Continuous Probability Functions
(Cont.)

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Outline

- **Continuous Gamma Distribution**
  PDF, Moments, CDF, Matlab

- **Continuous Exponential Distribution**
  PDF, Moments, CDF, Matlab

- **Continuous Chi-square Distribution**
  PDF, Moments, CDF, Matlab

Continuous Distributions

Gamma:

A random variable x has a continuous gamma distribution, x~gamma(α,β) if

$$f(x|\alpha,\beta)=\begin{cases} 0 & x\le 0 \\ x^{\alpha-1}\dfrac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & x>0 \end{cases},$$

where, $\alpha,\beta>0$.

$$\Gamma(\alpha)=\int_{t=0}^{\infty}t^{\alpha-1}e^{-t}dt$$

$$\Gamma(\alpha)=(\alpha-1)!$$

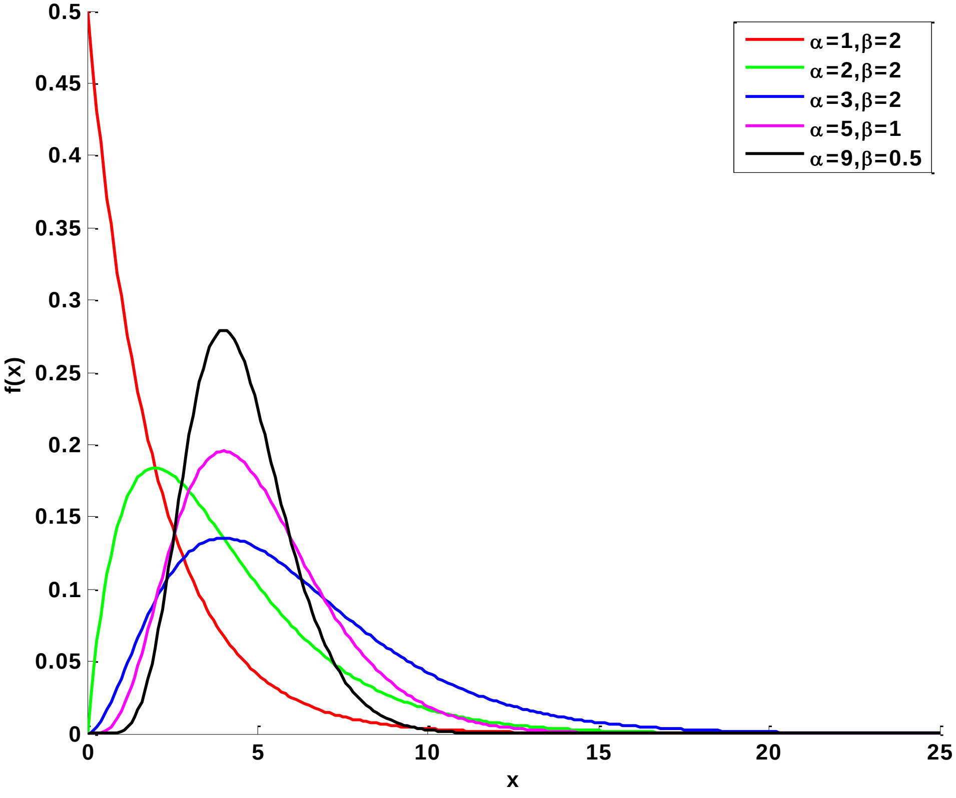
If α a pos integer

Continuous Distributions

Gamma:

$$f(x|\alpha,\beta) = x^{\alpha-1}\frac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

```
x=(0:.1:50)';
alpha=[1,2,3,5,9];, beta=[2,2,2,1,.5];
figure(1)
hold on
for count=1:5
    y = gampdf(x,alpha(count),beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 25])
```



Continuous Distributions

Gamma:

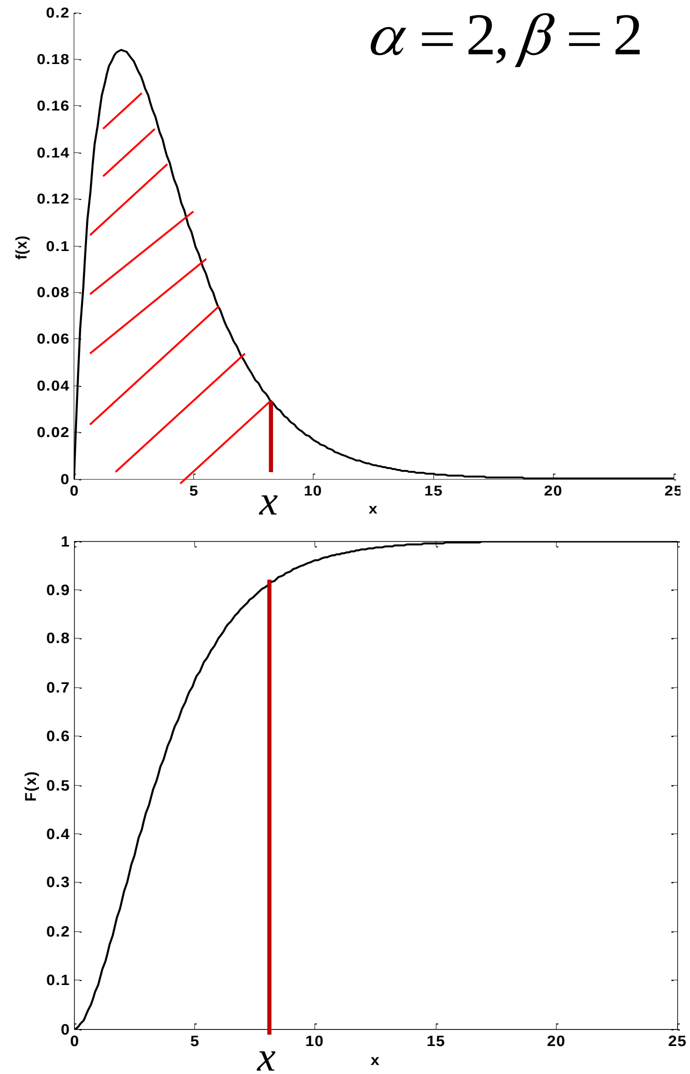
The CDF of the continuous gamma distribution is

$$F(x|\theta) = \int_{t=-\infty}^{x} f(t|\theta)dt$$

$$F(x|\alpha,\beta) = \int_{t=0}^{x} t^{\alpha-1}\frac{e^{-t/\beta}}{\beta^{\alpha}\Gamma(\alpha)}dt$$

$$= \begin{cases} 0 & x \le 0 \\ \dfrac{\gamma(\alpha, x/\beta)}{\Gamma(\alpha)} & x > 0 \end{cases}$$

Where $\gamma(\alpha,x/\beta)$ is the lower incomplete gamma function.

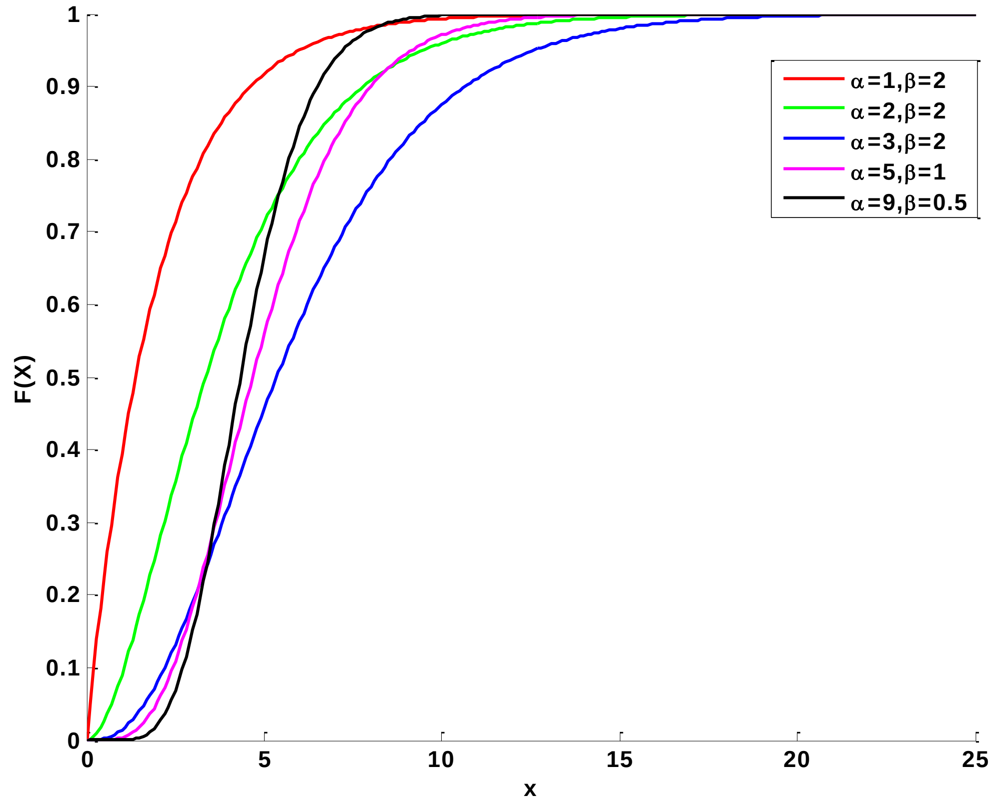


Continuous Distributions

Gamma:

$$f(x\,|\,\alpha,\beta) = x^{\alpha-1}\frac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

```
x=(0:.1:50)';
alpha=[1,2,3,5,9];, beta=[2,2,2,1,.5];
figure(1)
hold on
for count=1:5
    y = gamcdf(x,alpha(count),beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 25])
```



