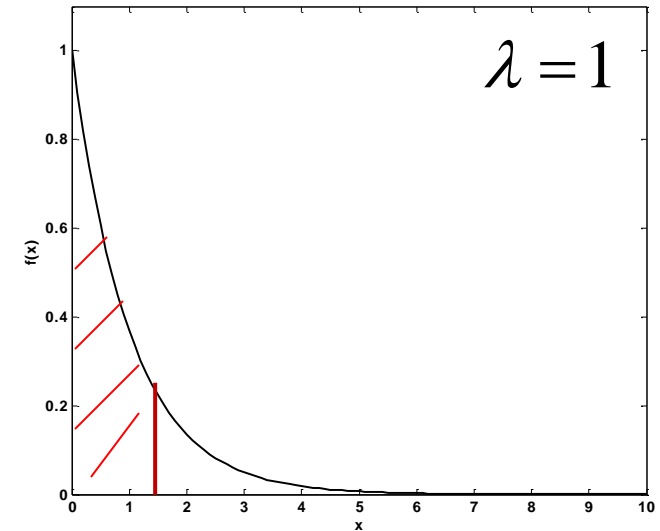
Continuous Distributions

Exponential: $f(x | \lambda) = \lambda e^{-\lambda x}$

The CDF of the continuous exponential distribution is

$$F(x | \lambda) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

Note that there is a closed form solution!



Continuous Distributions

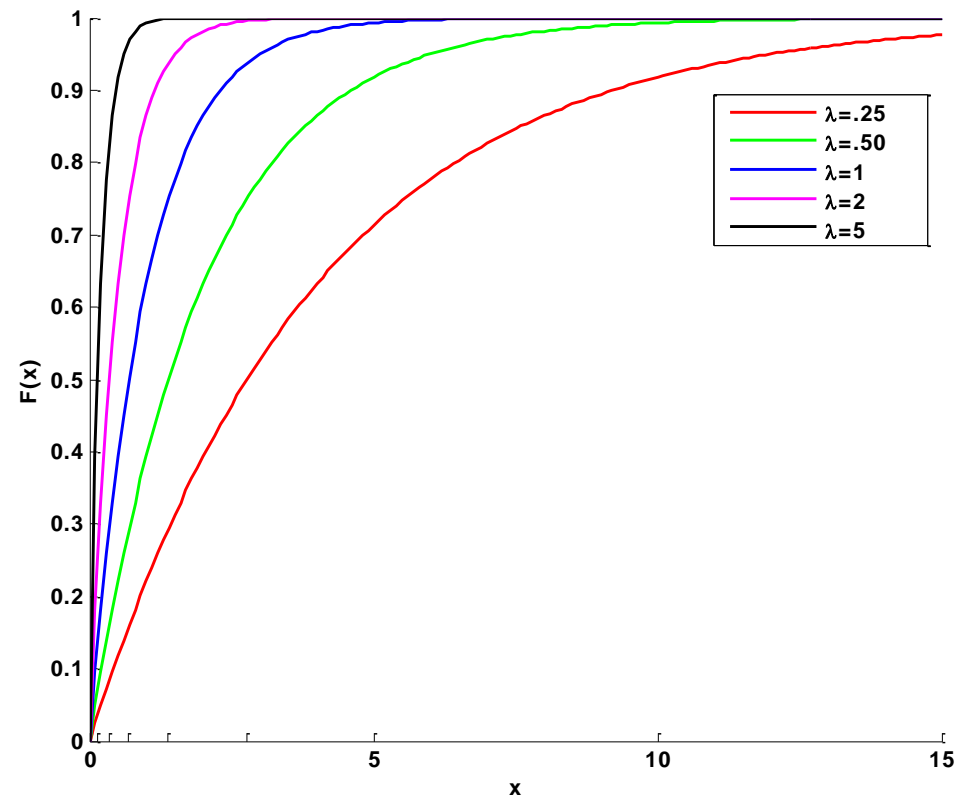
Exponential:

```

x=(0:.1:15)';
lambda=[.25,.5,1,2,5];, beta=1./lambda;
figure(1)
hold on
for count=1:length(lambda)
    y = expcdf(x,beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 15]),ylim([0 1])

