Math Review

Daniel B. Rowe, Ph.D.

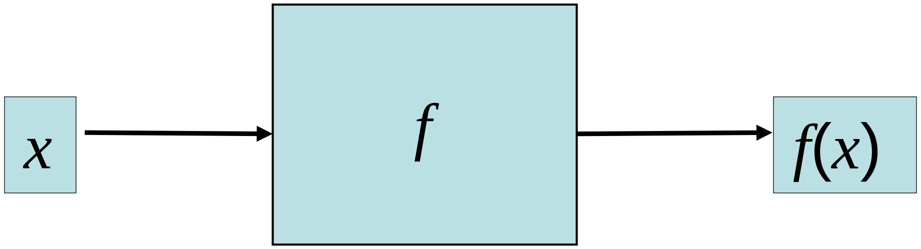
Professor
Department of Mathematical and Statistical Sciences

Outline

- Differentiation
  - Definition
  - Analytic Approach
  - Numerical Approach
- Integration
  - Definition
  - Analytic Approach
  - Numerical Approach
- Summary

Differentiation - Definition

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

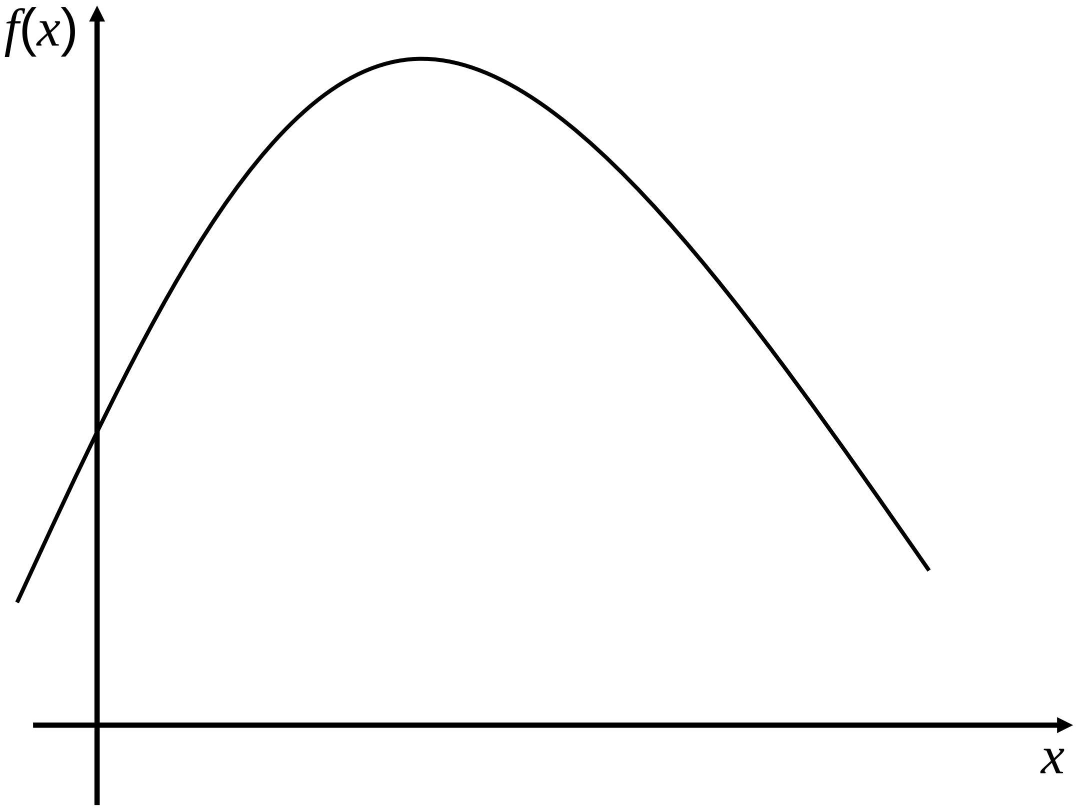


The set A is the domain of f

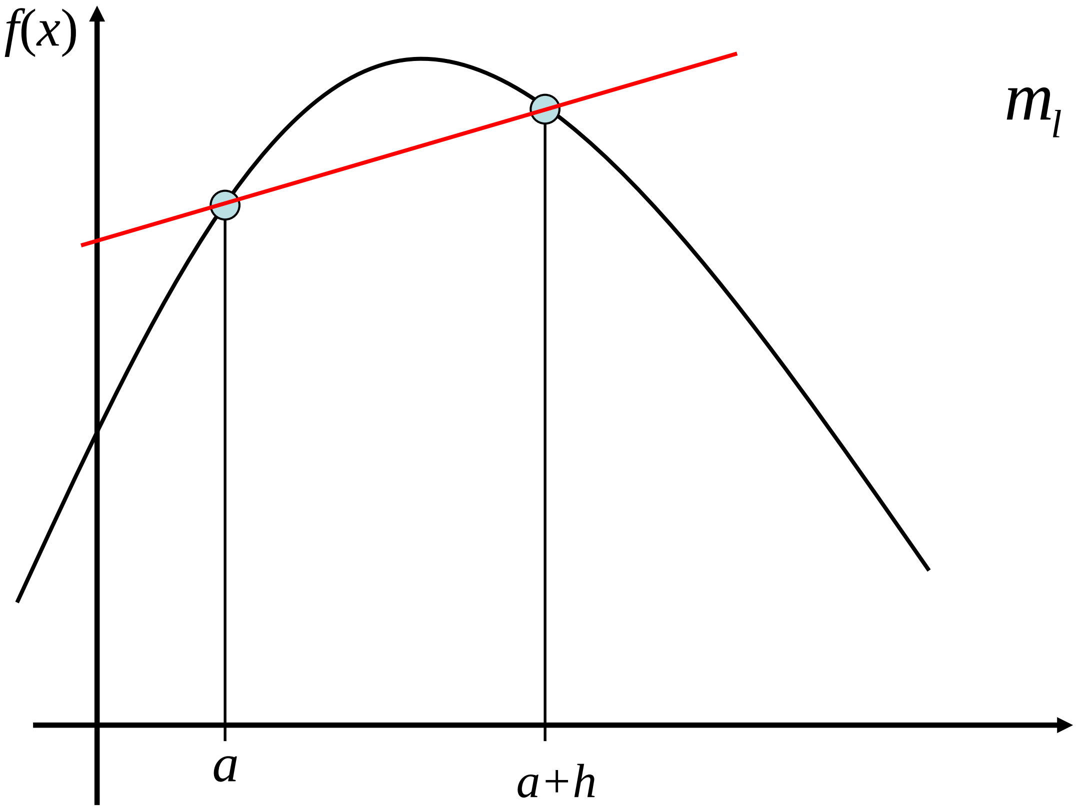
and

the set B is the range of f.