Continuous Distributions

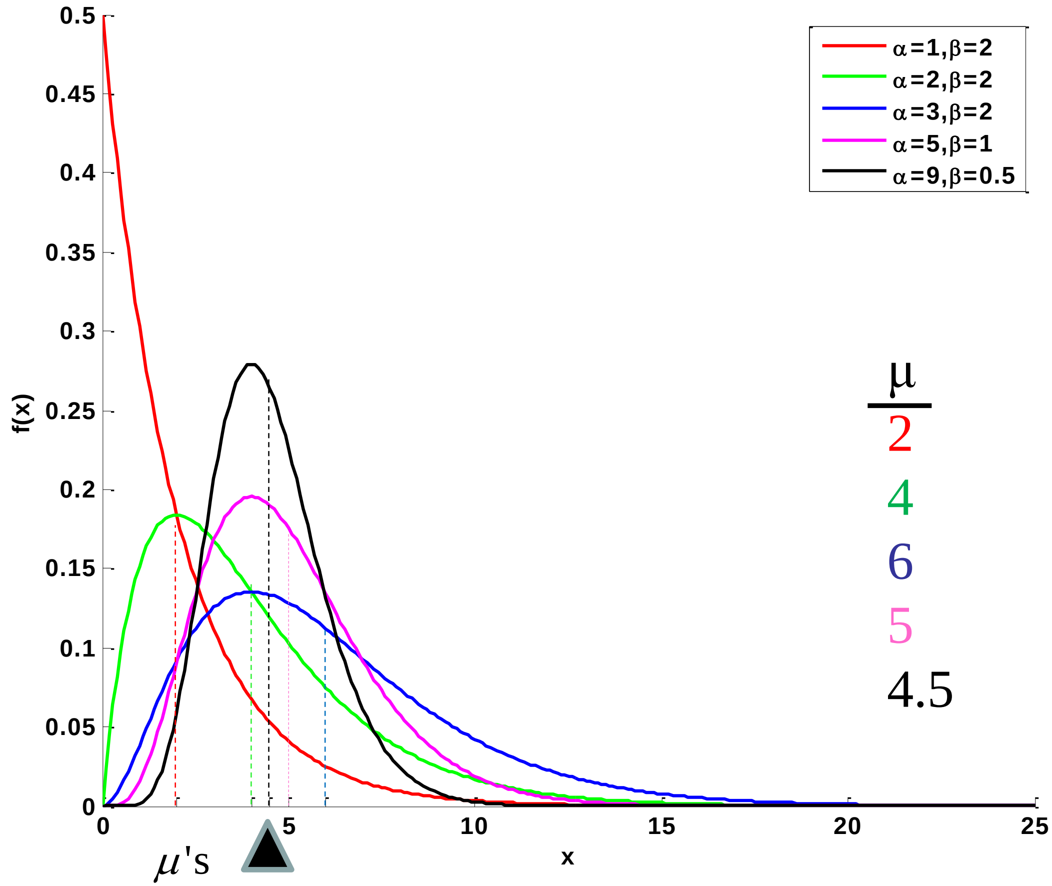
Gamma:

It can be shown that

$$\mu = \int_x x f(x|\theta) dx$$

$$= \int_{x=0}^{\infty} x \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} dx$$

$$= \alpha\beta$$



Continuous Distributions

Gamma:

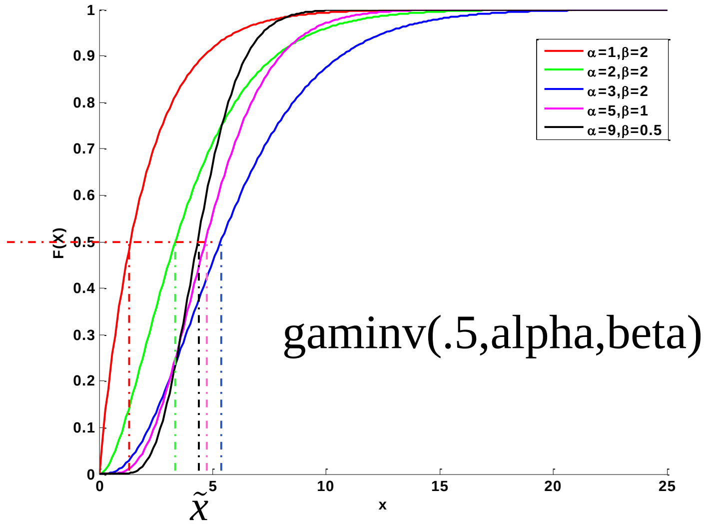
It can be shown that

median

$$\int_{x=0}^{\tilde{x}} f(x|\theta)dx = \frac{1}{2}$$

$\tilde{x}$ has no closed form solution

1/2



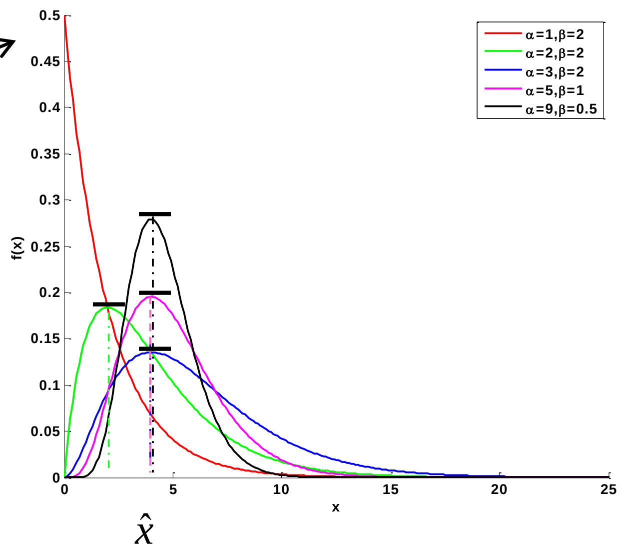
$\tilde{x}$
1.3863
3.3567
5.3481
4.6709
4.3345

mode

$$\left.\frac{\partial}{\partial x} f(x|\theta)\right|_{\hat{x}} = 0$$

$$\hat{x} = (\alpha - 1)\beta \qquad \alpha \geq 1$$

Check boundary points for max.



$\hat{x}$
0
2
4
4
4

Continuous Distributions

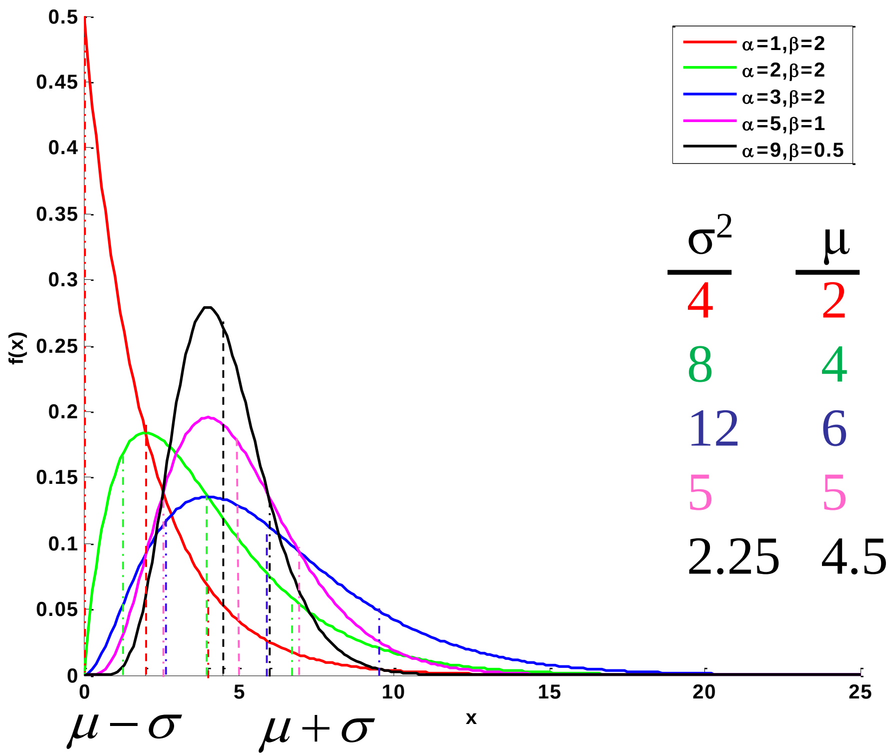
Gamma:

that

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=0}^{\infty} (x-\mu)^2 \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx$$