```

$$f(x | \lambda) = \lambda e^{-\lambda x}$$

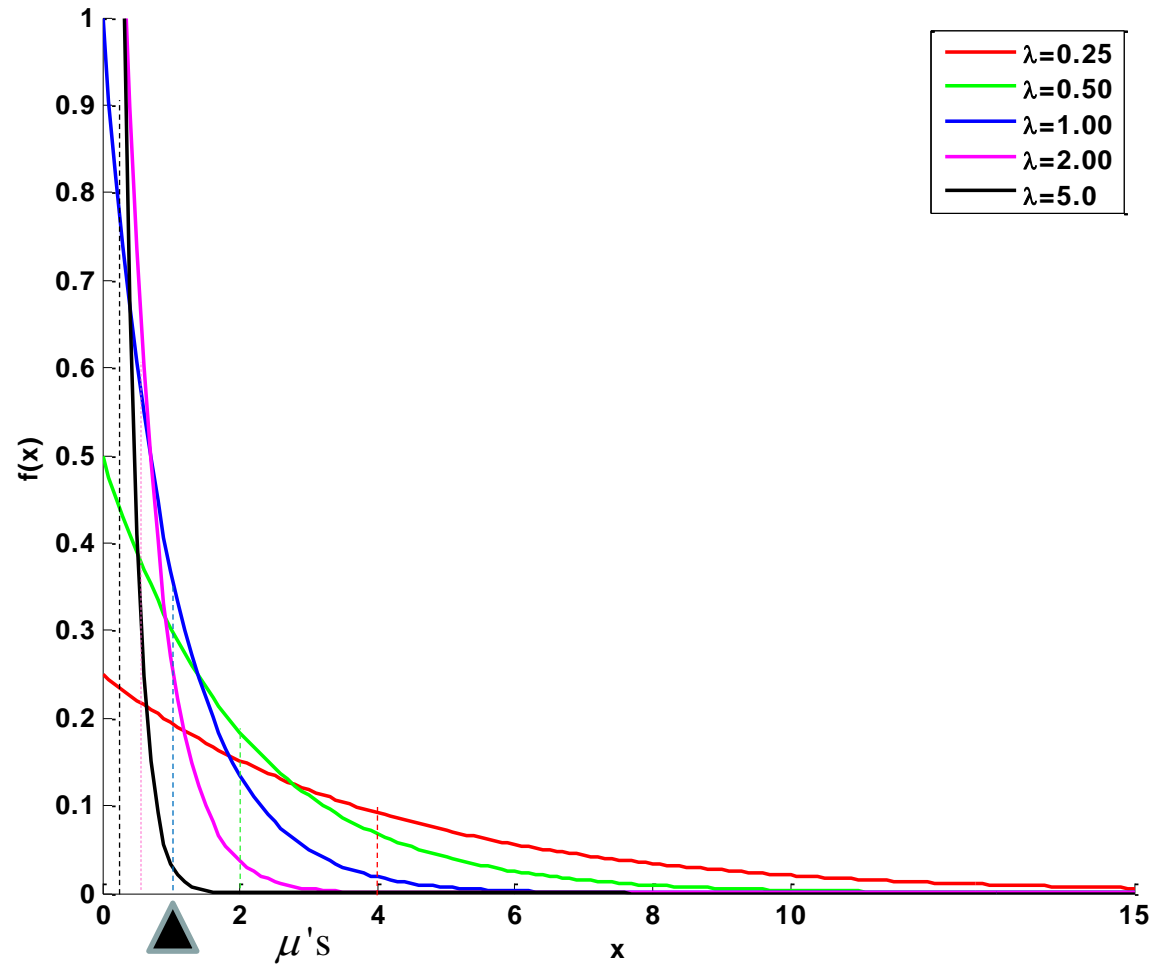


Continuous Distributions

Exponential:

It can be shown that

$$\begin{aligned}\mu &= \int_0^{\infty} xf(x|\theta)dx \\ &= \int_0^{\infty} x\lambda e^{-\lambda x} dx \\ &= \frac{1}{\lambda}\end{aligned}$$



Continuous Distributions

Exponential: $f(x | \lambda) = \lambda e^{-\lambda x}$

It can be shown that

median

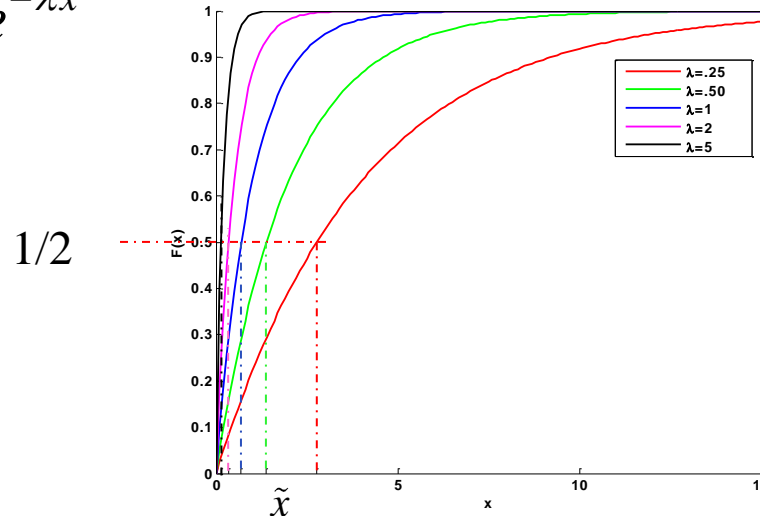
$$\int_{x=0}^{\tilde{x}} f(x | \theta) dx = \frac{1}{2}$$

$$\tilde{x} = \frac{\ln(2)}{\lambda}$$

mode

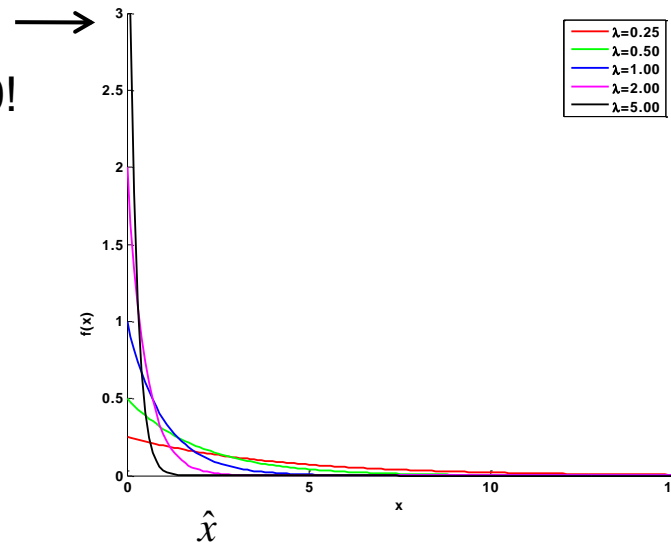
$$\left. \frac{\partial}{\partial x} f(x | \theta) \right|_{\hat{x}} = 0$$

$$\hat{x} = 0$$



\tilde{x}
2.7726
1.3863
0.6931
0.3466
0.1386

All at 0!