Differentiation - Definition



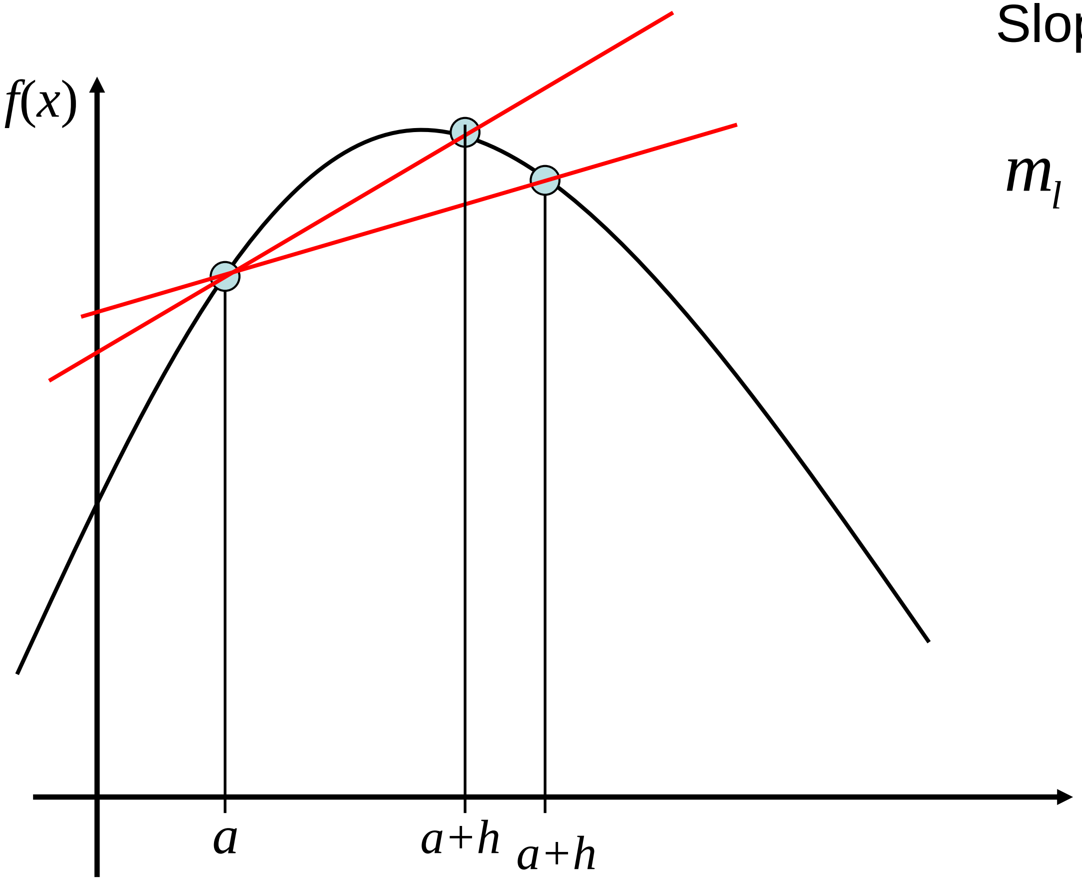
Differentiation - Definition



Slope of Line: $h=\Delta x$

$$m_l = \frac{f(a+h) - f(a)}{h}$$

Differentiation - Definition



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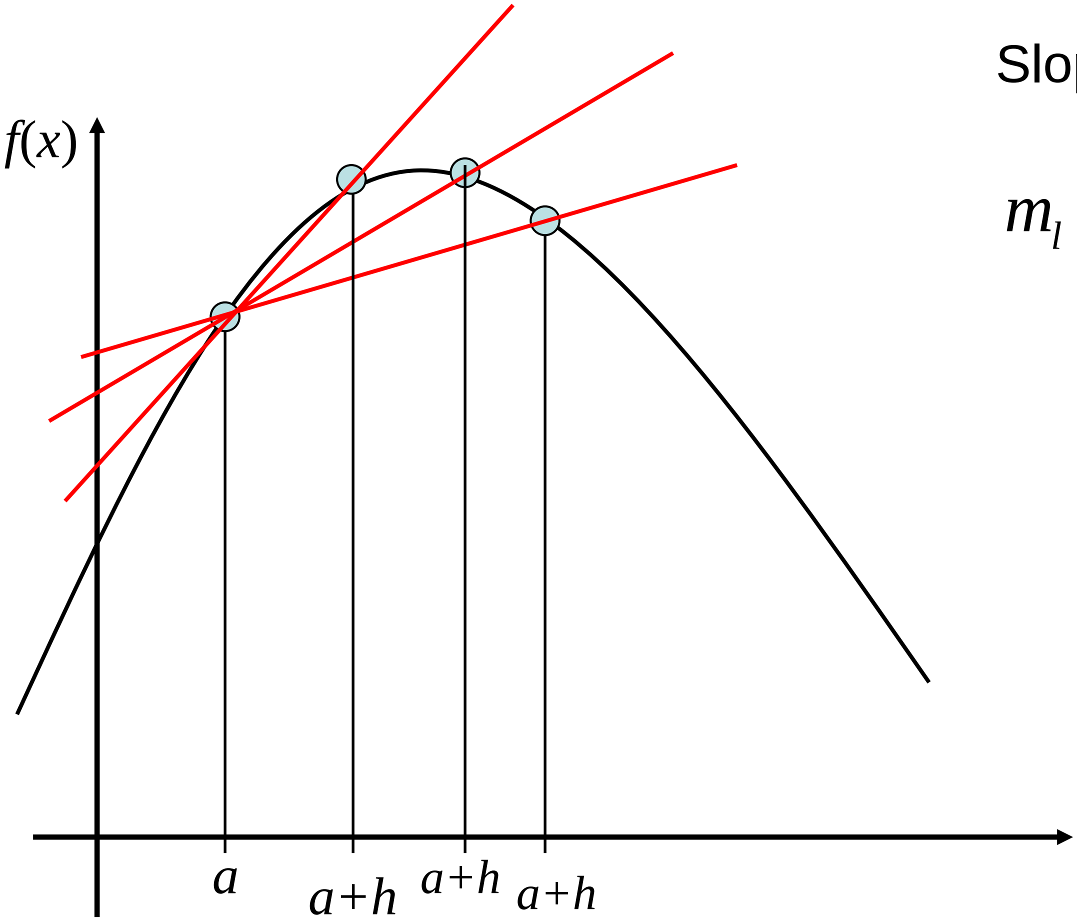
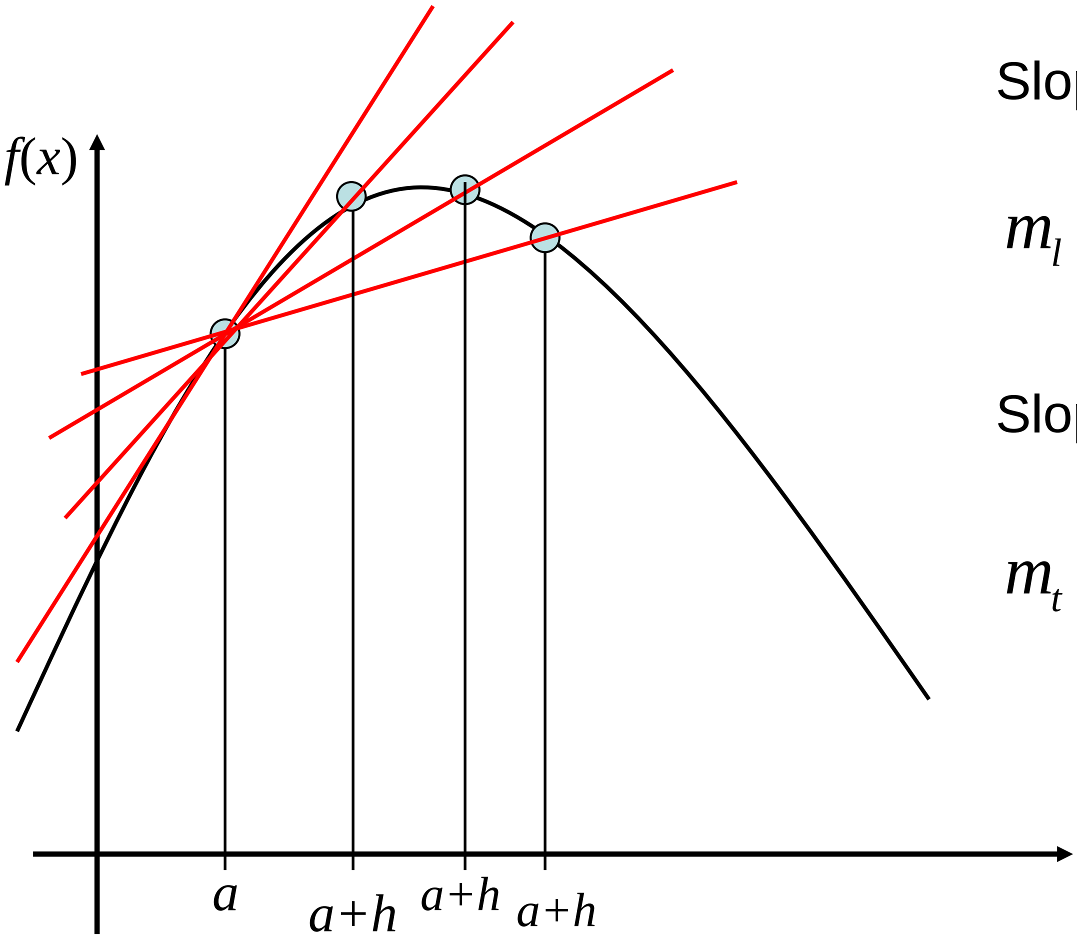
Differentiation - Definition



Slope of Line: $h=\Delta x$

$$m_l = \frac{f(a+h) - f(a)}{h}$$

Differentiation - Definition



Slope of Line: $h=\Delta x$

$$m_l = \frac{f(a+h)-f(a)}{h}$$

Slope of Tangent Line:

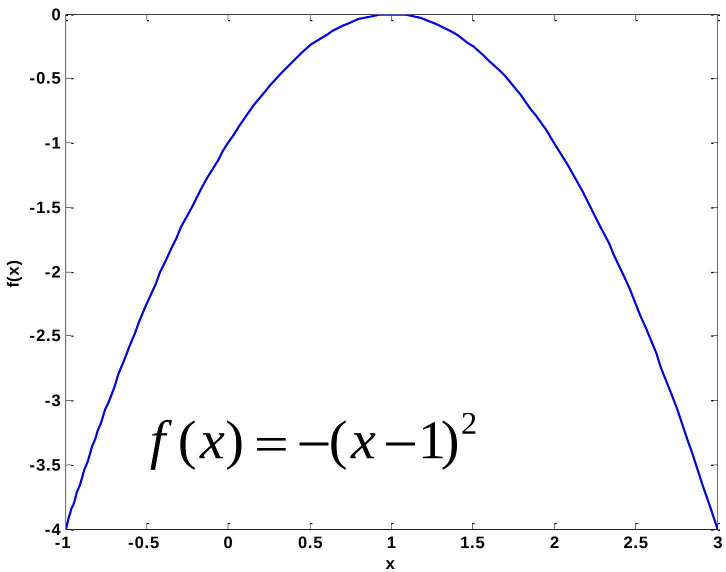
$$m_t = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

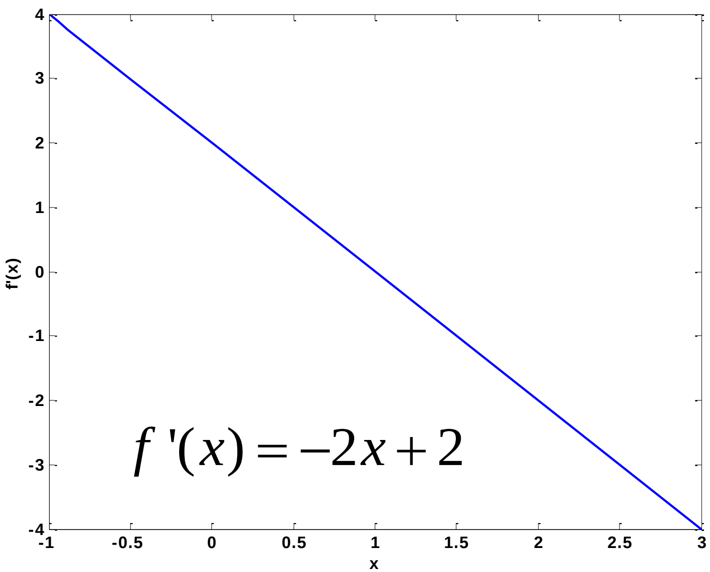
Differentiation - Analytic Approach

$$m_t = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \frac{df(x)}{dx}$$

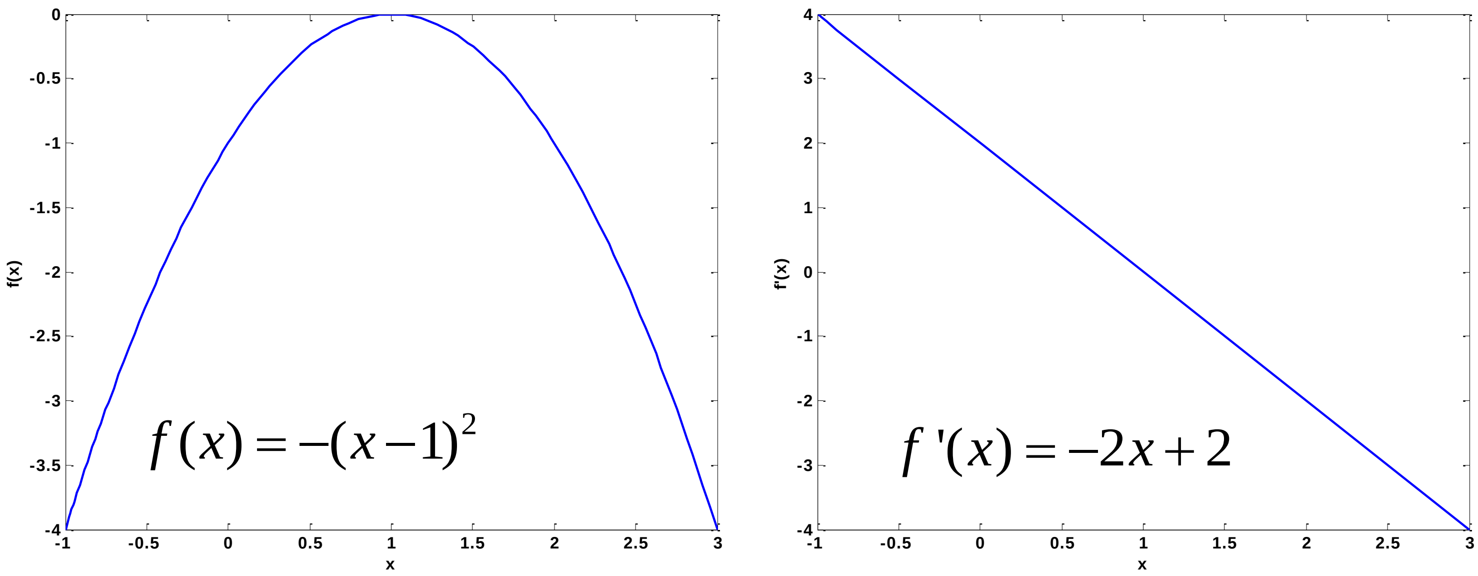
$$= f'(x)$$





Differentiation - Analytic Approach

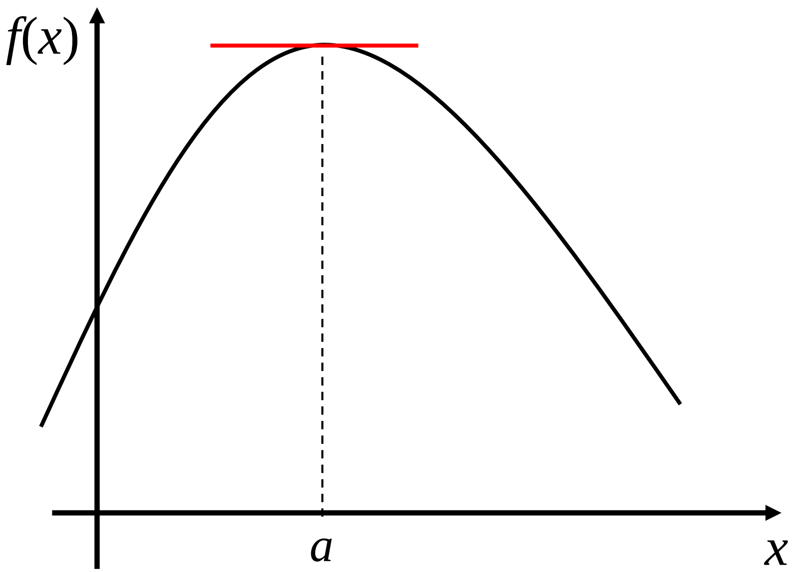
```
% analytical derivative

f='-(x-1)^2'
fprime='-2*x+2'

figure(1)
fplot(f,[-1 3],'b')
figure(2)
fplot(fprime,[-1 3],'b')
```