$$= \alpha\beta^2$$

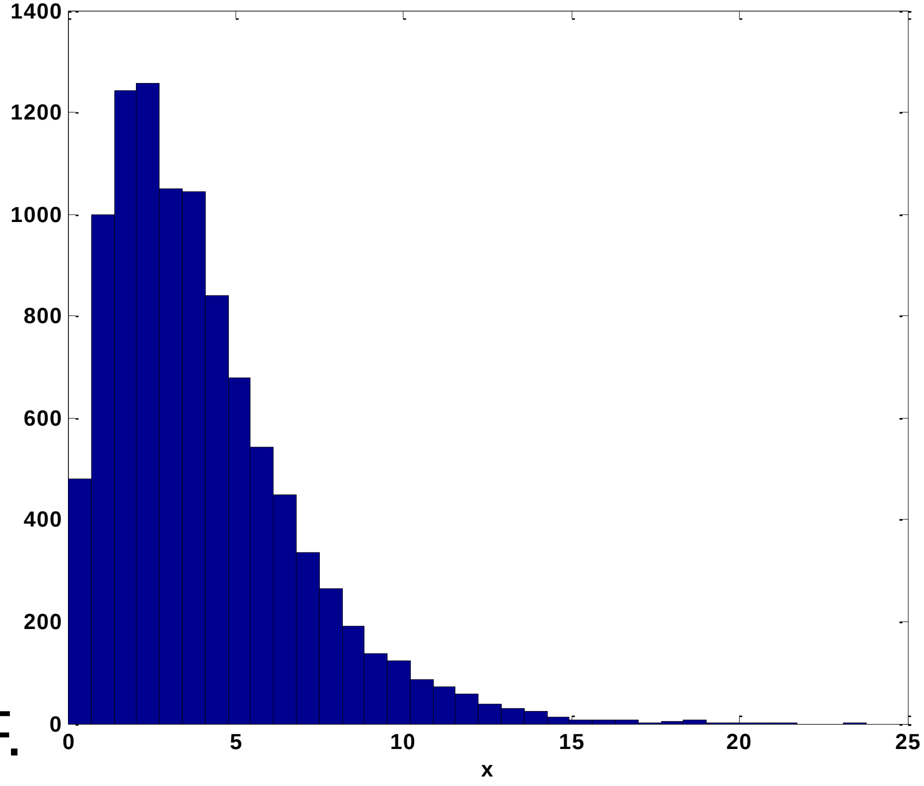


Continuous Distributions

Gamma: $$f(x\,|\,\alpha,\beta)=\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

```
alpha=2;,beta=2;,num=10^4;
x=gamrnd(alpha,beta,num,1);
mean(x)
var(x)
hist(x,35)
```

	True	Simulated
μ	4	3.9841
σ^2	8	7.7597

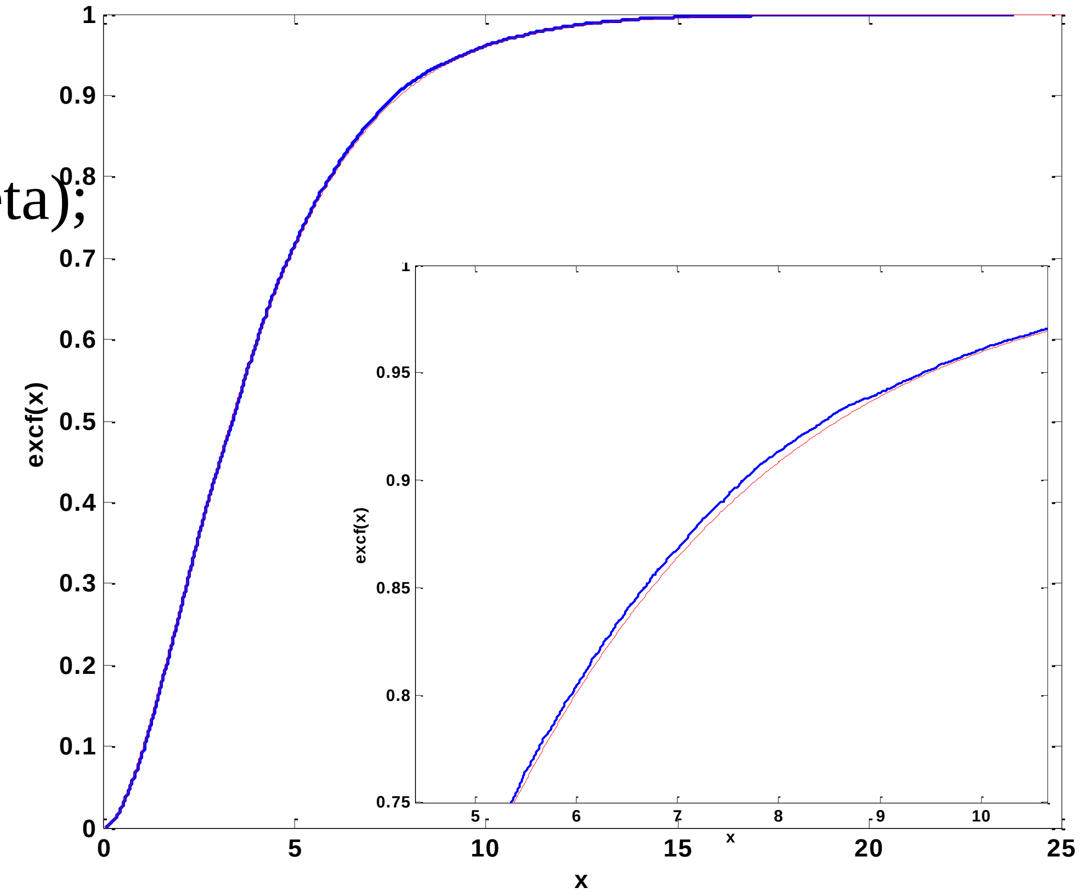


Can also find and plot ECDF.

Continuous Distributions

Gamma: $f(x \mid \alpha, \beta) = \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}$

```
alpha=2;,beta=2;
y=gamcdf((0:.01:25),alpha,beta);
plot((0:.01:25),y, 'r')
hold on
[F,xx]=ecdf(x);
stairs(xx,F,'LineWidth',2)
```



Continuous Distributions

Exponential:

A random variable x has a continuous exponential distribution, x~exponential(λ) if

$$f(x|\lambda)=\begin{cases} 0 & x\le 0 \\ \lambda e^{-\lambda x} & x>0 \end{cases}, \quad \text{where, } \lambda>0 .$$

Exponential distribution is a special case of the gamma distribution with α=1 and $\lambda\equiv 1/\beta$.

$$f(x|\alpha,\beta)=x^{\alpha-1}\frac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

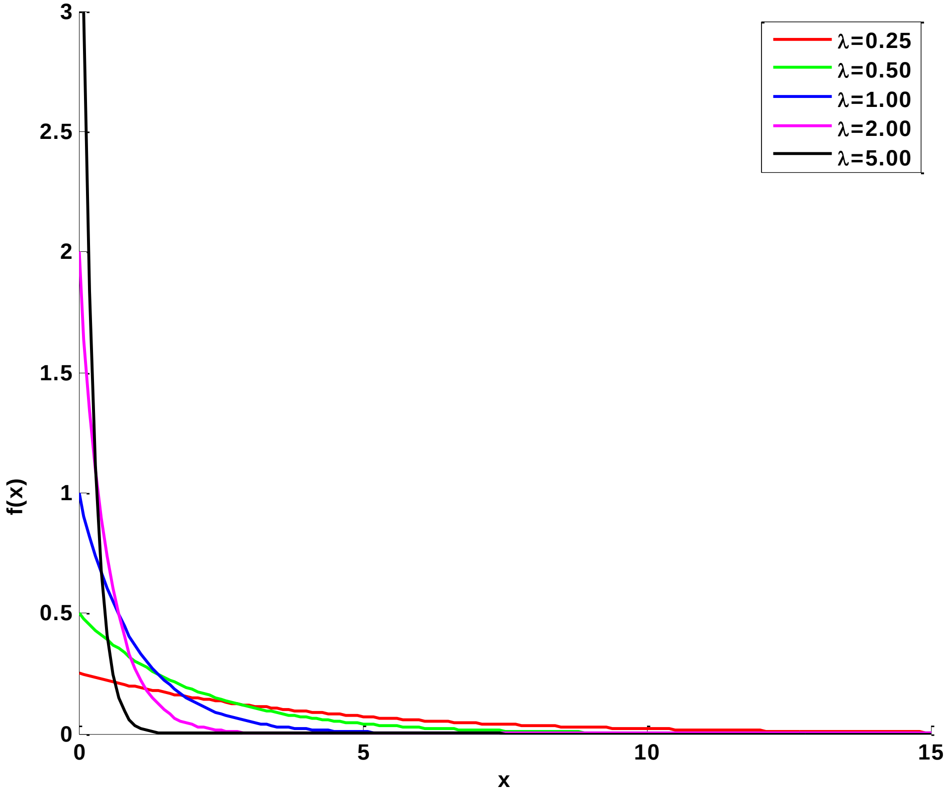
Continuous Distributions

Exponential:

Matlab uses gamma dist def

$$f(x \mid \lambda) = \lambda e^{-\lambda x}$$

```
x=(0:.1:15)';
lambda=[.25,.5,1,2,5];, beta=1./lambda;
figure(1)
hold on
for count=1:length(lambda)
    y = exppdf(x,beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
ylim([0 3]), xlim([0 15])
```