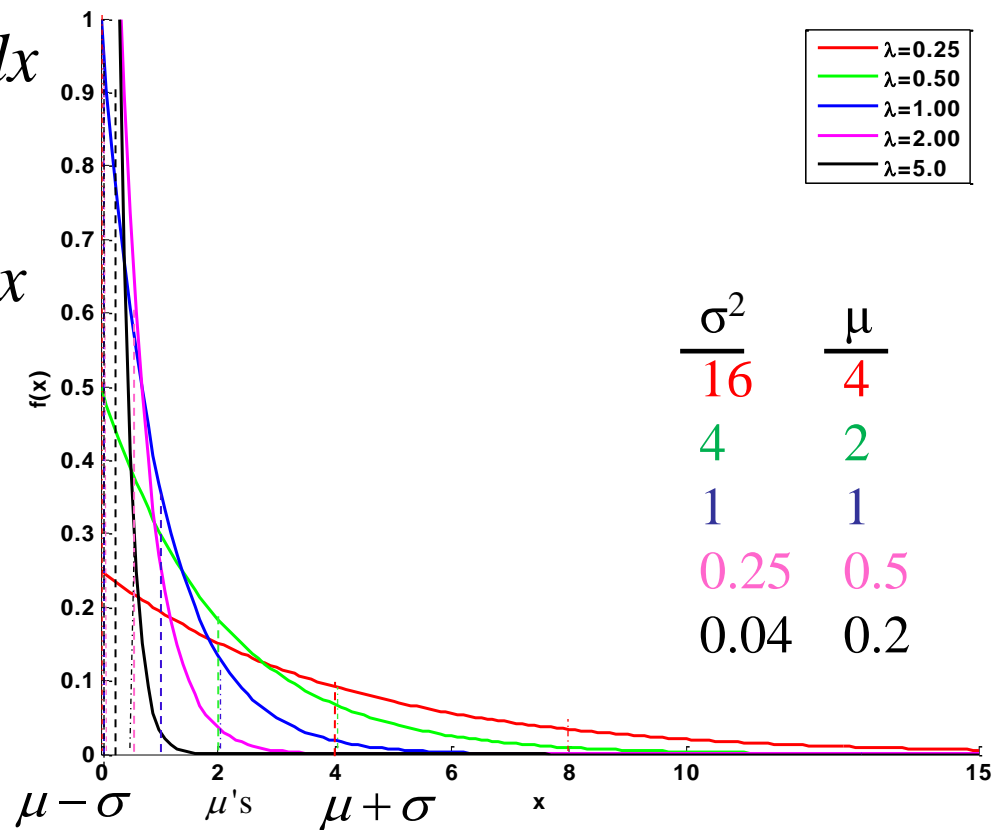
\hat{x}
0
0
0
0
0

Continuous Distributions

Exponential: $f(x | \lambda) = \lambda e^{-\lambda x}$

that

$$\begin{aligned}\sigma^2 &= \int_x (x - \mu)^2 f(x | \theta) dx \\ &= \int_{x=0}^{\infty} (x - \mu)^2 \lambda e^{-\lambda x} dx \\ &= \frac{1}{\lambda^2}\end{aligned}$$

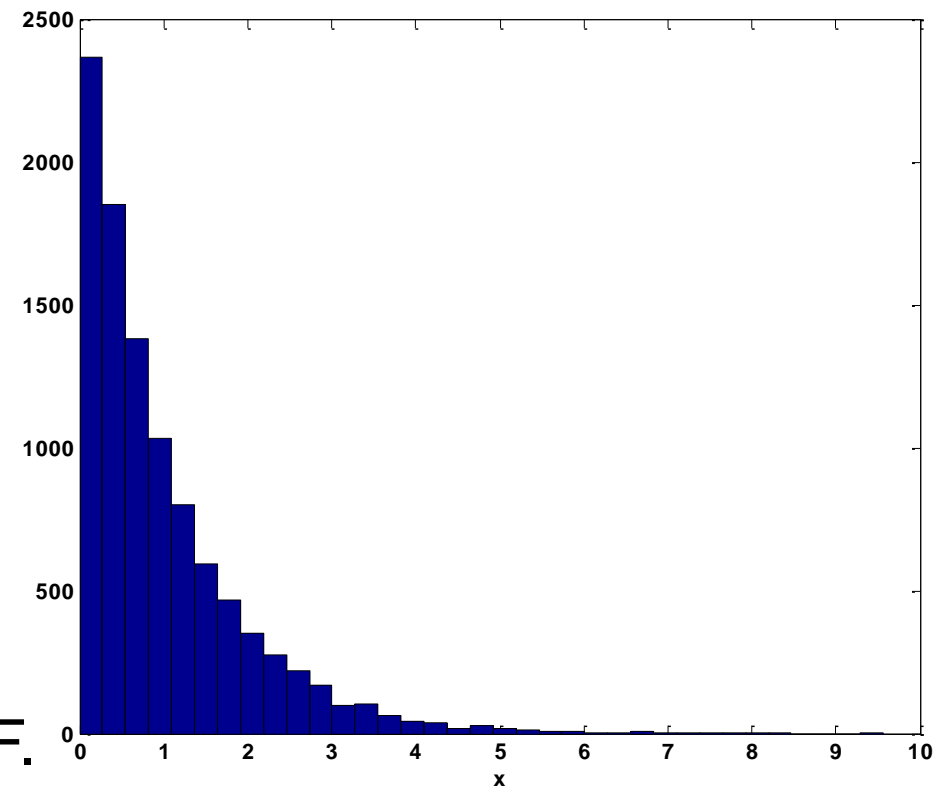


Continuous Distributions

Exponential: $f(x | \lambda) = \lambda e^{-\lambda x}$

```
lambda=1;,num=10^4;
x=exprnd(1/lambda,num,1);
mean(x)
var(x)
hist(x,35)
```