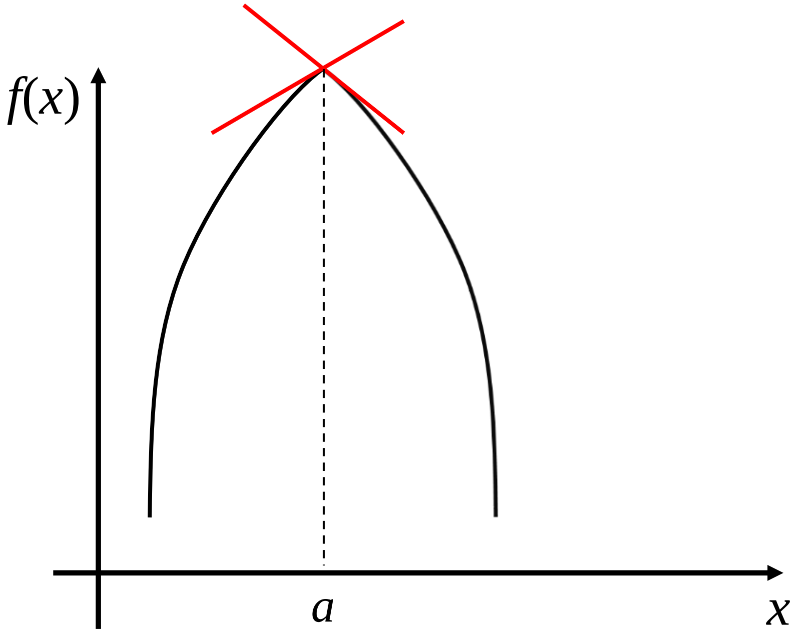
Differentiation - Analytic Approach

A function $f(x)$ is differentiable at $x=a$ if there exists only one unique tangent line to the graph of $f(x)$ at $x=a$.

Differentiable



Not Differentiable



Differentiation - Analytic Approach

$$\frac{d}{dx}c = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}e^x = e^x$$

Differentiation - Analytic Approach

Let $f'(x)$ and $g'(x)$ exist.

Linearity Rule:

$$\frac{d}{dx}\left[c_1 f(x) + c_2 g(x)\right] = c_1 f'(x) + c_2 g'(x)$$

Product Rule:

$$\frac{d}{dx}\left[cf(x)g(x)\right] = c\left[f'(x)g(x) + f(x)g'(x)\right]$$

Quotient Rule:

$$\frac{d}{dx}\left[c\frac{f(x)}{g(x)}\right] = c\frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$g(x) \neq 0$$

Chain Rule:

$$\frac{d}{dx}\left[cf(g(x))\right] = cf'(g(x))g'(x)$$

$f'(g(x))$ must exist

Differentiation - Analytic Approach

Examples:

Linearity Rule:

$$\frac{d}{dx}\left[c_1 x + c_2 x^2\right] = c_1 1 + c_2 2x$$

Product Rule:

$$\frac{d}{dx}\left[cx\sin(x)\right] = c\left[1\sin(x) + x\cos(x)\right]$$

Quotient Rule:

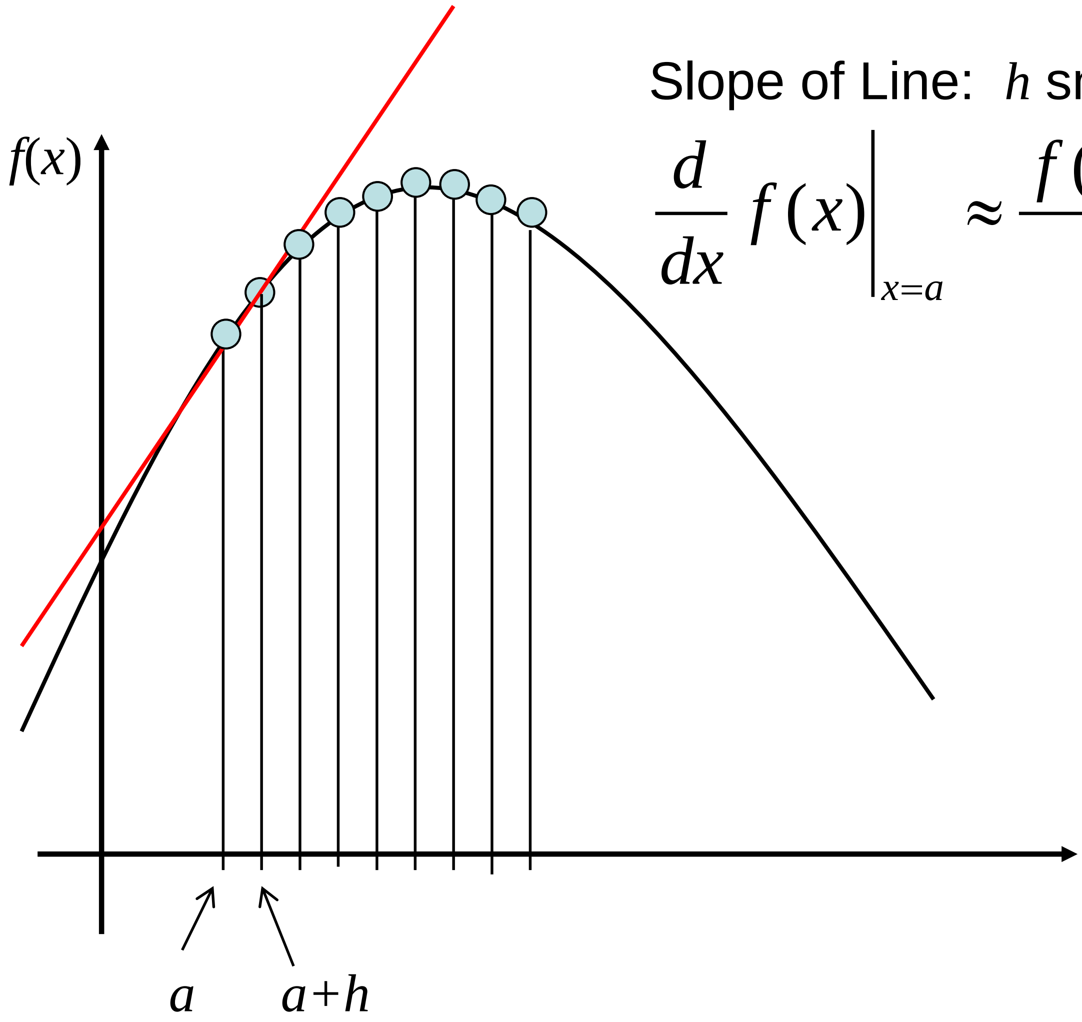
$$\frac{d}{dx}\left[c\frac{\sin(x)}{\cos(x)}\right] = c\frac{\cos(x)\cos(x) - \sin(x)[-\sin(x)]}{\left[\cos(x)\right]^2}$$

$$= c\sec^2(x)$$

Chain Rule:

$$\frac{d}{dx}\left[c(x^2+1)^{1/2}\right] = c\frac{1}{2}(x^2+1)^{-1/2}2x$$

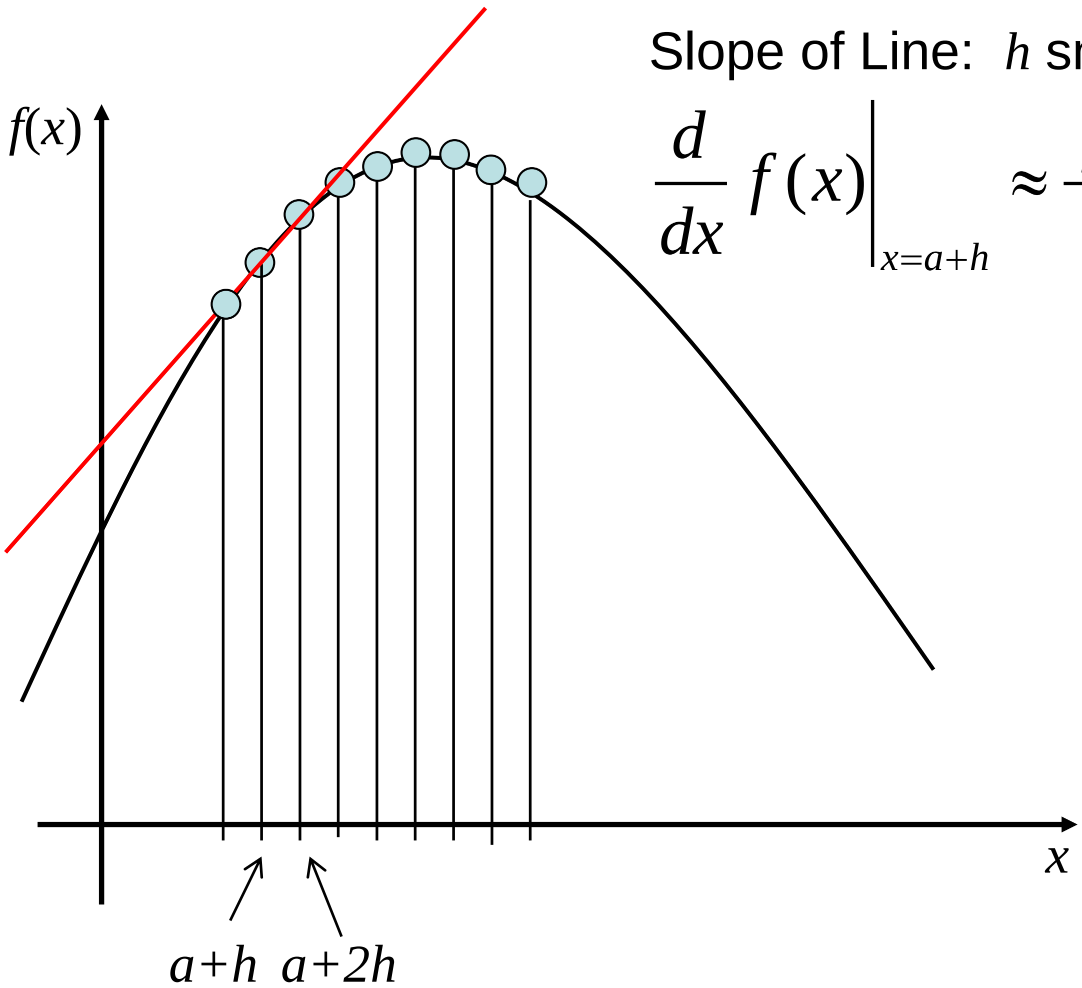
Differentiation - Numerical Approach



Slope of Line: h small

$$\frac{d}{dx} f(x)\bigg|_{x=a} \approx \frac{f(a+h) - f(a)}{h}$$

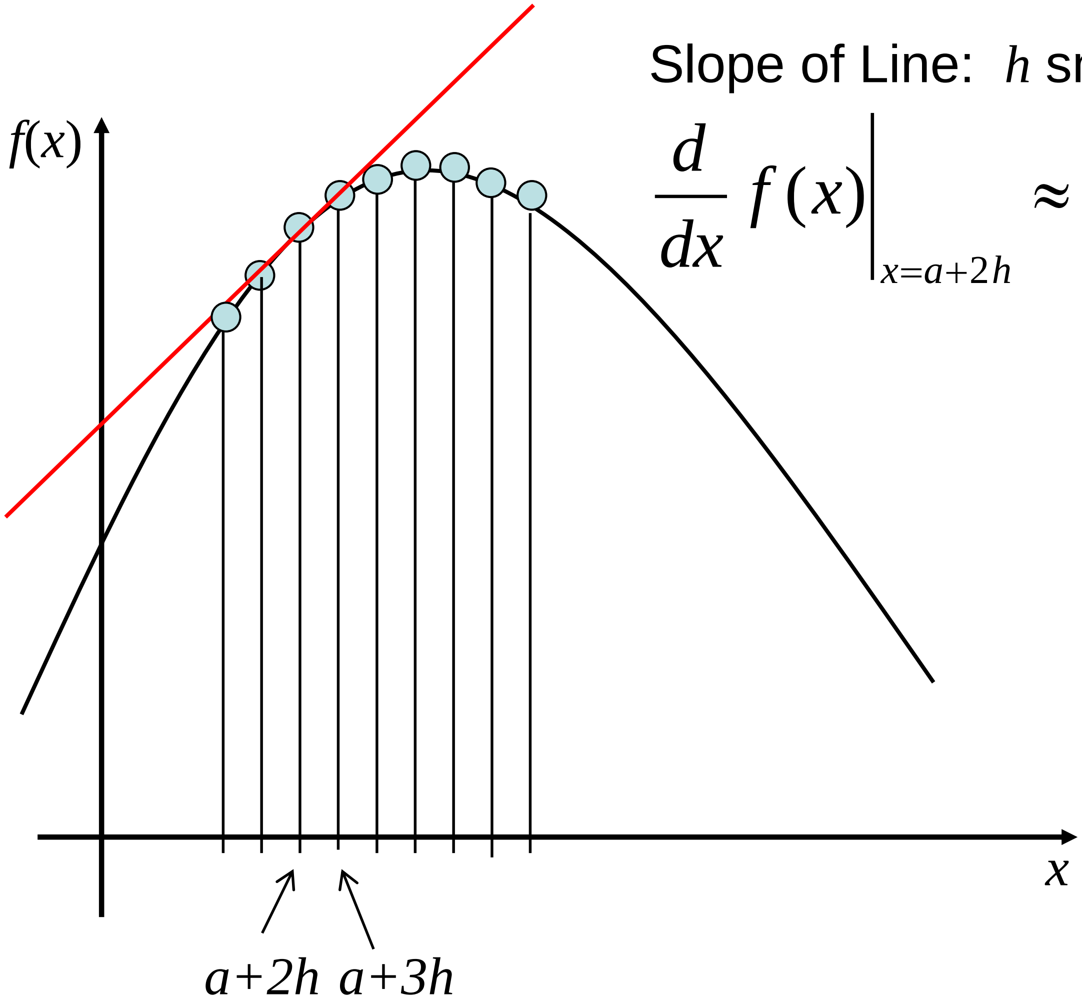
Differentiation - Numerical Approach



Slope of Line: h small

$$\frac{d}{dx} f(x)\bigg|_{x=a+h} \approx \frac{f(a+2h) - f(a+h)}{h}$$

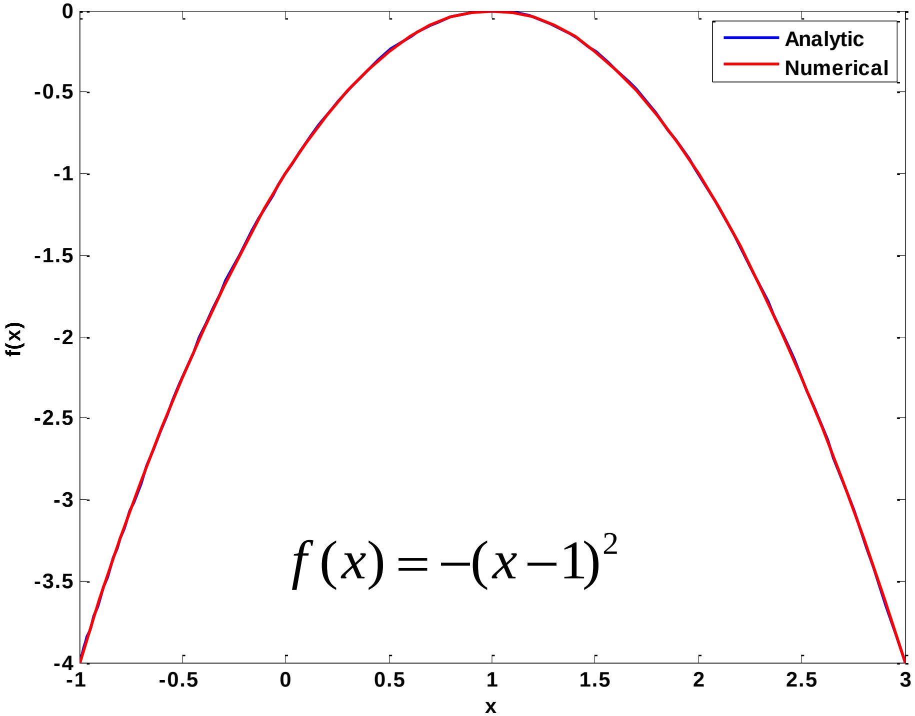
Differentiation - Numerical Approach



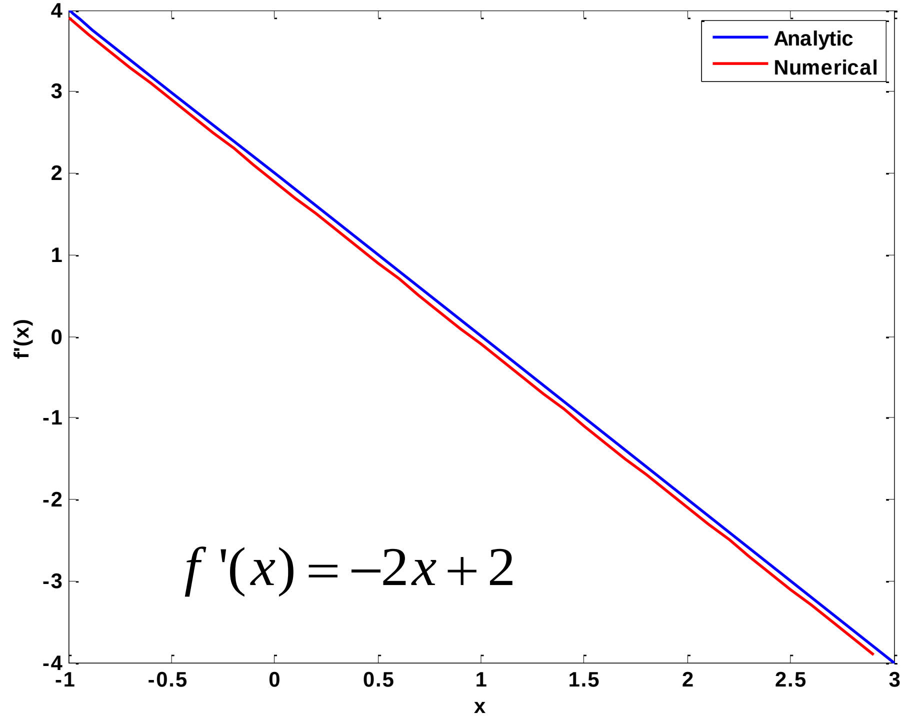
Slope of Line: h small

$$\left.\frac{d}{dx}f(x)\right|_{x=a+2h} \approx \frac{f(a+3h)-f(a+2h)}{h}$$

Differentiation - Numerical Approach



h=0.1



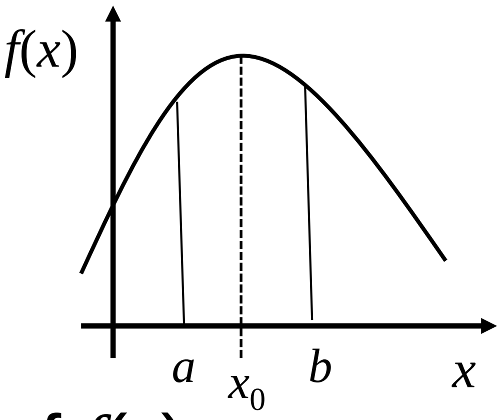

```
>>polyder([-1 2 -1])
ans= -2   2
```

If h=0.01, then lines look same!

Differentiation - Numerical Approach

```
% numerical derivative

f='-(x-1)^2'
fprime='-2*x+2'

xpts=(-1:.1:3)'
fpts=-(xpts-1).^2

figure(1)
fplot(f,[-1 3],'b')
hold on
plot(xpts,fpts,'r')

numder=zeros(length(xpts)-1,1);
for count=1:length(xpts)-1
    numder(count,1)=...
        ( fpts(count+1,1)-fpts(count,1) )/( xpts(count+1,1)-xpts(count,1) );
end

figure(2)
fplot(fprime,[-1 3],'b')
hold on
plot(xpts(1:length(xpts)-1,1),numder,'r')
```

Maximization - Analytic Approach



Given a function $f(x)$, if it has a maxima

in the interval $[a,b]$ at x_0, then the slope of $f(x)$

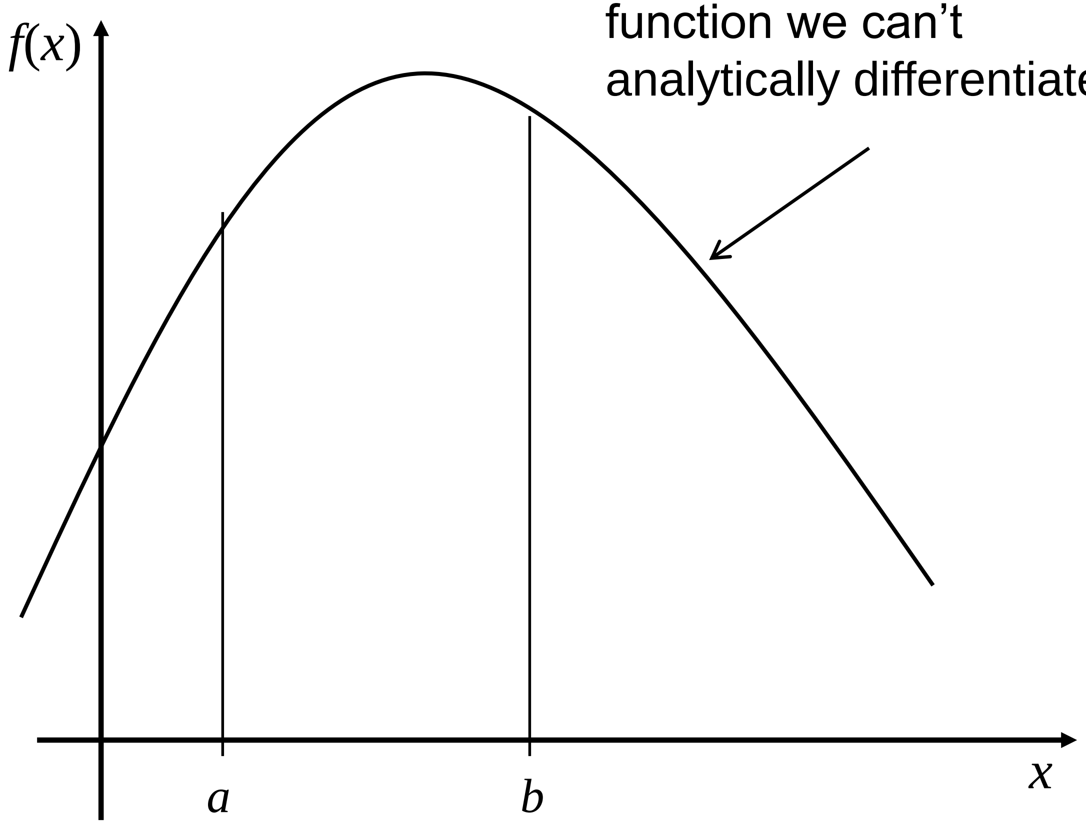
is zero at x_0. This means that $f'(x)=0$ at $x=x_0$.

x_0 is a global maxima if it is unique.