Continuous Distributions

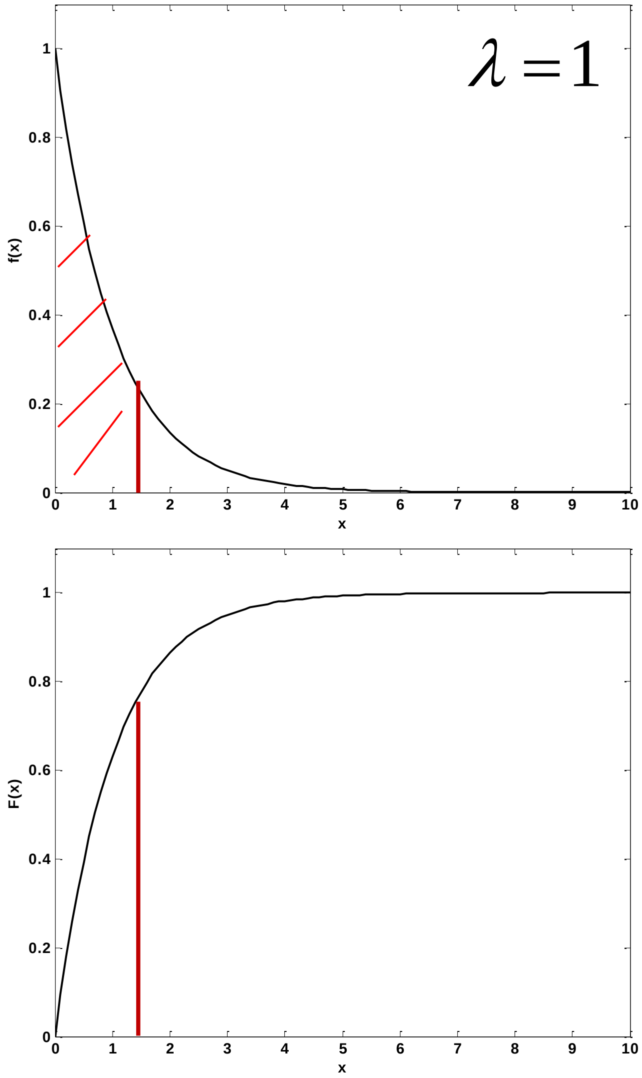
Exponential:

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

The CDF of the continuous exponential distribution is

$$F(x|\lambda) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

Note that there is a closed form solution!

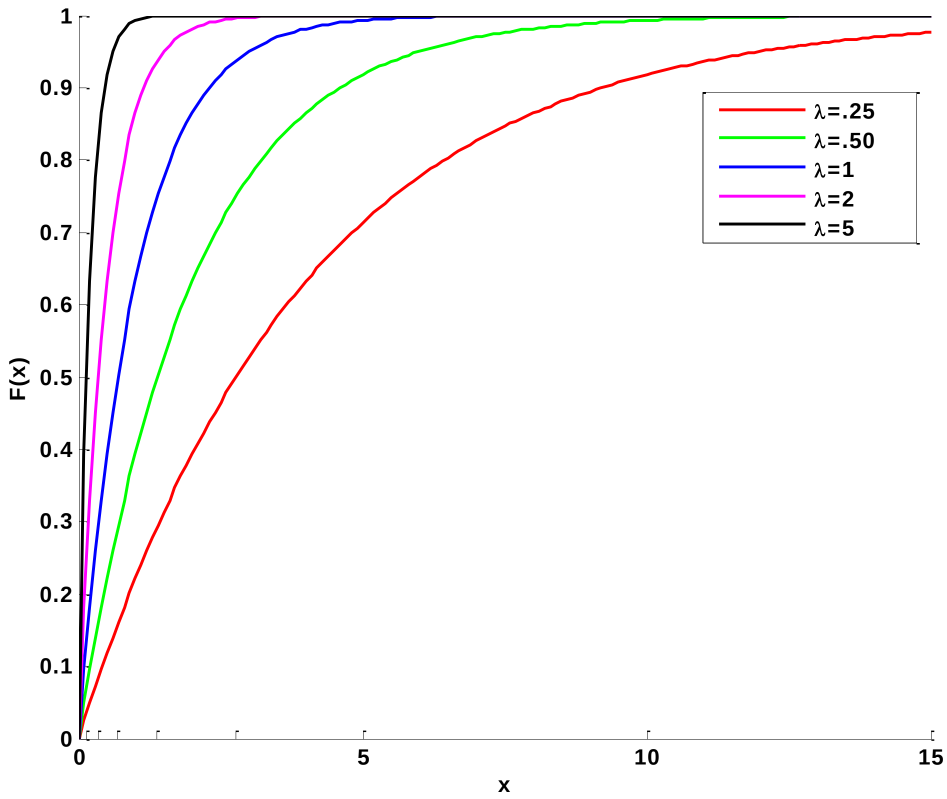


Continuous Distributions

Exponential:

$$f(x|\lambda)=\lambda e^{-\lambda x}$$

```
x=(0:.1:15)';
lambda=[.25,.5,1,2,5];, beta=1./lambda;
figure(1)
hold on
for count=1:length(lambda)
    y = expcdf(x,beta(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 15]) ,ylim([0 1])
```



Continuous Distributions

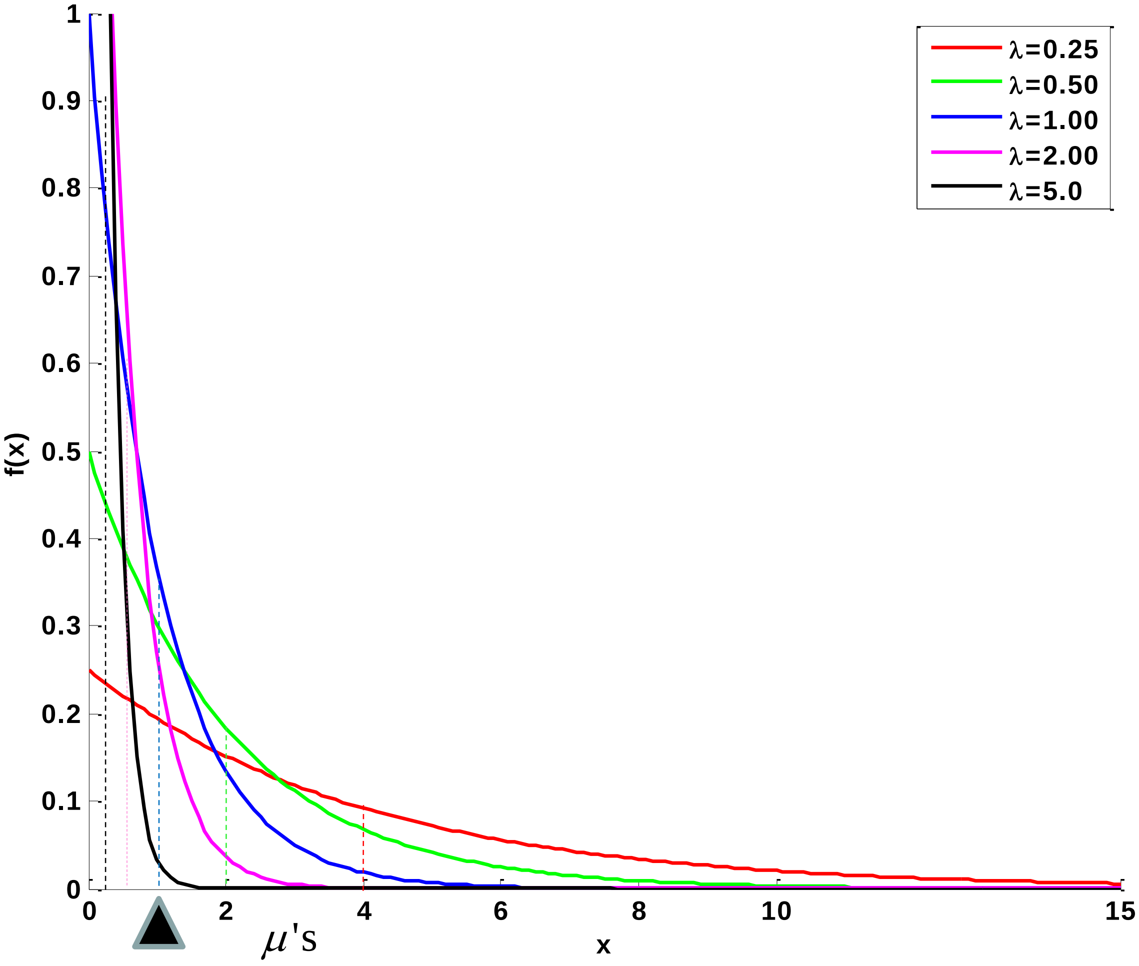
Exponential:

It can be shown that

$$\mu = \int_x x f(x|\theta) dx$$

$$= \int_{x=0}^{\infty} x\lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda}$$