	True	Simulated
μ	1	1.0018
σ^2	1	1.0127

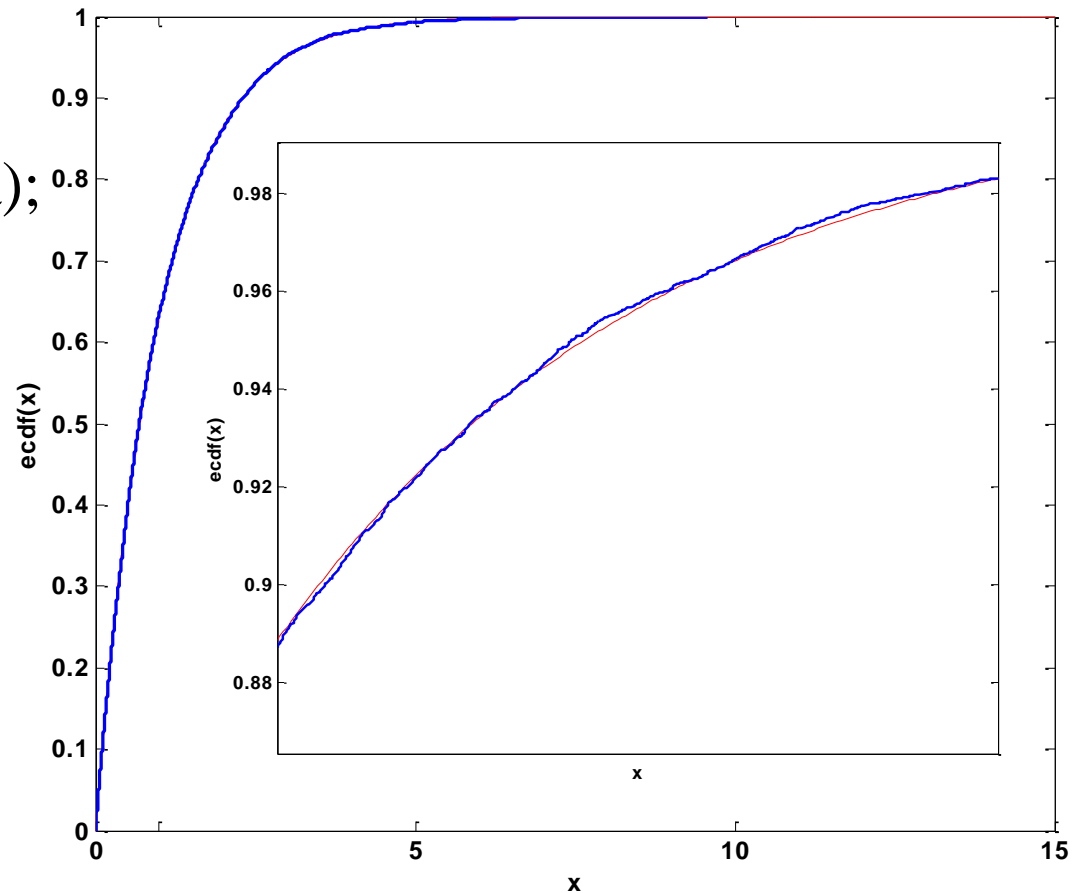


Can also find and plot ECDF.

Continuous Distributions

Exponential: $f(x | \lambda) = \lambda e^{-\lambda x}$

```
lambda=1;  
y=expcdf((0:.01:15),1/lambda);  
plot((0:.01:15),y, 'r')  
hold on  
[F,xx]=ecdf(x);  
stairs(xx,F,'LineWidth',2)
```



Continuous Distributions

Chi-Square:

A random variable x has a continuous chi-square distribution, $x \sim \text{chi-square}(\nu)$ if

$$f(x | \nu) = \begin{cases} 0 & x \leq 0 \\ \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}} & x > 0 \end{cases}, \text{ where, } \nu > 0 .$$

Chi-square distribution is a special case of the gamma distribution with $\nu=2\alpha$ and $\beta=2$.

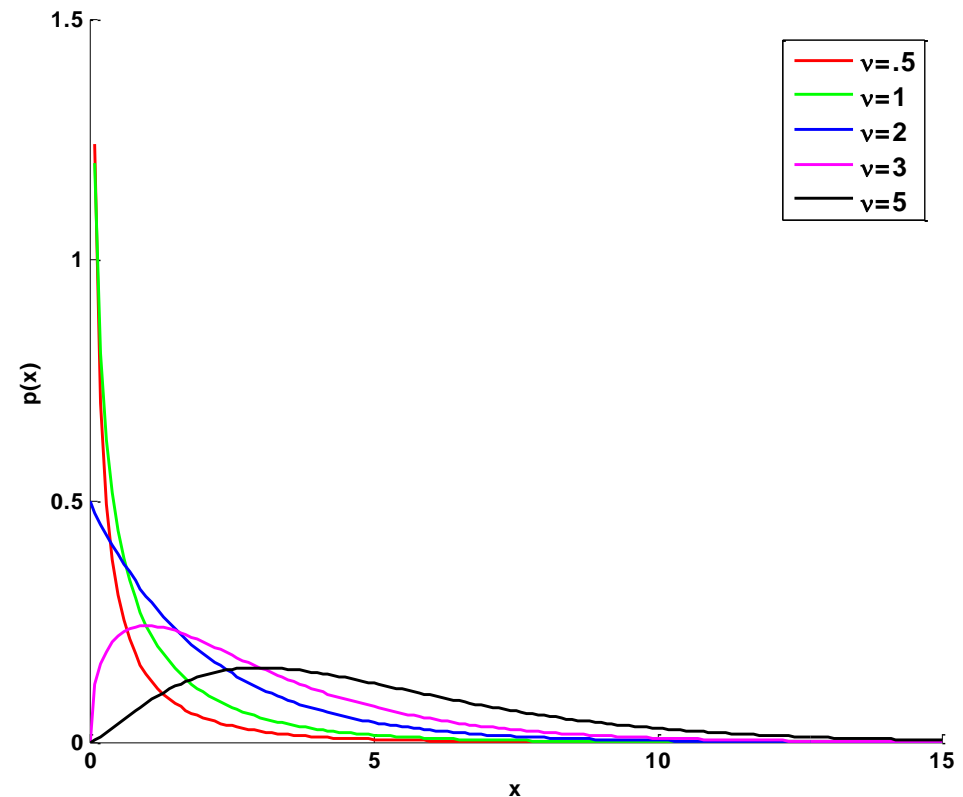
$$f(x | \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

