Bayesian Complex-Valued Latent Regression

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Outline

$$y = X\beta + \varepsilon$$

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Homework

1. Introduction

Linear Regression Refresher

- For $\beta \neq 0$

y = Observed Data

X = Design Matrix (Regressors)

β = Vector of Regression Coefficients

ε = Error (Residuals)

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i$$

$$y = X\beta + \varepsilon$$

$$\underset{n\times 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \underset{n\times(p+1)}{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{np} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix} \quad \underset{(p+1)\times 1}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \quad \underset{n\times 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

1. Introduction

No y-intercept

- For $\beta = 0$

y = Observed Data

X = Design Matrix (Regressors)

β = Vector of Regression Coefficients

ε = Error (Residuals)

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i$$

$$y = X\beta + \varepsilon$$

$$\underset{n\times 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \underset{n\times p}{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{np} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix} \qquad \underset{p\times 1}{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} \qquad \underset{n\times 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

1. Introduction

Objective

y = Observed Data

X = Design Matrix (Regressors)

β = Vector of Regression Coefficients

ε = Error (Residuals)

Model 1

Real Numbers

$n = 1, p = 1$

X is observed

$y: 1 \times 1$
$X: 1 \times 1$
$\beta: 1 \times 1$

Model 9

Complex Numbers

$n = 4, p = 3$

X is unobserved

$y: 2n \times 1$
$X: 2n \times 2p$
$\beta: 2p \times 1$

$$\underset{n\times 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \underset{n\times p}{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{np} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix} \quad \underset{p\times 1}{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} \quad \underset{n\times 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$y = X\beta + \varepsilon,\ \varepsilon \sim \mathrm{N}\left(0, \sigma^2 I_n\right)$$

1. Introduction

Model 1-9

Models 1,4,7	Models 2,5,8	Models 3,6,9
$n = 1$	$n = 1$	$n = 4$
$p = 1$	$p = 3$	$p = 3$
$y_1 = x_1\beta_1 + \varepsilon_1$	$y_1 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \varepsilon_1$	$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$

	Real (1-3)			Real (4-6)			Complex (7-9)		
y	1×1	1×1	$\mathrm{n} \times 1$	1×1	1×1	$\mathrm{n} \times 1$	1×1	1×1	$\mathrm{n} \times 1$
X	1×1 (observed)	$1 \times p$ (observed)	$\mathrm{n} \times p$ (observed)	1×1 (unobserved)	$1 \times p$ (unobserved)	$\mathrm{n} \times p$ (unobserved)	1×1 (unobserved)	$1 \times p$ (unobserved)	$\mathrm{n} \times p$ (unobserved)
β	1×1	$p \times 1$	$p \times 1$	1×1	$p \times 1$	$p \times 1$	1×1	$p \times 1$	$p \times 1$

2. Models 1-3

Overview of Models 1-3

$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$

X is Observable

Real-valued parameters

Models 1

$n = 1$

$p = 1$

$$y_1 = x_1\beta_1 + \varepsilon_1$$

Models 2

$n = 1$

$p = 3$

$$y_1 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \varepsilon_1$$

Models 3

$n = 4$

$p = 3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

	Model 1	**Model 2**	**Model 3**
y	1×1	1×1	$\mathrm{n} \times 1$
X	1×1 (observed)	$1 \times p$ (observed)	$\mathrm{n} \times p$ (observed)
β	1×1	$p \times 1$	$p \times 1$

- Known Parameters: y, X
- Parameters to estimate:
  - β, σ^2
- "Friendly" Marginals
- Can be solved using calculus (or other numerical techniques)

2. Model 1

Likelihood, Priors, Posterior

$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$

X is Observable

Parameter	Dimensions
y	1×1
X	1×1
β	1×1

Data Likelihood:

$$f\left(y|X,\beta,\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)^2\right]$$ *(normal)*

Priors:

$$f\left(\beta|\beta_0,\sigma^2,n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)^2\right]$$ *(normal)*

$$f\left(\sigma^2|\alpha,\gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right]$$ *(inverse gamma)*

Posterior:

$$f\left(\beta,\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-\left(\frac{2\alpha+2}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)^2 + n_\beta(\beta - \beta_0)^2 + 2\gamma$$