Maximization - Numerical Approach

Define values of x want to find max of $f(x)$ for, a to b.

Assume complicated function we can't analytically differentiate.

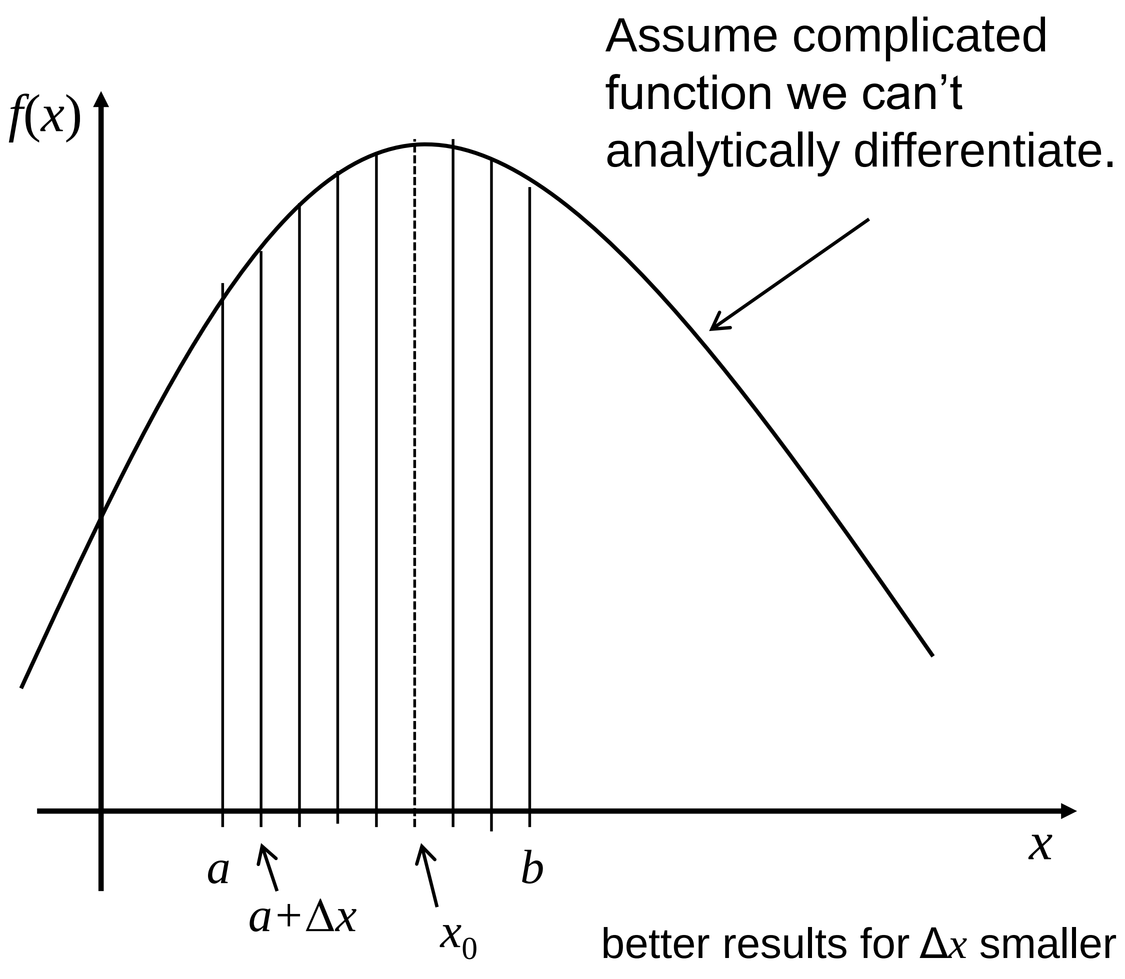


Maximization - Numerical Approach

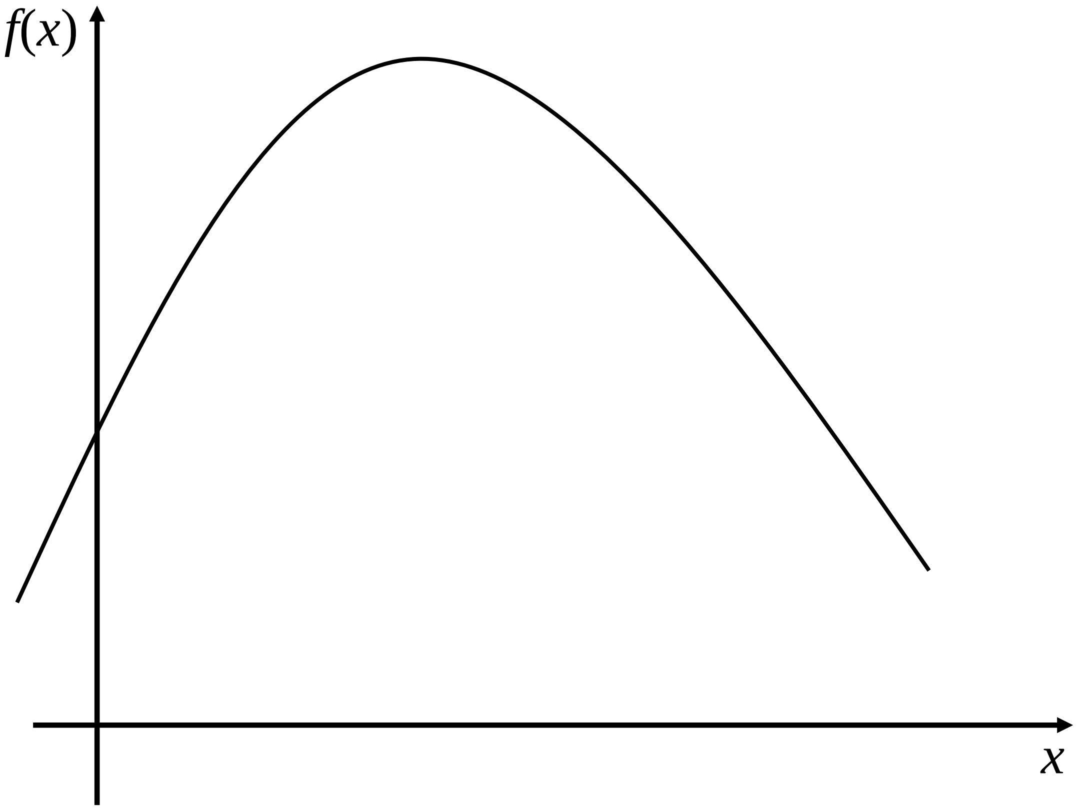
Define values of x want to find max of $f(x)$ for, a to b.

Set Δx to evaluate $f(x)$ at increments.

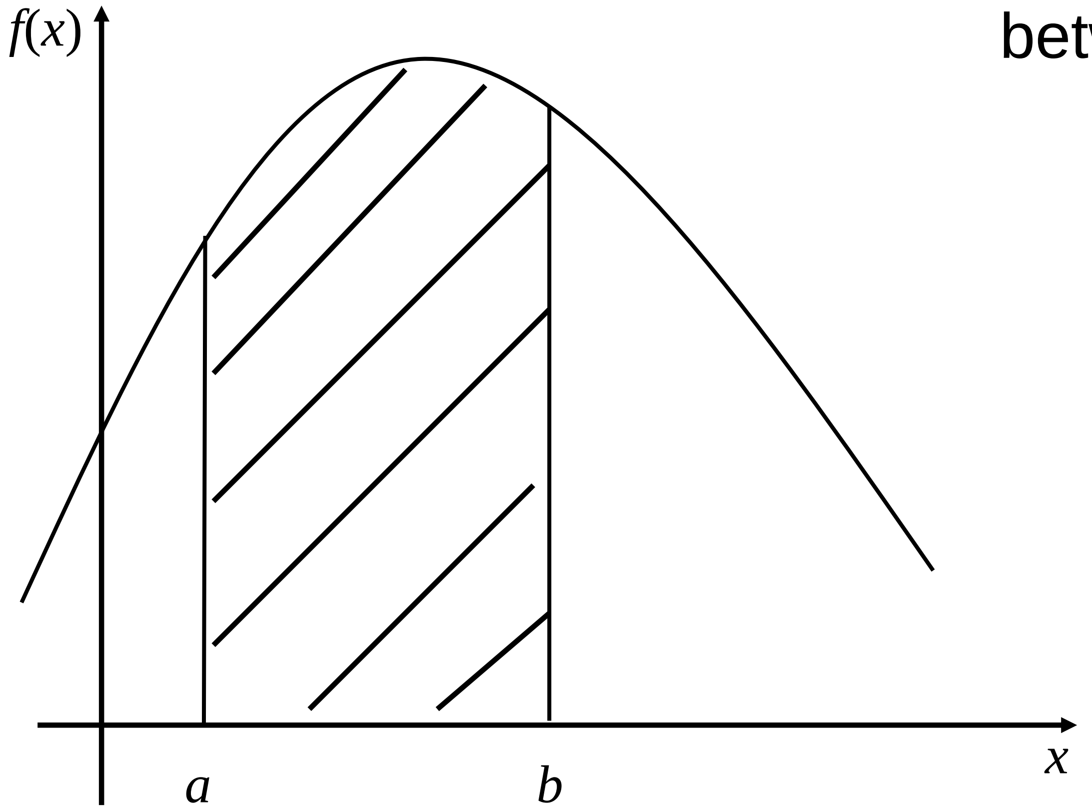
x_0 is value that makes $f(x)$ largest.