Continuous Distributions

Chi-Square:

```
x=(0:.1:15)'; nu=[.5,1,2,3,5];
figure(1)
hold on
for count=1:length(nu)
    y = chi2pdf(x,nu(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
ylim([0 1.5]), xlim([0 15])
```

$$f(x|\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$$



Continuous Distributions

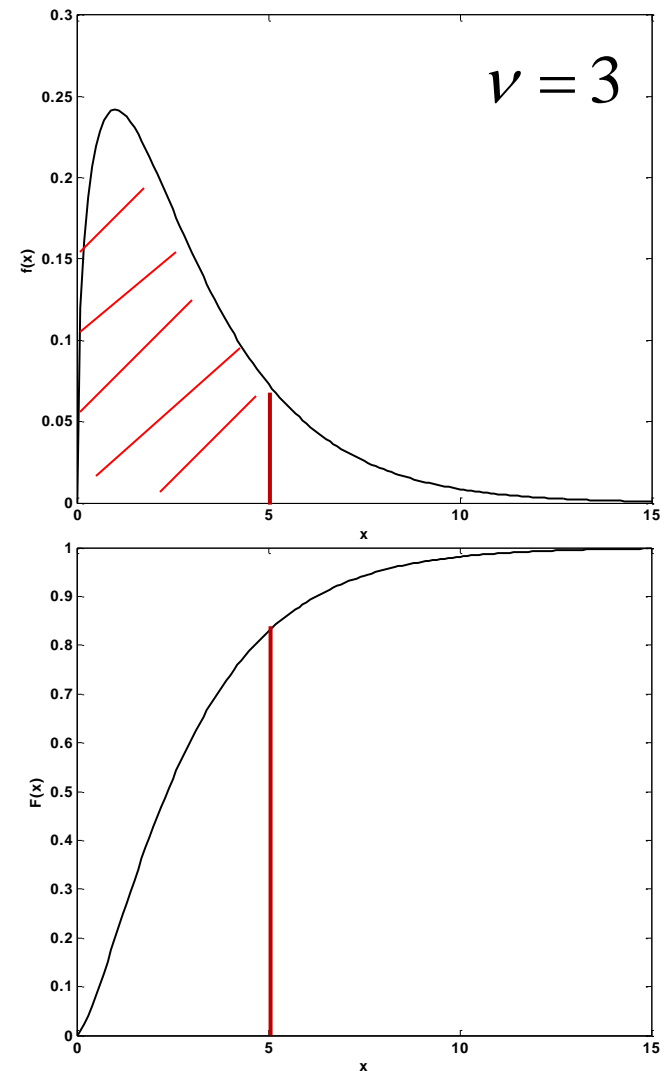
Chi-Square:

$$f(x | \nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$$

The CDF of the continuous chi-square distribution is

$$F(x | \nu) = \begin{cases} 0 & x \leq 0 \\ \frac{\gamma(\nu/2, x/2)}{\Gamma(\nu/2)} & x > 0 \end{cases}$$

Where $\gamma(\nu, x/2)$ is the lower incomplete gamma function.



Continuous Distributions

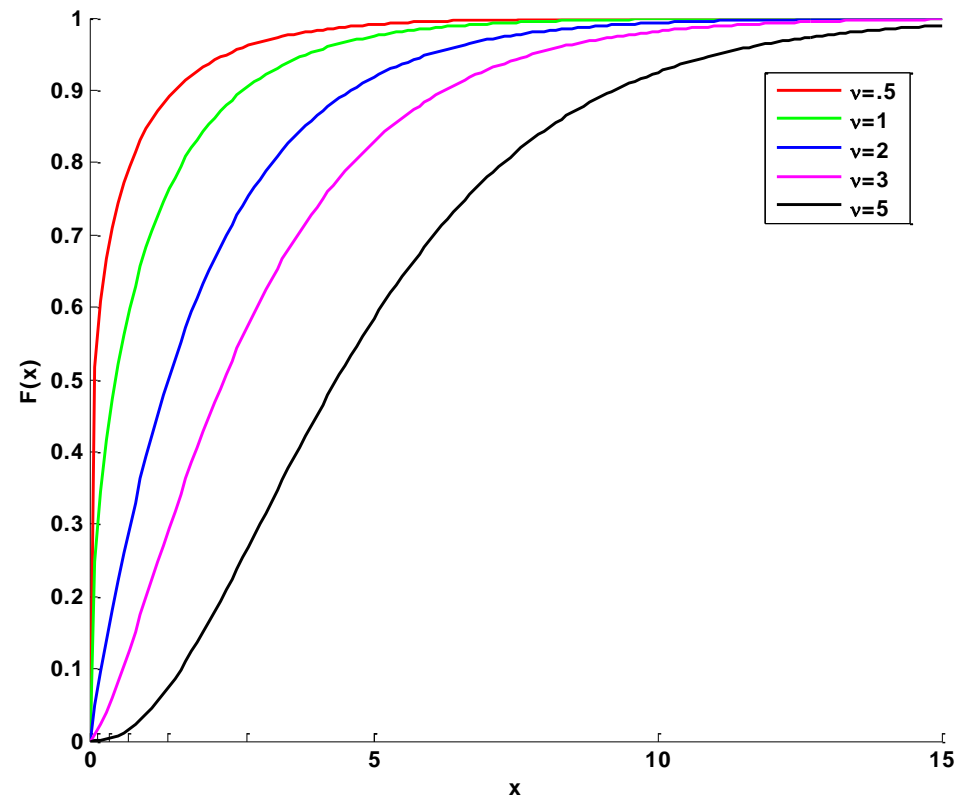
Chi-Square:

```

x=(0:.1:15)'; nu=[.5,1,2,3,5];
figure(1)
hold on
for count=1:length(nu)
    y = chi2cdf(x,nu(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 15]) , ylim([0 1])