2. Model 1

Marginals & Conditionals

$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$

X is Observable

Parameter	Dimensions
y	1×1
X	1×1
β	1×1

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu}\left(\frac{\beta-\hat{\beta}}{\tau}\right)^2\right]^{-\frac{\nu+1}{2}}$$

$$\beta|X,y \sim t\left(\nu,\hat{\beta},\tau^2\right), E(\beta|X,y) = \hat{\beta} = \left(X^2 + n_\beta\right)^{-1}\left(Xy + n_\beta\beta_0\right), \quad Var(\beta|X,y) = \frac{\nu}{\nu-2}\tau^2$$

$$f\left(\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma_*}{\sigma^2}\right]$$

$$\sigma^2\,|X,y \sim IG\left(\alpha_* = \frac{1+2\alpha}{2}, \gamma_* = \frac{\phi}{2}\right), \quad E\left(\sigma^2|X,y\right) = \frac{\gamma_*}{\alpha_*-1}, \quad Var\left(\sigma^2|X,y\right) = \frac{\gamma_*^2}{(\alpha_*-1)^2(\alpha_*-2)}$$

Posterior Conditionals:

$$f\left(\beta|X,\sigma^2,y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\Sigma_\beta\right)^{-1}\left(\beta-\hat{\beta}\right)^2\right]$$

$$\beta|X,\sigma^2,y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X^2+n_\beta\right)^{-1}\right)$$

$$f\left(\sigma^2|\beta,X,y\right) \propto \left(\sigma^2\right)^{-(\alpha_*+1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2|\beta,X,y \sim IG(\alpha_*,\gamma_{**})$$

Gibbs Sampler:

$$\beta_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(1)} \downarrow \sigma^2_{(2)}$$

$$\nu = 2\alpha + 1$$

$$\tau^2 = \frac{\phi}{\nu}\left(X^2 + n_\beta\right)^{-1}$$

$$\phi = y^2 - \left(X^2+n_\beta\right)\hat{\beta}^2 + n_\beta\beta_0^2 + 2\gamma$$

$$\gamma_{**} = \frac{1}{2}\left[\left(X^2+n_\beta\right)\left(\beta-\hat{\beta}\right)^2 + \phi\right]$$

2. Model 2

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$$

X is Observable

Parameter	Dimensions
y	1×1
X	$1 \times p$
β	$p \times 1$

Data Likelihood:

$$f\left(y|X,\beta,\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right] \qquad \textit{(normal)}$$

Priors:

$$f\left(\beta|\beta_0,\sigma^2,n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right] \qquad \textit{(normal)}$$

$$f\left(\sigma^2|\alpha,\gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right] \qquad \textit{(inverse gamma)}$$

Posterior:

$$f\left(\beta,\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-\left(\frac{p+2\alpha+1}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + 2\gamma$$

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2. Model 2

Marginals & Conditionals

$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$

X is Observable

Parameter	Dimensions
y	1×1
X	$1 \times p$
β	$p \times 1$

Gibbs Sampler:

$\beta_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(1)} \downarrow \sigma^2_{(2)}$

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu}(\beta - \hat{\beta})' \left[\frac{\nu(X'X + n_\beta I_p)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu+p}{2}}$$

$$\beta|X,y \sim t\left(\nu, \hat{\beta}, \mathrm{T}\right),\ E(\beta|X,y) = \hat{\beta} = \left(X'X + n_\beta I_p\right)^{-1}\left(X'y + n_\beta \beta_0\right),\ Var(\beta|X,y) = \frac{\nu}{\nu-2}\mathrm{T}$$

$$f\left(\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma_*}{\sigma^2}\right]$$

$$\sigma^2\,|X,y \sim IG\left(\alpha_* = \frac{p+2\alpha+1}{2}, \gamma_* = \frac{\phi}{2}\right),\quad E\left(\sigma^2|X,y\right) = \frac{\gamma_*}{\alpha_*-1},\ Var\left(\sigma^2|X,y\right) = \frac{\gamma_*^2}{(\alpha_*-1)^2(\alpha_*-2)}$$

Posterior Conditionals:

$$f\left(\beta|X,\sigma^2,y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta - \hat{\beta})'\left(\Sigma_\beta\right)^{-1}(\beta - \hat{\beta})\right]$$

$$\beta|X,\sigma^2,y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_p\right)^{-1}\right)$$

$$f\left(\sigma^2|\beta,X,y\right) \propto \left(\sigma^2\right)^{-(\alpha_*+1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2|\beta,X,y \sim IG(\alpha_*, \gamma_{**})$$

$$\nu = 2\alpha + 1$$

$$\tau^2 = \frac{\phi}{\nu}\left(X'X + n_\beta I_p\right)^{-1}$$

$$\phi = y'y - \hat{\beta}'\left(X'X + n_\beta I_p\right)\hat{\beta} + n_\beta \beta_0{}'\beta_0 + 2\gamma$$

$$\gamma_{**} = \frac{1}{2}\left[(\beta - \hat{\beta})'\left(X'X + n_\beta I_p\right)(\beta - \hat{\beta}) + \phi\right]$$