Continuous Distributions

Exponential: $f(x|\lambda)=\lambda e^{-\lambda x}$

It can be shown that

median

$$\int_{x=0}^{\tilde{x}} f(x|\theta)dx=\frac{1}{2}$$

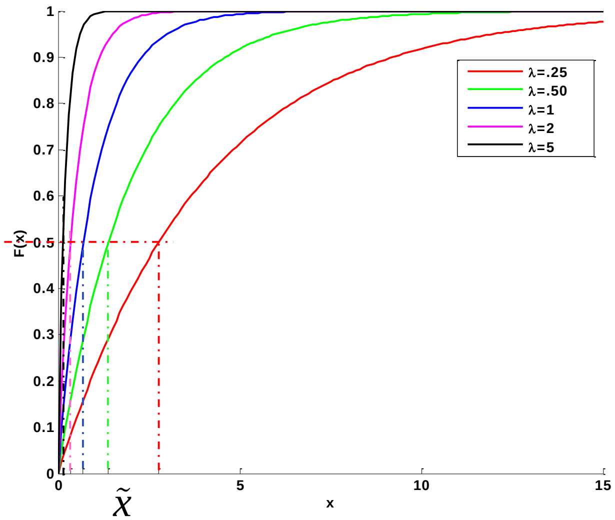
$$\tilde{x}=\frac{\ln(2)}{\lambda}$$

mode

$$\frac{\partial}{\partial x} f(x|\theta)\Big|_{\hat{x}}=0$$

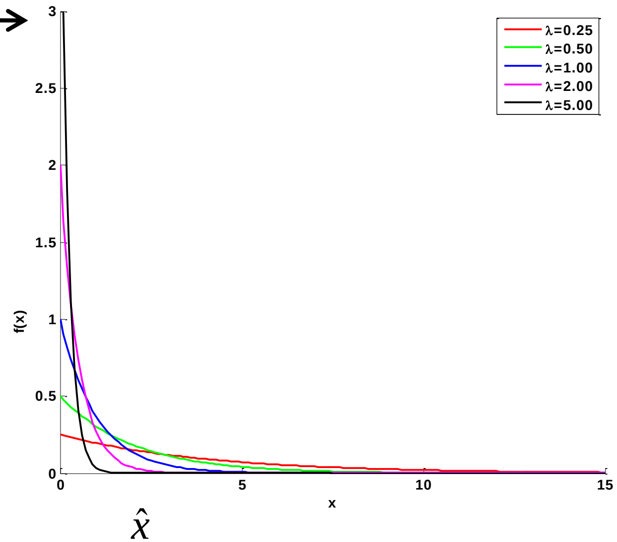
$$\hat{x}=0$$

1/2



$\tilde{x}$
2.7726
1.3863
0.6931
0.3466
0.1386

All at 0! →



$\hat{x}$
0
0
0
0
0

Continuous Distributions

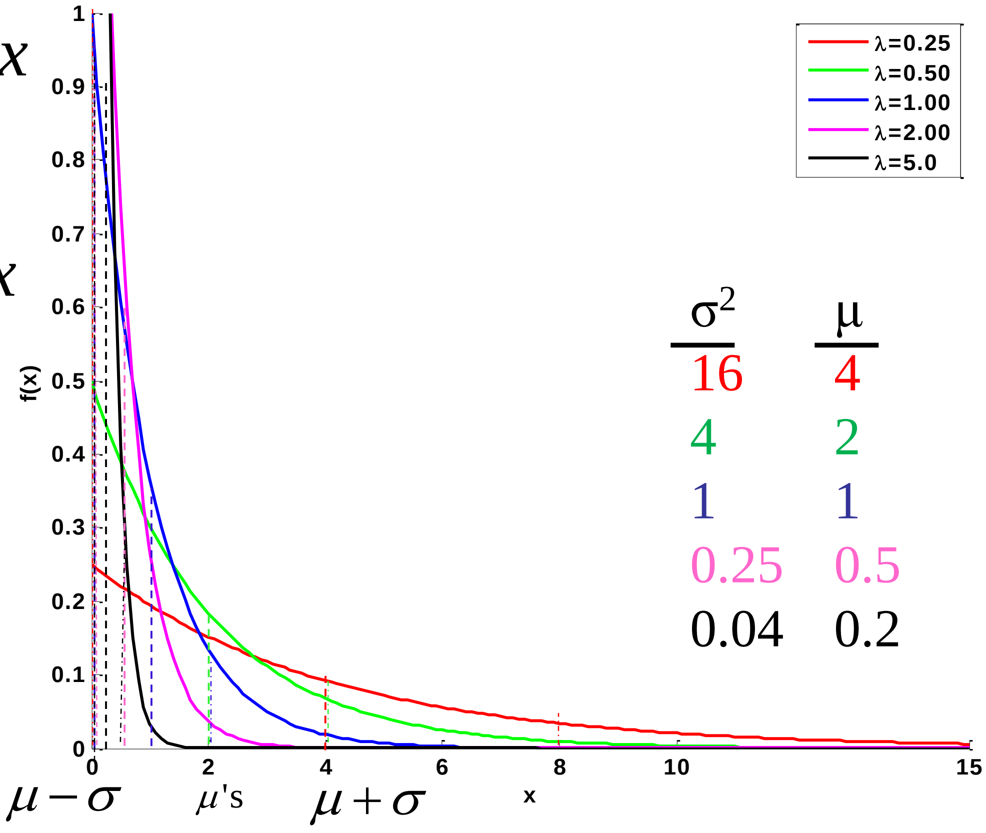
Exponential: $f(x|\lambda)=\lambda e^{-\lambda x}$

that

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=0}^{\infty} (x-\mu)^2 \lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda^2}$$



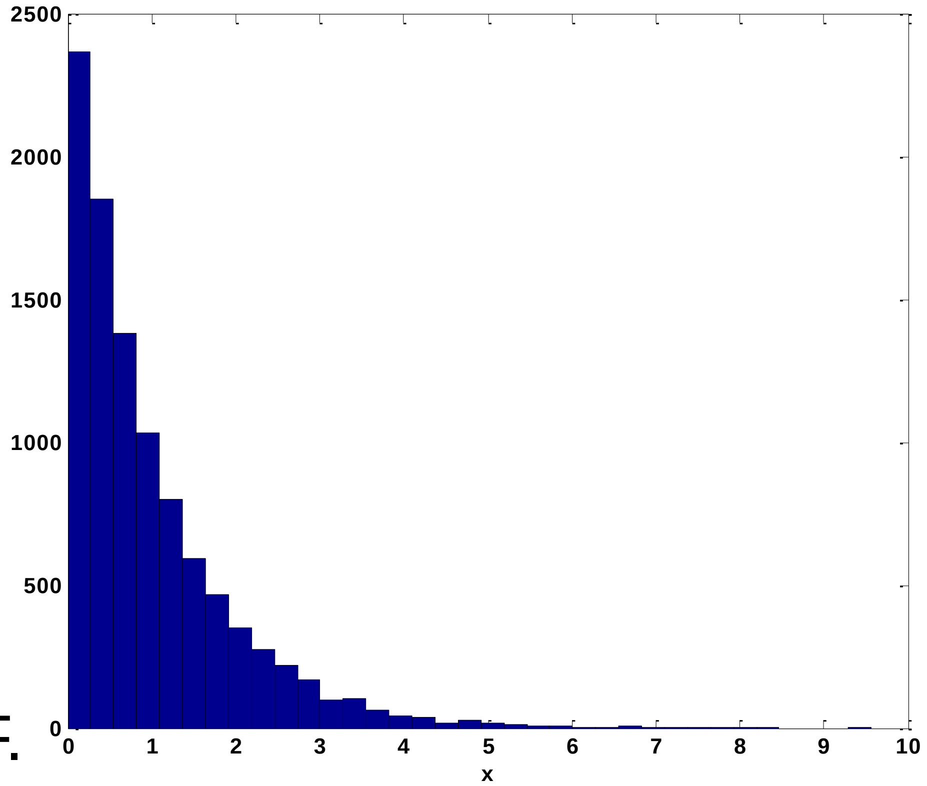
σ^2	μ
16	4
4	2
1	1
0.25	0.5
0.04	0.2

Continuous Distributions

Exponential: $f(x \mid \lambda) = \lambda e^{-\lambda x}$

```
lambda=1;,num=10^4;
x=exprnd(1/lambda,num,1);
mean(x)
var(x)
hist(x,35)
```