```

$$f(x | \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

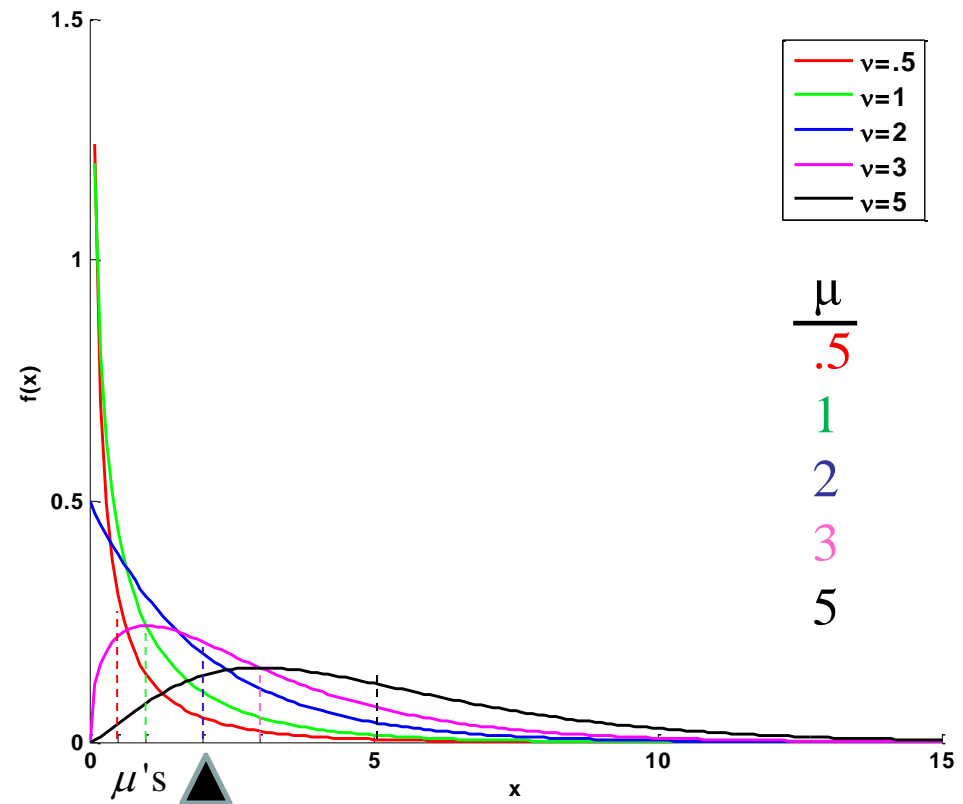


Continuous Distributions

Chi-Square:

It can be shown that

$$\begin{aligned}\mu &= \int_x x f(x | \theta) dx \\ &= \int_{x=0}^{\infty} x \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}} dx \\ &= \nu\end{aligned}$$



Continuous Distributions

Chi-Square: $f(x|\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$

It can be shown that

median

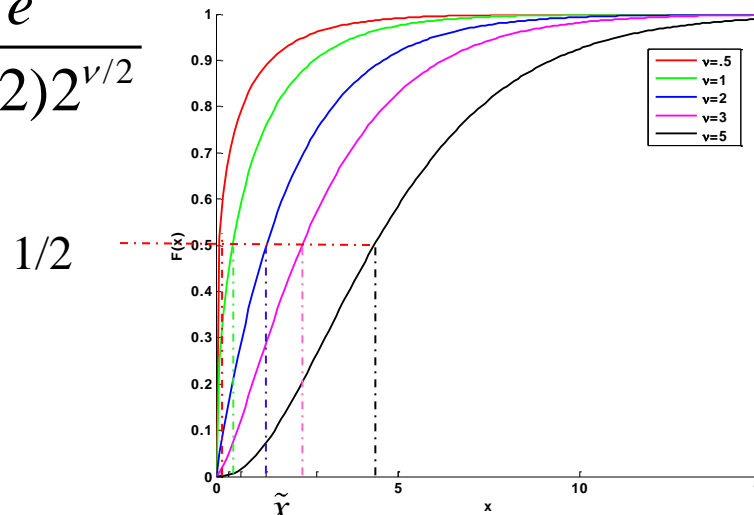
$$\int_{x=0}^{\tilde{x}} f(x|\theta) dx = \frac{1}{2}$$