Integration - Area Under Curve



Integration - Area Under Curve

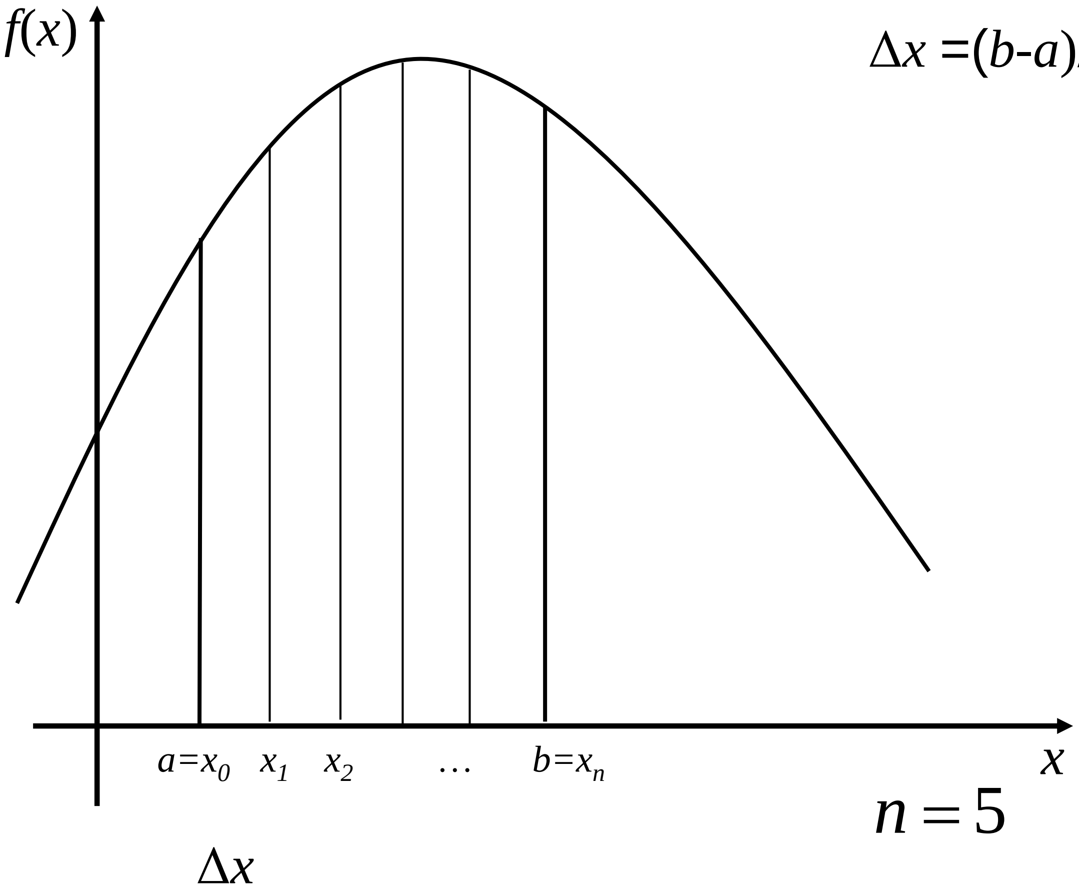


Area under curve between a and b.

Integration - Area Under Curve

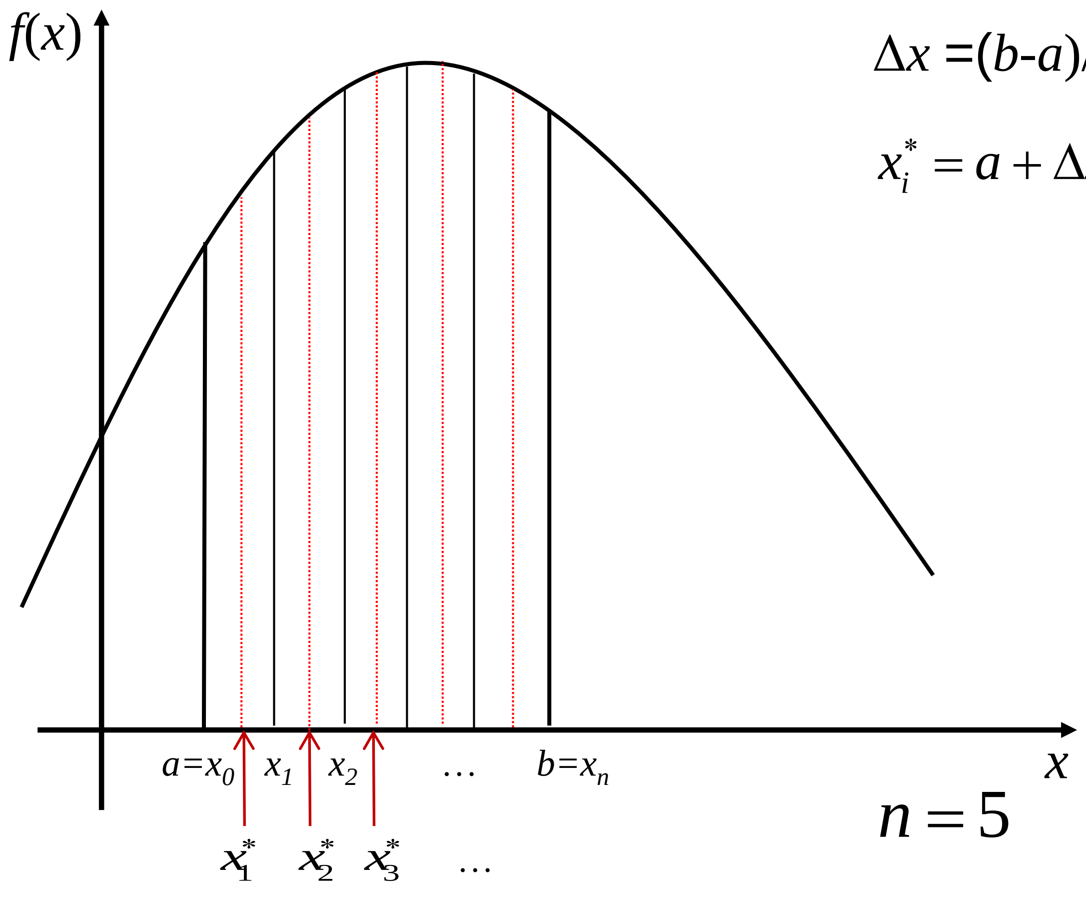
Divide into intervals: Δx small

$\Delta x = (b-a)/n$ $\Delta x = x_i - x_{i-1}$



$n = 5$

Integration - Area Under Curve



Find midpoints: Δx small

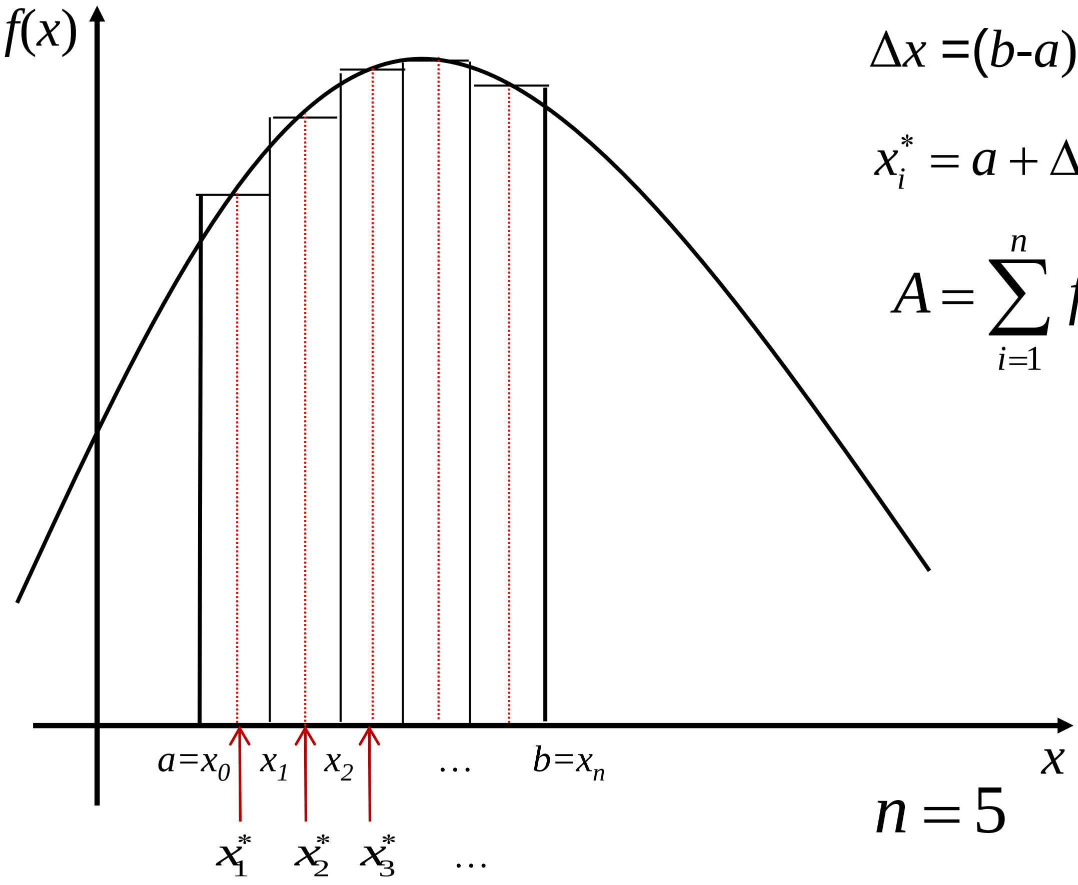
$\Delta x = (b-a)/n \qquad \Delta x = x_i - x_{i-1}$

$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$

$$n = 5$$

Integration - Area Under Curve



Area by rectangles: Δx small

$\Delta x = (b\text{-}a)/n$ $\qquad \Delta x = x_i - x_{i-1}$

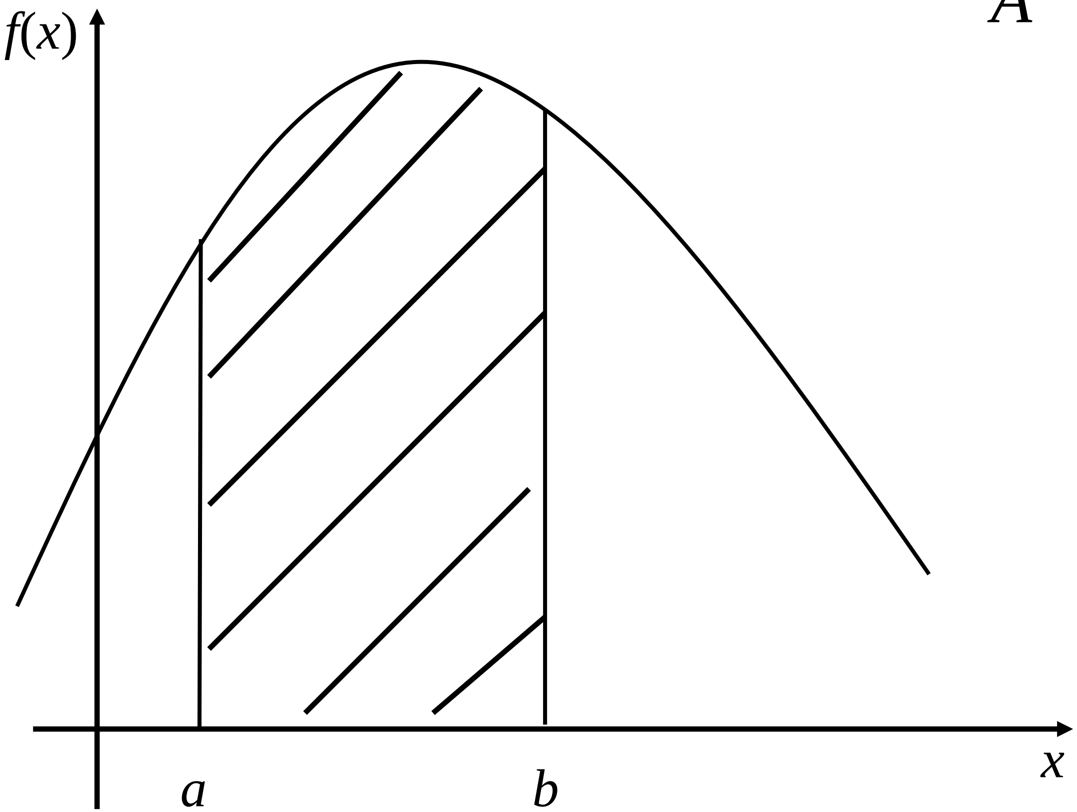
$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$n = 5$$