	True	Simulated
μ	1	1.0018
σ^2	1	1.0127

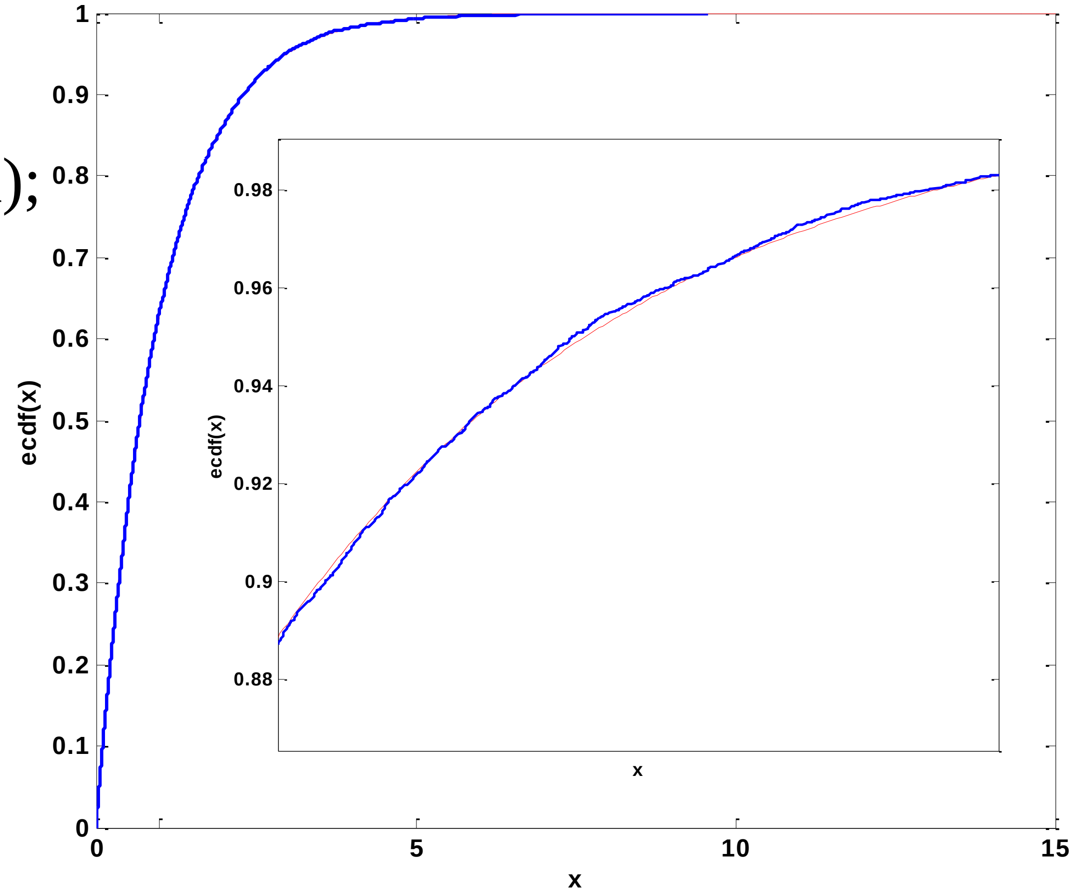


Can also find and plot ECDF.

Continuous Distributions

Exponential: $f(x\,|\,\lambda) = \lambda e^{-\lambda x}$

```
lambda=1;
y=expcdf((0:.01:15),1/lambda);
plot((0:.01:15),y, 'r')
hold on
[F,xx]=ecdf(x);
stairs(xx,F,'LineWidth',2)
```



Continuous Distributions

Chi-Square:

A random variable x has a continuous chi-square distribution, x~chi-square(ν) if

$$f(x|\nu)=\begin{cases} 0 & x\le 0 \\ \dfrac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}} & x>0 \end{cases}$$

, where, $\nu>0$.

Chi-square distribution is a special case of the gamma distribution with $\nu=2\alpha$ and $\beta=2$.

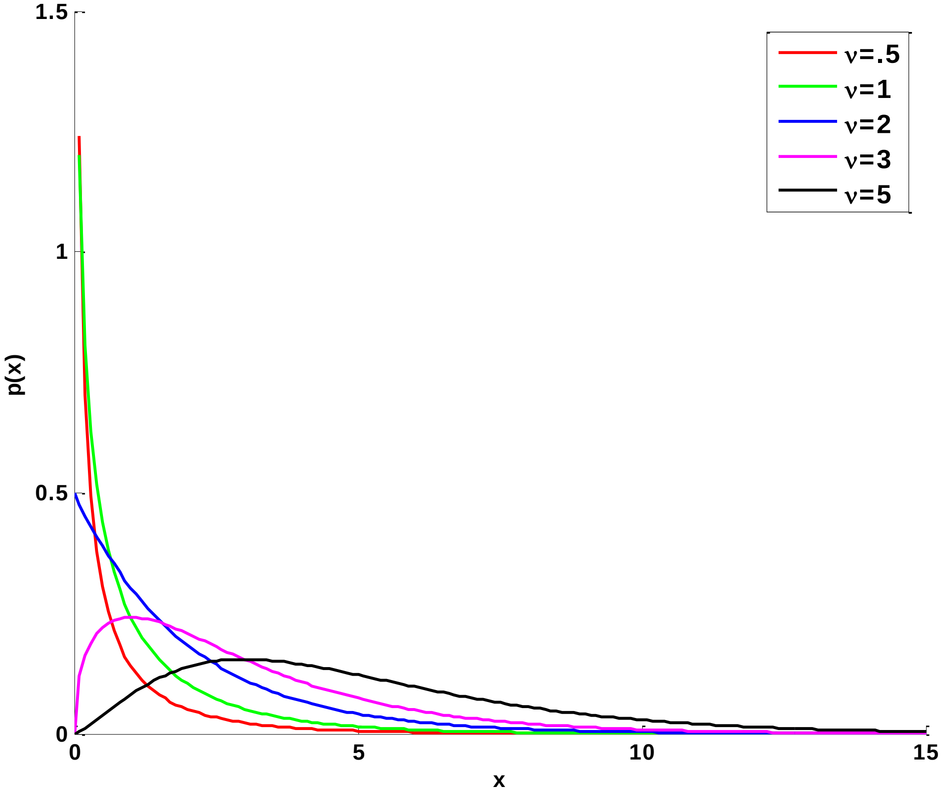
$$f(x|\alpha,\beta)=x^{\alpha-1}\frac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

Continuous Distributions

Chi-Square:

$$f(x|\nu)=\frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

```
x=(0:.1:15)';, nu=[.5,1,2,3,5];
figure(1)
hold on
for count=1:length(nu)
    y = chi2pdf(x,nu(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
ylim([0 1.5]), xlim([0 15])
```



Continuous Distributions

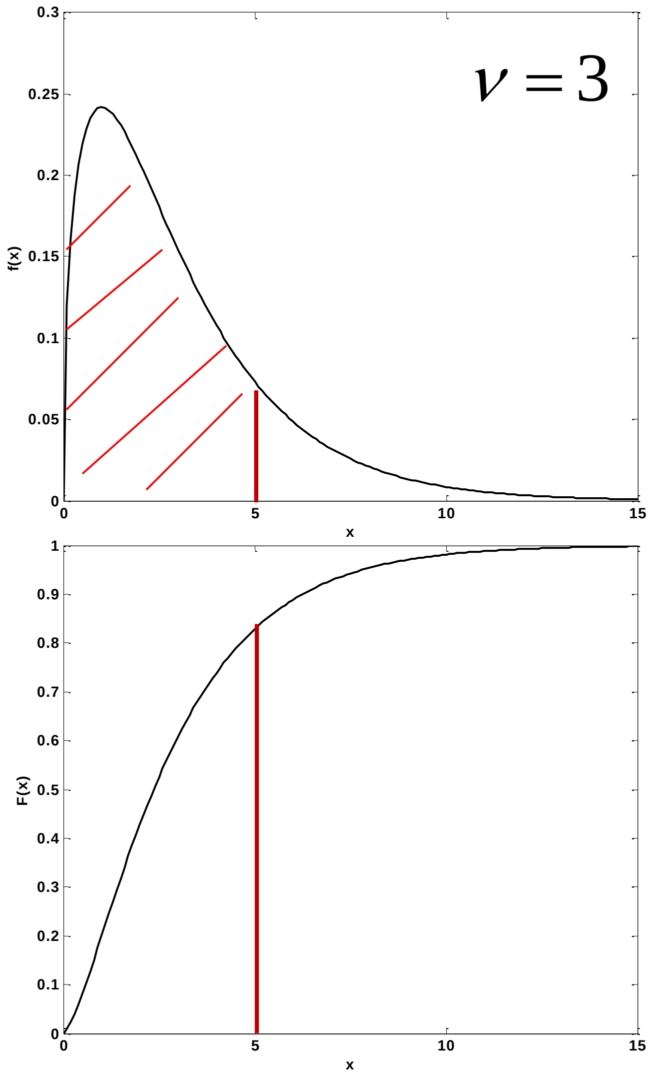
Chi-Square:

$$f(x|\nu)=\frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

The CDF of the continuous chi-square distribution is

$$F(x|\nu)=\begin{cases}0 & x\le 0\\ \dfrac{\gamma(\nu/2,x/2)}{\Gamma(\nu/2)} & x>0\end{cases}$$

Where $\gamma(\nu,x/2)$ is the lower incomplete gamma function.