$$\tilde{x} \approx \nu - \frac{2}{3} + \frac{4}{27\nu} - \frac{8}{729\nu^2}$$

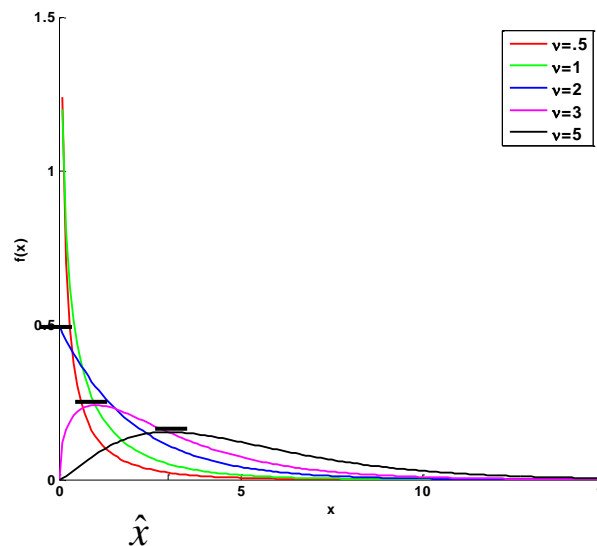
mode

$$\left. \frac{\partial}{\partial x} f(x|\theta) \right|_{\hat{x}} = 0$$

$$\hat{x} = \nu - 2 \quad \nu \geq 2$$



\tilde{x}
0.0857
0.4705
1.4047
2.3815
4.3625



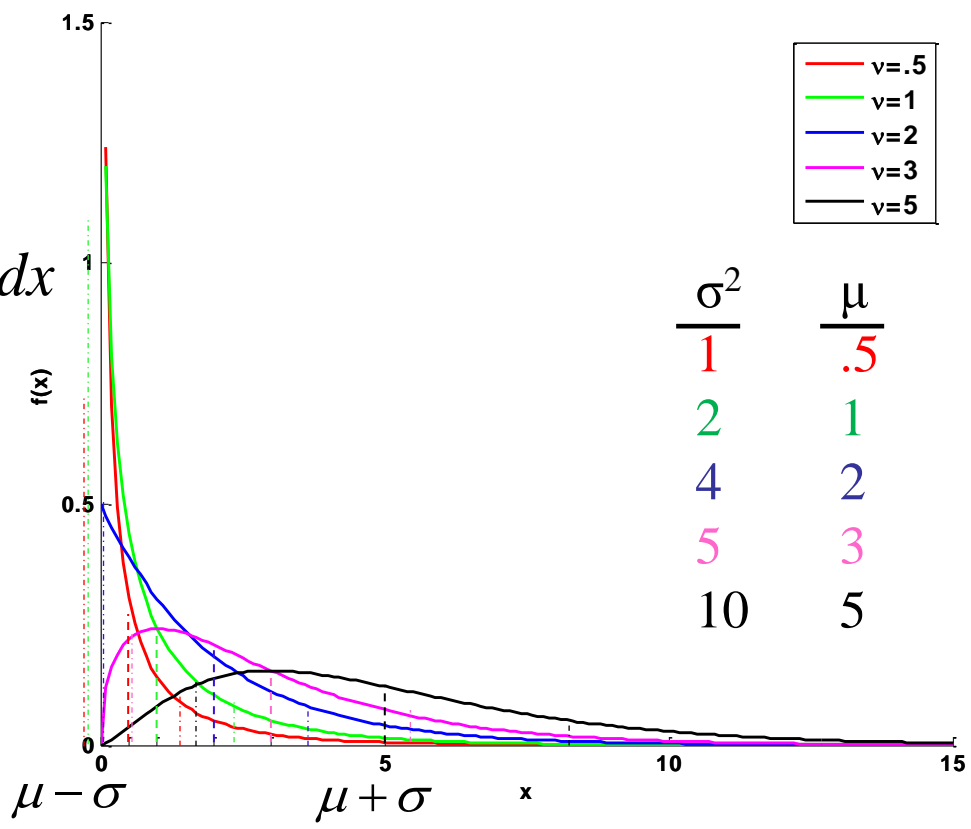
\hat{x}
-
-
0
1
3

Continuous Distributions

Chi-Square: $f(x|\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$

that

$$\begin{aligned} \sigma^2 &= \int_x (x - \mu)^2 f(x|\theta) dx \\ &= \int_{x=0}^{\infty} (x - \mu)^2 \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}} dx \\ &= 2\nu \end{aligned}$$



Continuous Distributions

Chi-Square:
$$f(x|\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$$

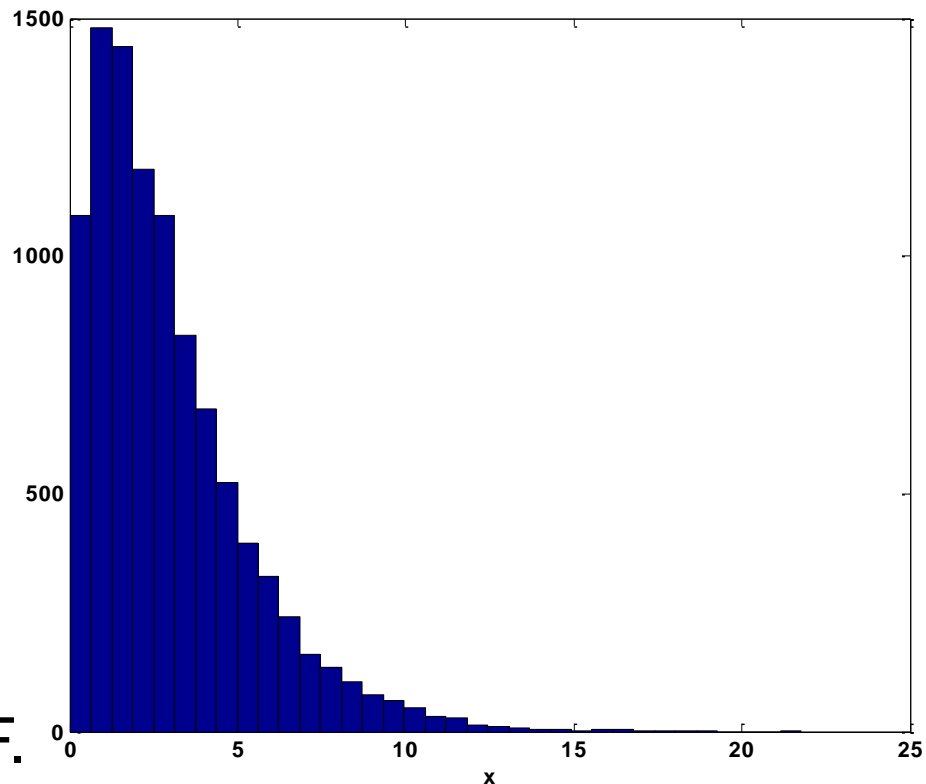
```

nu=3;,num=10^4;
x=chi2rnd(nu,num,1);
mean(x)
var(x)
hist(x,35)