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2. Model 3

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_n\right)$$

X is Observable

Parameter	Dimensions
y	$n \times 1$
X	$n \times p$
β	$p \times 1$

Data Likelihood:

$$f\left(y|X,\beta,\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$ *(normal)*

Priors:

$$f\left(\beta|\beta_0,\sigma^2,n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right]$$ *(normal)*

$$f\left(\sigma^2|\alpha,\gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right]$$ *(inverse gamma)*

Posterior:

$$f\left(\beta,\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-\left(\frac{n+p+2\alpha}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + 2\gamma$$

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2. Model 3

Marginals & Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_n\right)$$

X is Observable

Parameter	Dimensions
y	$n \times 1$
X	$n \times p$
β	$p \times 1$

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu}(\beta - \hat{\beta})' \left[\frac{\nu(X'X + n_\beta I_p)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu+p}{2}}$$

$$\beta|X,y \sim t\left(\nu, \hat{\beta}, \mathrm{T}\right),\ E(\beta|X,y) = \hat{\beta} = \left(X'X + n_\beta I_p\right)^{-1}\left(X'y + n_\beta \beta_0\right),\ Var(\beta|X,y) = \frac{\nu}{\nu-2}\mathrm{T}$$

$$f\left(\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma_*}{\sigma^2}\right]$$

$$\sigma^2\,|X,y \sim IG\left(\alpha_* = \frac{n+p+2\alpha}{2}, \gamma_* = \frac{\phi}{2}\right),\quad E\left(\sigma^2|X,y\right) = \frac{\gamma_*}{\alpha_*-1},\ Var\left(\sigma^2|X,y\right) = \frac{\gamma_*^2}{(\alpha_*-1)^2(\alpha_*-2)}$$

Posterior Conditionals:

$$f\left(\beta|X,\sigma^2,y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta - \hat{\beta})'\left(\Sigma_\beta\right)^{-1}(\beta - \hat{\beta})\right]$$

$$\beta|X,\sigma^2,y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_p\right)^{-1}\right)$$

$$f\left(\sigma^2|\beta,X,y\right) \propto \left(\sigma^2\right)^{-(\alpha_*+1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2|\beta,X,y \sim IG(\alpha_*, \gamma_{**})$$

Gibbs Sampler:

$$\beta_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(1)} \downarrow \sigma^2_{(2)}$$

$$\nu = 2\alpha + n$$

$$\tau^2 = \frac{\phi}{\nu}\left(X'X + n_\beta I_p\right)^{-1}$$

$$\phi = y'y - \hat{\beta}'\left(X'X + n_\beta I_p\right)\hat{\beta} + n_\beta \beta_0{}'\beta_0 + 2\gamma$$

$$\gamma_{**} = \frac{1}{2}\left[(\beta - \hat{\beta})'\left(X'X + n_\beta I_p\right)(\beta - \hat{\beta}) + \phi\right]$$

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

X is Observable

2. Models 1-3

Quick Review

Posterior: $f(\beta, \sigma^2 | X, y) \propto (\sigma^2)^{-\left(\frac{n+p+2\alpha}{2}+1\right)} \exp\left[-\frac{(y-X\beta)'(y-X\beta)+n_\beta(\beta-\beta_0)'(\beta-\beta_0)+2\gamma}{2\sigma^2}\right]$

Marginal for $\boldsymbol{\beta}$: $f(\beta | X, y) \propto \left[1 + \frac{1}{\nu}(\beta-\hat{\beta})'\left[\frac{\nu(X'X+n_\beta I_p)}{\phi}\right](\beta-\hat{\beta})\right]^{-\frac{\nu+p}{2}}$

Conditional for $\boldsymbol{\beta}$: $f(\beta | X, \sigma^2, y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\beta-\hat{\beta})'(X'X+n_\beta I_p)^{-1}(\beta-\hat{\beta})\right]$

	Model 1 $\boldsymbol{n=1, p=1}$	**Model 2** $\boldsymbol{n=1, p=3}$	**Model 3** $\boldsymbol{n=4, p=3}$
α_*	$\frac{2\alpha+1+1}{2}$	$\frac{2\alpha+p+1}{2}$	$\frac{2\alpha+p+n}{2}$
ν	$2\alpha+1$	$2\alpha+1$	$2\alpha+n$
β Marginal exponent	$\frac{\nu+1}{2}$	$\frac{\nu+p}{2}$	$\frac{\nu+p}{2}$