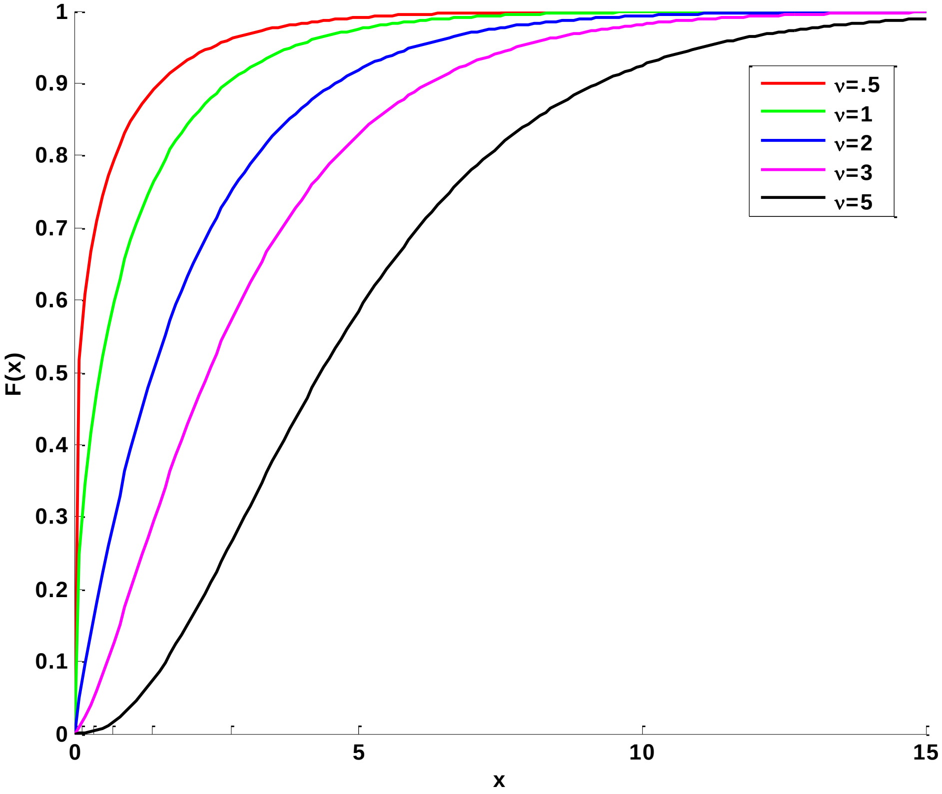


Continuous Distributions

Chi-Square:

$$f(x \mid \alpha, \beta) = x^{\alpha-1} \frac{e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$$

```
x=(0:.1:15)';, nu=[.5,1,2,3,5];
figure(1)
hold on
for count=1:length(nu)
    y = chi2cdf(x,nu(count));
    if count==1
        plot(x,y,'r','LineWidth',2)
    elseif count==2
        plot(x,y,'g','LineWidth',2)
    elseif count==3
        plot(x,y,'b','LineWidth',2)
    elseif count==4
        plot(x,y,'m','LineWidth',2)
    elseif count==5
        plot(x,y,'k','LineWidth',2)
    end
end
xlim([0 15]) , ylim([0 1])
```



Continuous Distributions

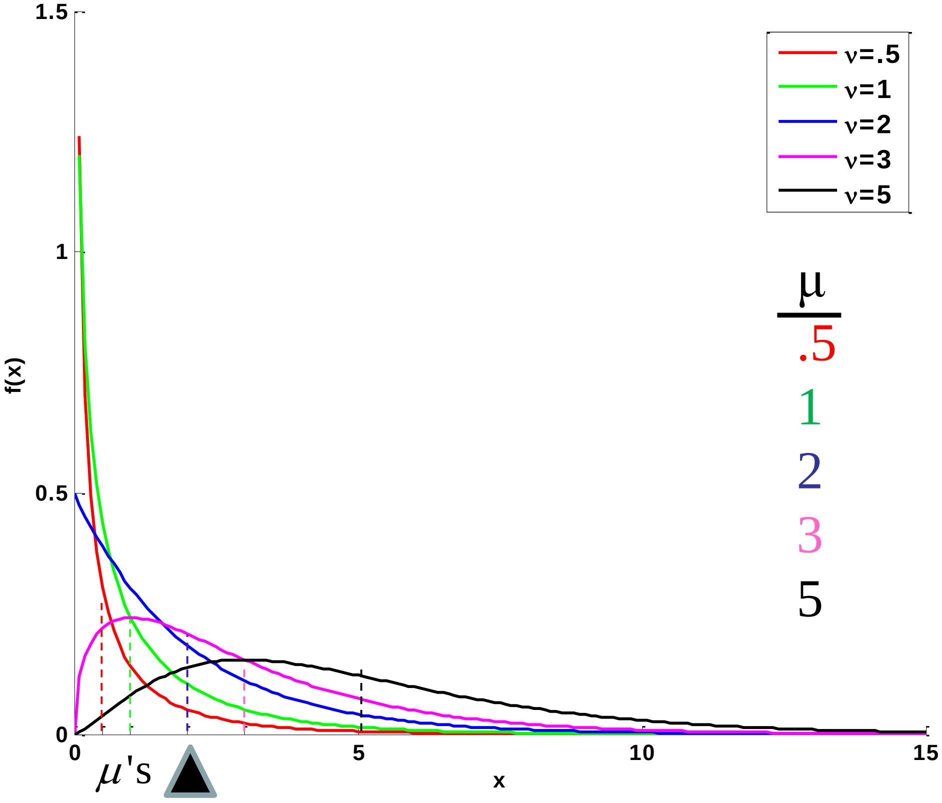
Chi-Square:

It can be shown that

$$\mu = \int_{x} x f(x|\theta) dx$$

$$= \int_{x=0}^{\infty} x \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}} dx$$

$$= \nu$$



Continuous Distributions

Chi-Square:

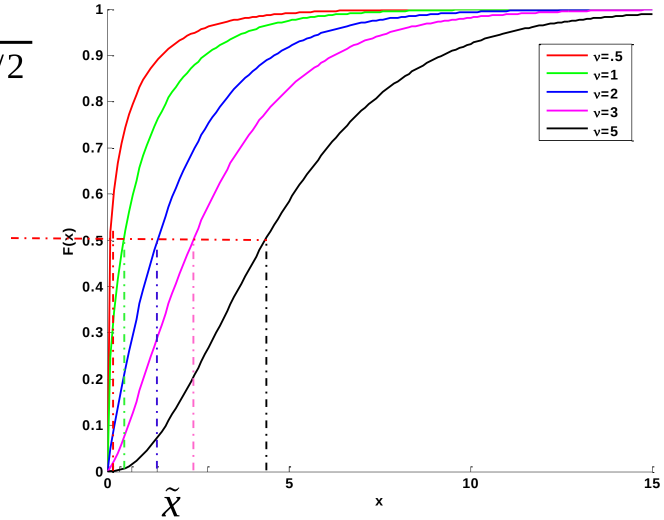
$$f(x|\nu)=\frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

It can be shown that

median

$$\int_{x=0}^{\tilde{x}} f(x|\theta)dx=\frac{1}{2}$$

$$\tilde{x}\approx\nu-\frac{2}{3}+\frac{4}{27\nu}-\frac{8}{729\nu^2}$$

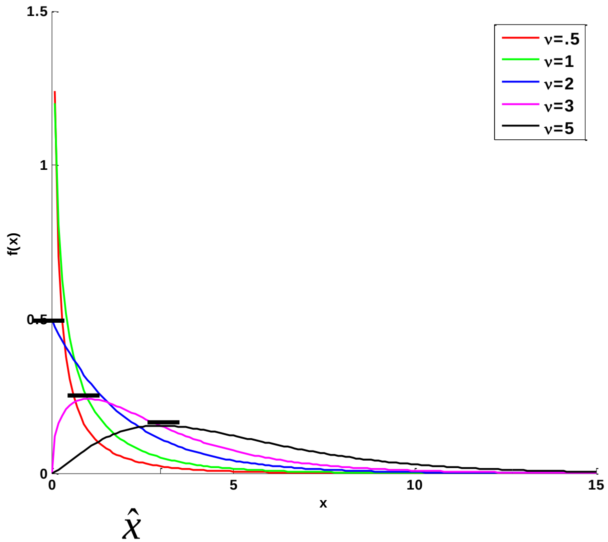


$\tilde{x}$
0.0857
0.4705
1.4047
2.3815
4.3625

mode

$$\frac{\partial}{\partial x}f(x|\theta)\bigg|_{\hat{x}}=0$$

$$\hat{x}=\nu-2 \qquad \nu\geq 2$$



$\hat{x}$
-
-
0
1
3

Continuous Distributions

Chi-Square:

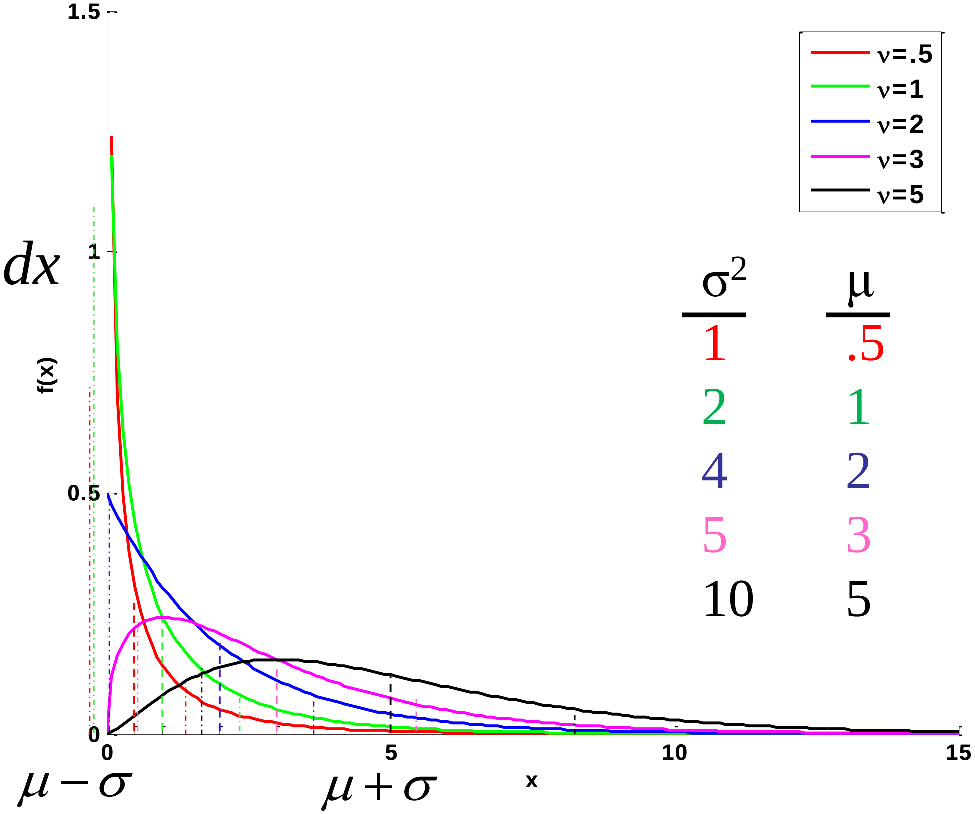
$$f(x|\nu)=\frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