Integration - Area Under Curve



$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \int_{x=a}^{b} f(x)dx$$

$$\Delta x = (b-a)/n$$

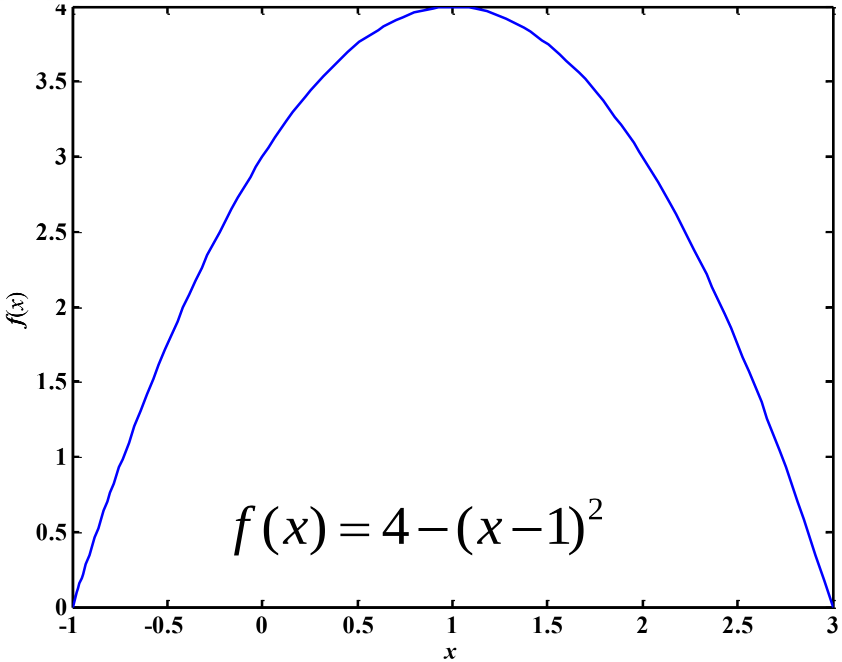
$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

Integration - Analytic Approach

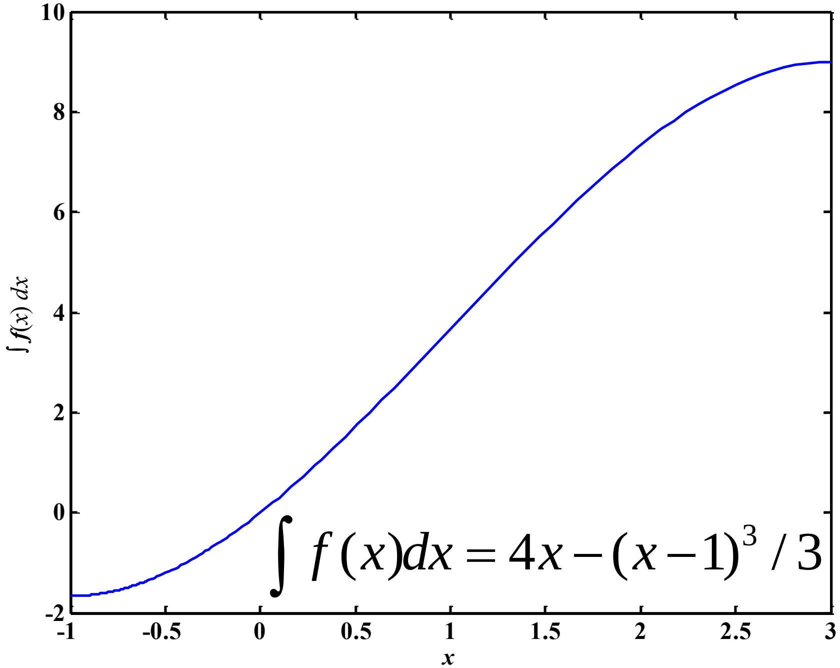
```
% analytical integral

f='4-(x-1)^2'
fint='4*x-(x-1)^3/3'

figure(1)
fplot(f,[-1 3],'b')
figure(2)
fplot(fint,[-1 3],'b')
```

>> polyint([-1 2 3])

ans = -1/3 1 3 0



$f(x) = 4-(x-1)^2$



$\int f(x)dx = 4x-(x-1)^3/3$

Integration - Analytic Approach

$$\int c dx = cx + C$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int e^x dx = e^x + C$$

Integration - Analytic Approach

Linearity:

$$\int c_1 f(x) + c_2 g(x) dx = c_1 \int f(x) dx + c_2 \int g(x) dx$$

By Parts:

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

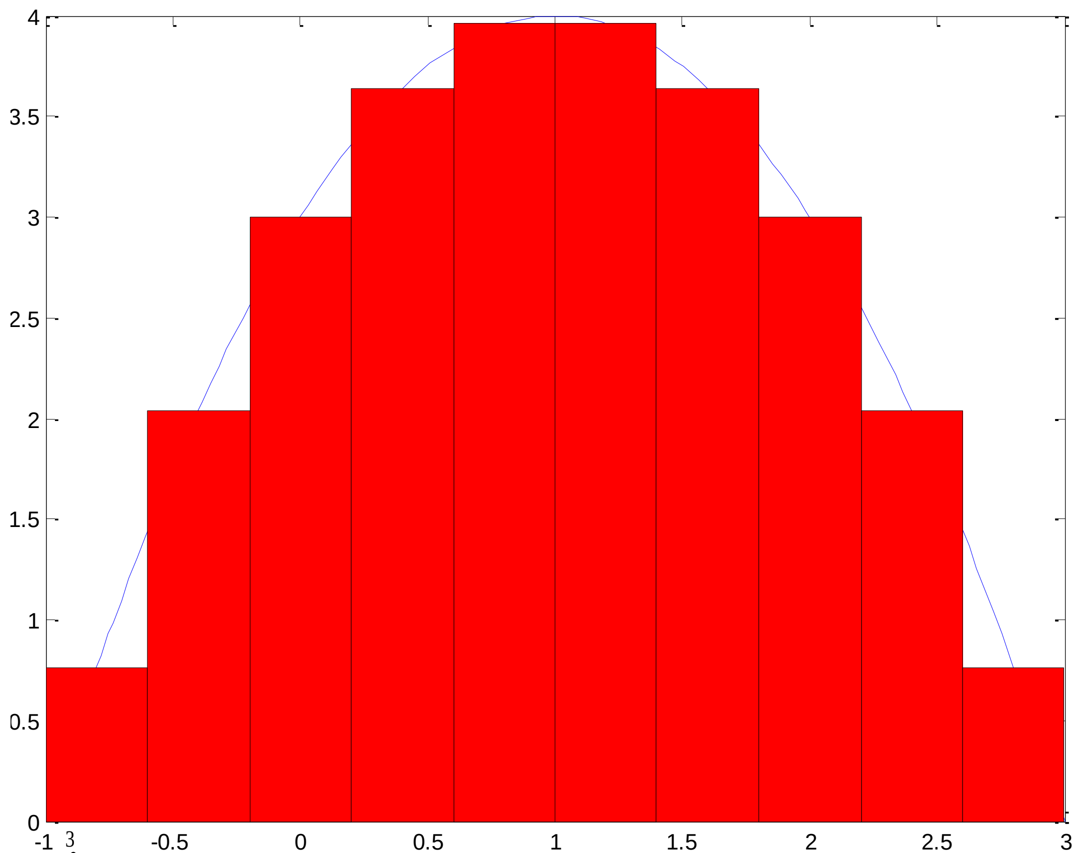
Assuming $f'(x)$ and $g'(x)$ exist.

Other methods:

Trigonometric Substitution

Partial Fractions

Integration - Numerical Approach

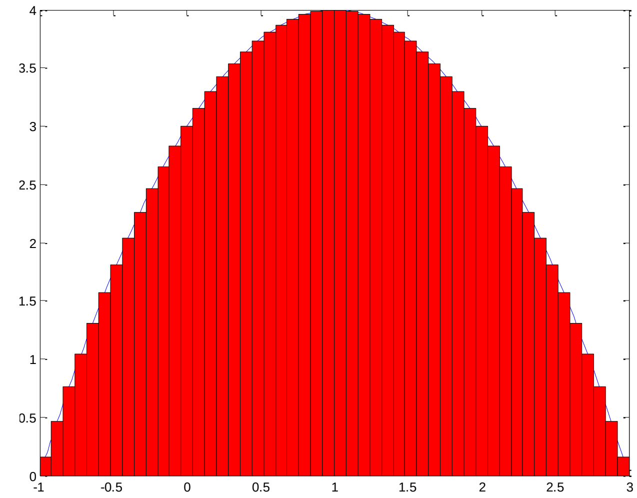


numerical

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\leftarrow$ n=10, Δx=0.4

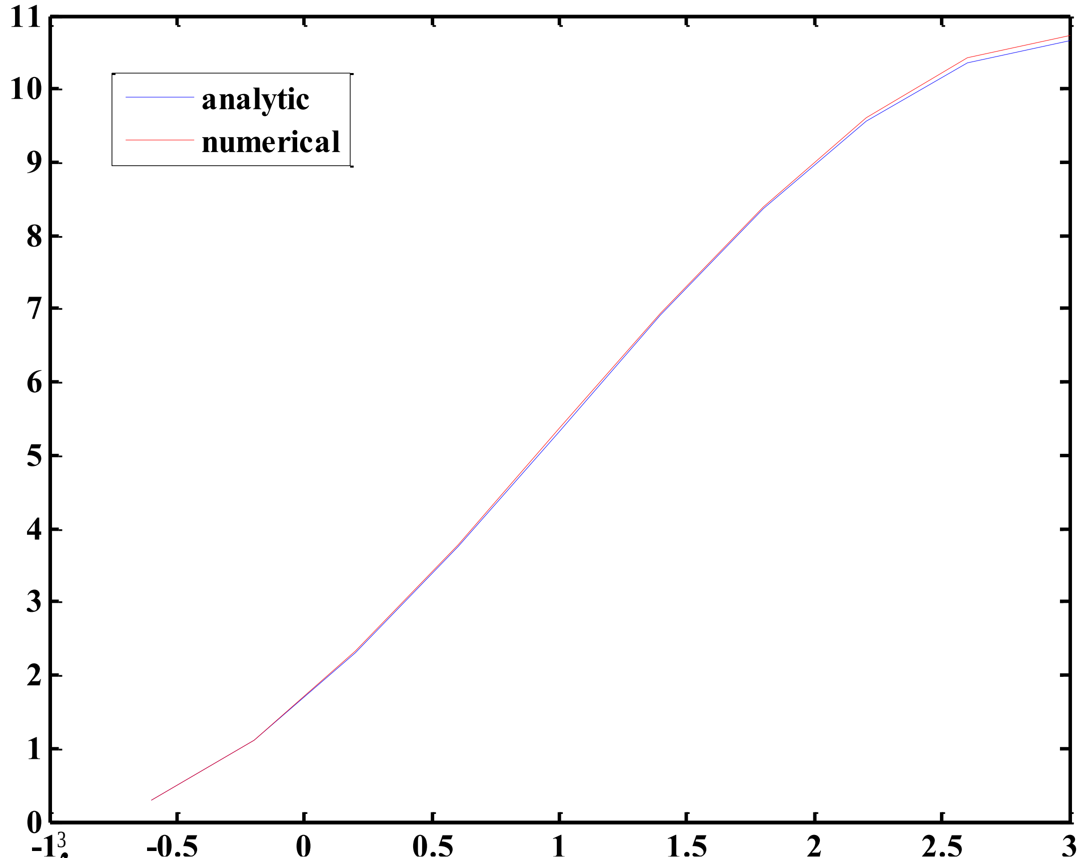
$\downarrow$ n=50, Δx=0.08



$$\int_{x=-1}^{3} [4-(x-1)^2]dx = 10.6667$$

$\leftarrow$ analytic

Integration - Numerical Approach



$$\int_{x=-1}^{3} [4-(x-1)^2]dx = 10.6667$$

← analytic

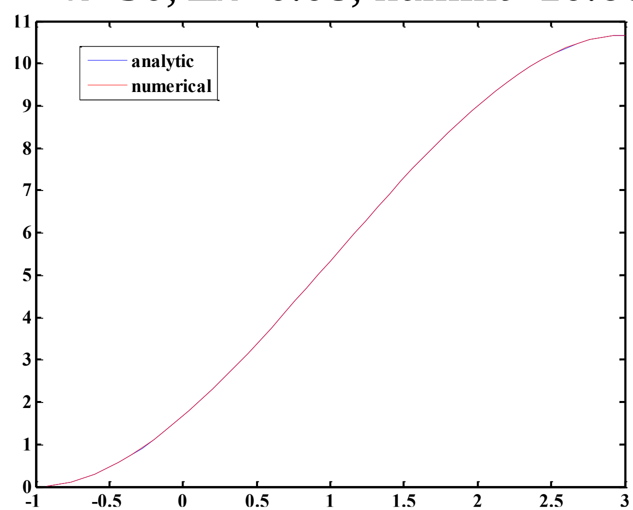
numerical

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

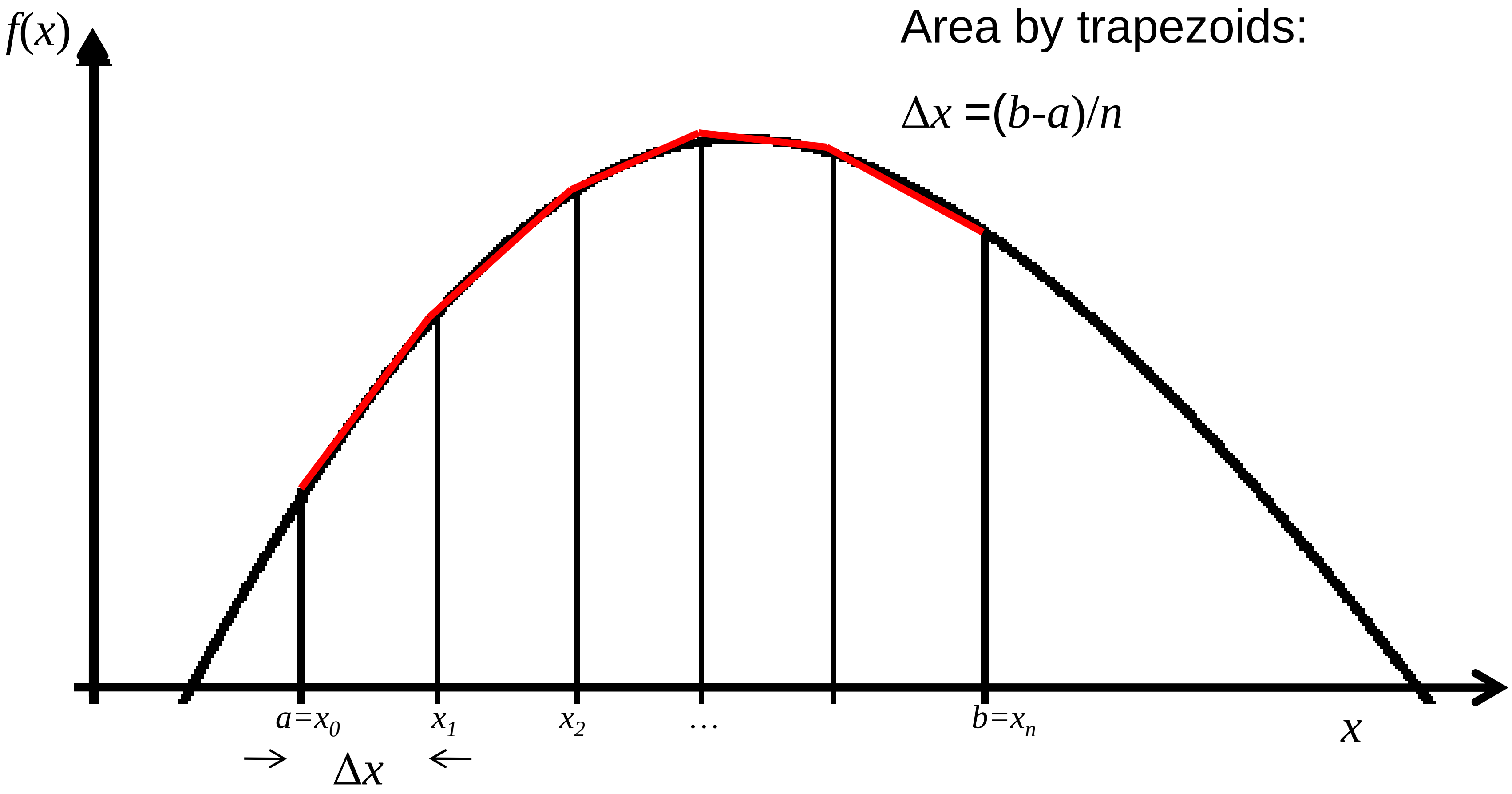
← n=10, Δx=0.4

numint=10.7200

↓ n=50, Δx=0.08, numint=10.6688



Integration - Area Under Curve

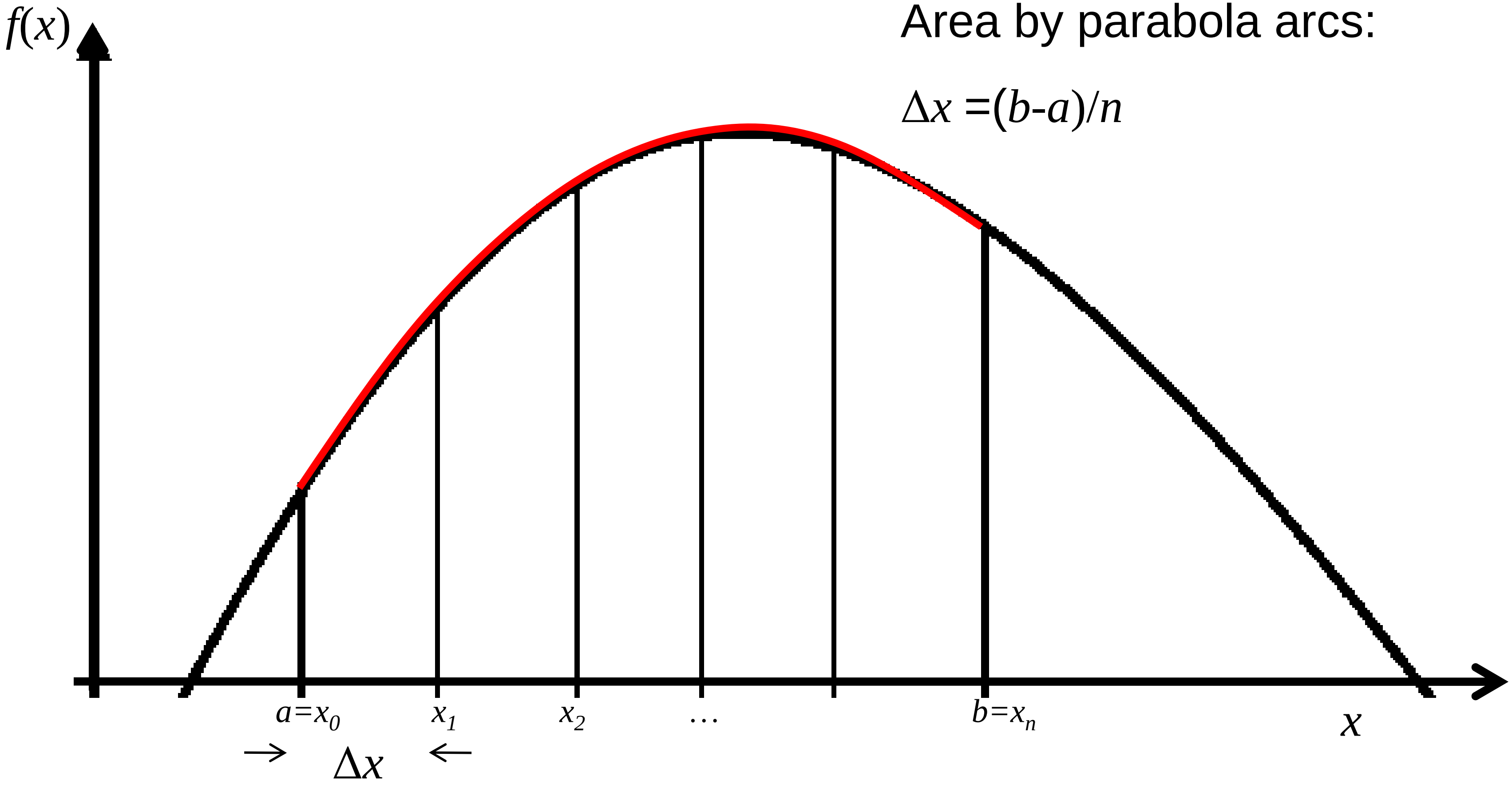


Trapezoidal Rule:

Area by trapezoids:

$\Delta x = (b-a)/n$

Integration - Area Under Curve



Simpson's Rule:

Area by parabola arcs:

$\Delta x = (b-a)/n$

Summary

- Differentiation

  Definition

  Analytic Approach

  Numerical Approach
- Integration

  Definition

  Analytic Approach

  Numerical Approach

Homework 1:

1) For the function $f(x) = x^3, x \in \mathbb{R}$

a) Differentiate analytically and evaluate at a=-1 and b=1.

b) Differentiate by hand numerically with Δx=0.5.

c) Write a Matlab program for 2) then let Δx=1/100.

c) Integrate analytically and evaluate from a=-1 to b=1.

c) Integrate by hand numerically using n=4.

$$\Delta x = 0.5 \qquad \left(x_1^*, x_2^*, x_3^*, x_4^*\right) = \left(-0.75, -0.25, 0.25, 0.75\right)$$

e) Write a Matlab program for 5) then let n=100.

Homework 1:

2) For the function $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}, z \in \mathbb{R}$ (normal dist.)

It is not possible to integrate analytically which is why there are tables (see next slide).

a) Integrate numerically with Matlab using rectangles and reproduce the 1.9 row of the table.

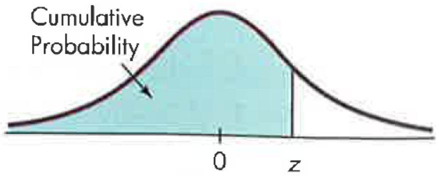
b) Use Matlab's random normal number generator "z=randn(n,1);" to generate random observations. Make a histogram and count the fraction below each number in the 1.9 row. Use "n=10^6;".

c) Compare your results in b) and c) to the table.

Homework 1:

Cumulative Areas of the Standard Normal Distribution (*continued*)

The entries in this table are the cumulative probabilities for the standard normal distribution z (that is, the normal distribution with mean 0 and standard deviation 1). The shaded area under the curve of the standard normal distribution represents the cumulative probability to the left of a z-value in the **left-hand tail**.



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5754
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7258	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7518	0.7549
0.7	0.7580	0.7612	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7996	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9485	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767