```

	True	Simulated
μ	3	2.9802
σ^2	6	5.7426

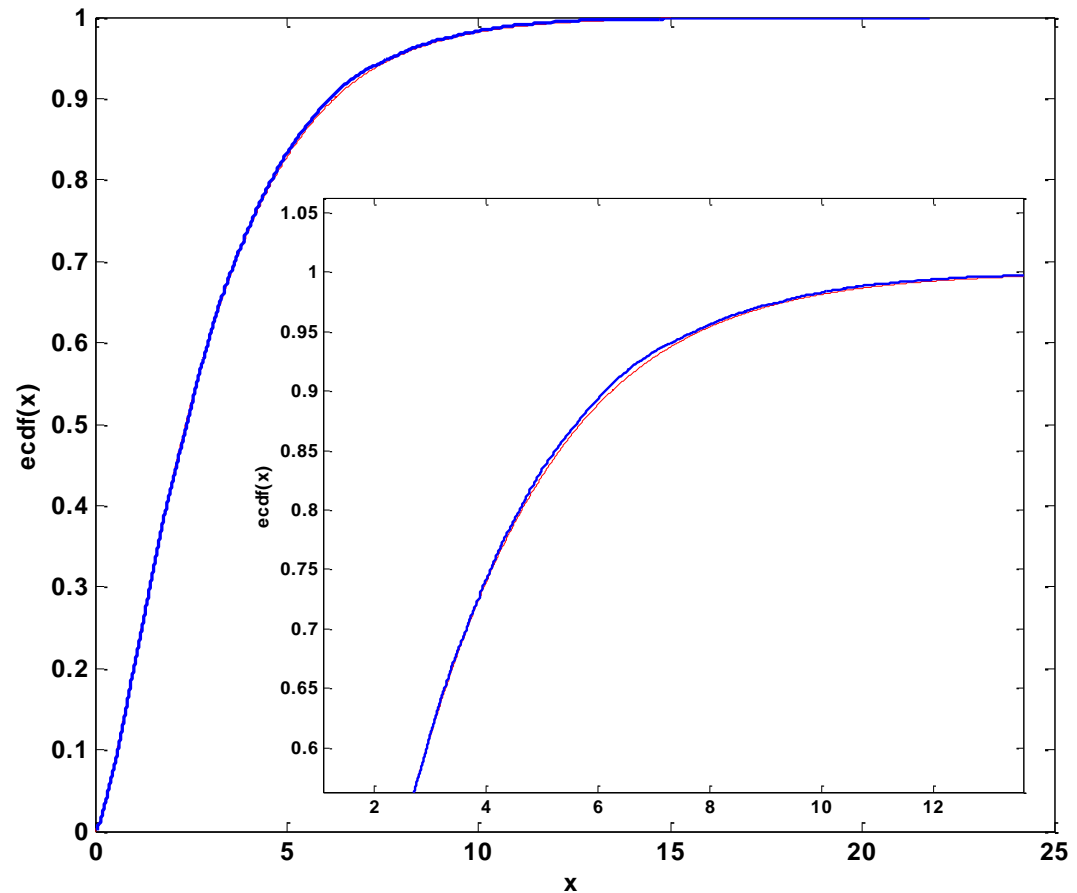
Can also find and plot ECDF.



Continuous Distributions

Chi-Square:
$$f(x|\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$$

```
nu=3;  
y=chi2cdf((0:.01:15),nu);  
plot((0:.01:15),y, 'r')  
hold on  
[F,xx]=ecdf(x);  
stairs(xx,F,'LineWidth',2)
```



Homework 4:

- 1) Let $x \sim \text{gamma}(2,2)$, using pencil and paper find $P(\mu - \sigma \leq x \leq \mu + \sigma)$.
- 2) Numerically integrate the $\text{gamma}(2,2)$ pdf to find $P(\mu - \sigma \leq x \leq \mu + \sigma)$. Use 100 rectangles.
- 3) Let $x \sim \text{gamma}(2,2)$, using pencil and paper find μ .
- 4) Numerically integrate the $\text{gamma}(2,2)$ pdf to find μ . Use 100 rectangles. Compare to answer in 3).

Homework 4:

5) Show that the mode of the gamma distribution is

$$\hat{x} = (\alpha - 1)\beta . \text{ Pencil and paper.}$$

6) Can you numerically determine an estimate of the mode? Use $\alpha=2, \beta=2$. Compare to 5).

7) Numerically integrate the gamma pdf to find the 50th and 99th percentiles. i.e. find x_0 such that

$$P(x \leq x_0) = 0.50 \text{ and } P(x \leq x_0) = 0.99 \text{ for } \alpha=2, \beta=2.$$

Compare median in 7) to mean in 3,4) & mode in 5,6).

Homework 4:

- 8) Generate 10^6 gamma $\alpha=2, \beta=2$ random variables. Empirically determine the 50th and 99th percentiles. i.e. Find the $.50 \cdot 10^6$ and $.99 \cdot 10^6$ largest values x_0 such that $P(x \leq x_0) \approx 0.50$ and $P(x \leq x_0) \approx 0.99$.

Compare to values to 7) and the approximate median $\alpha\beta \frac{3\alpha - 0.8}{3\alpha + 0.2}$.

Homework 4:

- 9) Make a histogram of the random variables in 8) with at least 50 bins. Divide the count in each bin by the total number 10^6 . Compare to or superimpose the true pdf.
- 10) Make a empirical CDF from the values in 8).
- 11) Using pencil and paper derive the mean, median, and variance for the exponential distribution.

Homework 4:

12) Can you analytically derive the approximate median for the chi-square distribution?