that

$$\sigma^2 = \int_x (x-\mu)^2 f(x|\theta)dx$$

$$= \int_{x=0}^{\infty} (x-\mu)^2 \frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}dx$$

$$= 2\nu$$



σ^2	μ
1	.5
2	1
4	2
5	3
10	5

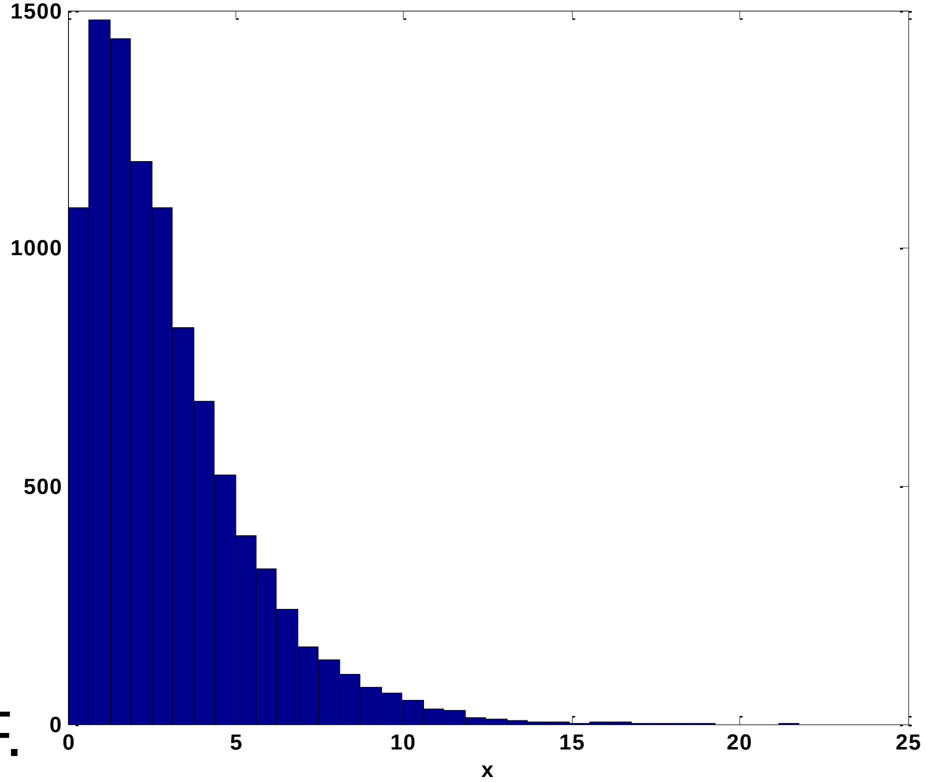
Continuous Distributions

Chi-Square:

$$f(x\,|\,\nu) = \frac{x^{\nu/2-1} e^{-x/2}}{\Gamma(\nu/2) 2^{\nu/2}}$$

```
nu=3;,num=10^4;
x=chi2rnd(nu,num,1);
mean(x)
var(x)
hist(x,35)
```

	True	Simulated
μ	3	2.9802
σ^2	6	5.7426



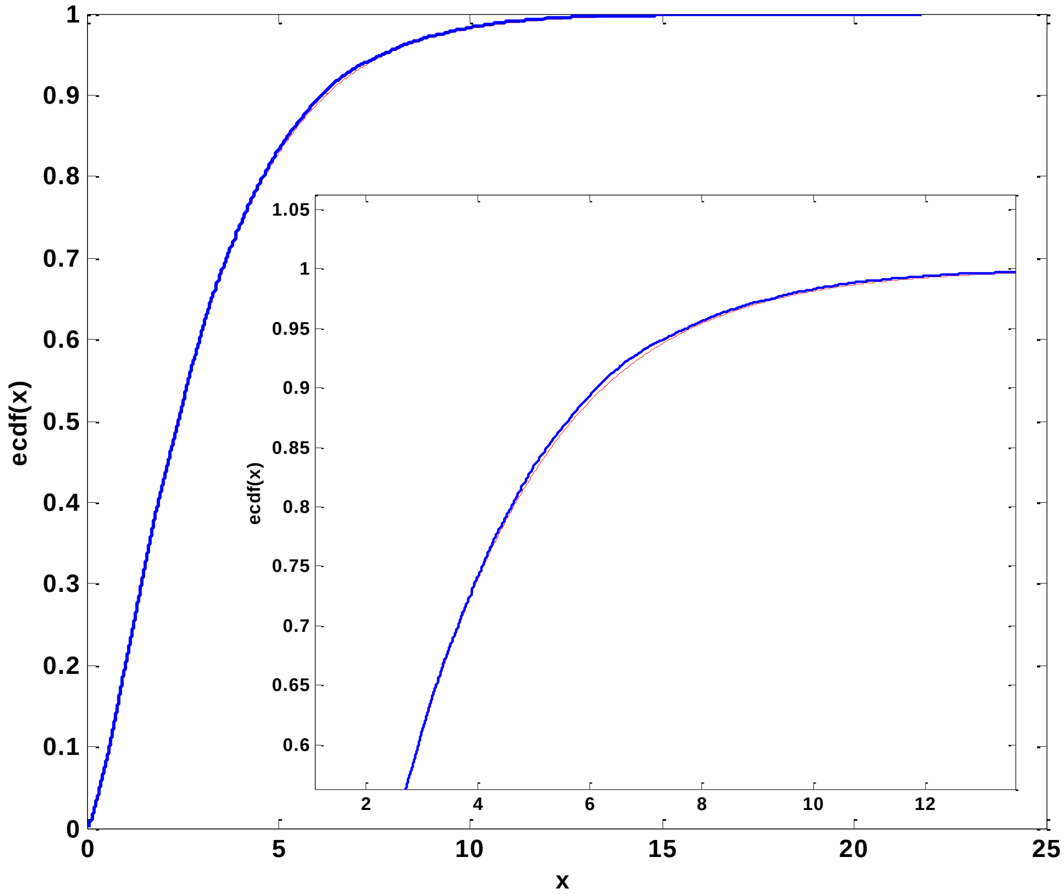
Can also find and plot ECDF.

Continuous Distributions

Chi-Square:

$$f(x|\nu) = \frac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

```
nu=3;
y=chi2cdf((0:.01:15),nu);
plot((0:.01:15),y, 'r')
hold on
[F,xx]=ecdf(x);
stairs(xx,F,'LineWidth',2)
```



Homework 4:

1) Let x~gamma(2,2), using pencil and paper find $P(\mu\text{-}\sigma \le x \le \mu+\sigma)$.

2) Numerically integrate the gamma(2,2) pdf to find $P(\mu\text{-}\sigma \le x \le \mu+\sigma)$. Use 100 rectangles.

3) Let x~gamma(2,2), using pencil and paper find μ.

4) Numerically integrate the gamma(2,2) pdf to find μ. Use 100 rectangles. Compare to answer in 3).

Homework 4:

5) Show that the mode of the gamma distribution is $\hat{x} = (\alpha - 1)\beta$. Pencil and paper.

6) Can you numerically determine an estimate of the mode? Use α=2, β=2. Compare to 5).

7) Numerically integrate the gamma pdf to find the 50th and 99th percentiles. i.e. find x_0 such that $P(x \leq x_0)=0.50$ and $P(x \leq x_0)=0.99$ for α=2, β=2.

Compare median in 7) to mean in 3,4) & mode in 5,6).

Homework 4:

8) Generate 10^6 gamma $\alpha=2, \beta=2$ random variables. Empirically determine the 50th and 99th percentiles. i.e. Find the $.50*10^6$ and $.99*10^6$ largest values x_0. such that $P(x \leq x_0) \approx 0.50$ and $P(x \leq x_0) \approx 0.99$. Compare to values to 7) and the approximate median $\alpha\beta \dfrac{3\alpha - 0.8}{3\alpha + 0.2}$.

Homework 4:

9) Make a histogram of the random variables in 8) with at least 50 bins. Divide the count in each bin by the total number 10^6. Compare to or superimpose the true pdf.

10) Make a empirical CDF from the values in 8).

11) Using pencil and paper derive the mean, median, and variance for the exponential distribution.

Homework 4:

12) Can you analytically derive the approximate median for the chi-square distribution?