3. Models 4-6

Overview of Models 4-6

$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_n\right)$

X is Unobservable

Real-valued parameters

Model 4

$n = 1$

$p = 1$

$$y_1 = x_1\beta_1 + \varepsilon_1$$

Model 5

$n = 1$

$p = 3$

$$y_1 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \varepsilon_1$$

Model 6

$n = 4$

$p = 3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

	Model 4	Model 5	Model 6
y	1×1	1×1	$\mathrm{n} \times 1$
X	1×1 (observed)	$1 \times p$ (observed)	$\mathrm{n} \times p$ (observed)
β	1×1	$p \times 1$	$p \times 1$

- Known Parameters: y
- Parameters to estimate:
  - X, β, σ^2
- Marginals not "Friendly"
- Can still integrate out σ^2
- Need to use numerical methods to find expectations of the marginals

3. Model 4

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$$

X is Unobservable

Parameter	Dimensions
y	1×1
X	1×1
β	1×1

Data Likelihood:

$$f\left(y|X, \beta, \sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)^2\right] \quad \textit{(normal)}$$

Priors:

$$f\left(\beta|\beta_0, \sigma^2, n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)^2\right] \quad \textit{(normal)}$$

$$f\left(X|X_0, \sigma^2, n_x\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{n_x}{2\sigma^2}(X - X_0)^2\right] \quad \textit{(normal)}$$

$$f\left(\sigma^2|\alpha, \gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right] \quad \textit{(inverse gamma)}$$

Posterior:

$$f\left(\beta, \sigma^2|X, y\right) \propto \left(\sigma^2\right)^{-\left(\frac{2\alpha+3}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)^2 + n_\beta(\beta - \beta_0)^2 + n_X(X - X_0)^2 + 2\gamma$$

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3. Model 4

Marginals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$$

X is Unobservable

Parameter	Dimensions
y	1×1
X	1×1
β	1×1

Gibbs Sampler:

$\beta_{(0)}$
$\downarrow$
$X_{(1)}$
$\downarrow$
$\beta_{(1)}$
$\downarrow$
$X_{(2)}$

Posterior Marginals:

$$f(\beta|X, y) \propto \left[1 + \frac{1}{\nu_\beta}\left(\frac{\beta - \hat{\beta}}{\tau}\right)^2\right]^{-\frac{\nu_\beta + 1}{2}}$$

$$\beta|X, y \sim t\left(\nu_\beta, \hat{\beta}, \tau^2\right), \quad E(\beta|X, y) = \hat{\beta} = \left(X^2 + n_\beta\right)^{-1}\left(Xy + n_\beta \beta_0\right), \qquad Var(\beta|X, y) = \frac{\nu_\beta}{\nu_\beta - 2}\tau^2$$

$$f(X|\beta, y) \propto \left[1 + \frac{1}{\nu_x}\left(\frac{X - \hat{X}}{\tau}\right)^2\right]^{-\frac{\nu_x + 1}{2}}$$

$$X|\beta, y \sim t\left(\nu_x, \hat{X}, \delta^2\right), \quad E(X|\beta, y) = \hat{X} = \left(\beta^2 + n_x\right)^{-1}\left(\beta y + n_x X_0\right), \qquad Var(X|\beta, y) = \frac{\nu_x}{\nu_x - 2}\delta^2$$

- Integrate out σ^2
- Can't integrate out X or β
- Can use Gibbs sampling to estimate X and β
- Mainly use marginals to check conditionals

$$\nu_\beta = 2\alpha,\ \nu_x = 2\alpha$$

$$\tau^2 = \frac{\phi}{\nu_\beta}\left(X^2 + n_\beta\right)^{-1},\ \delta^2 = \frac{\theta}{\nu_x}\left(\beta^2 + n_x\right)^{-1}$$

$$\phi = y^2 - \left(X^2 + n_\beta\right)\hat{\beta}^2 + n_\beta \beta_0^2 + n_x\left(X - X_0\right) + 2\gamma$$

$$\theta = y^2 - \left(\beta^2 + n_x\right)\hat{X}^2 + n_x X_0^2 + n_\beta\left(\beta - \beta_0\right) + 2\gamma$$

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3. Model 4

Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$$

X is Unobservable

Parameter	Dimensions
y	1×1
X	1×1
β	1×1

Posterior Conditionals:

$$f\left(\beta | X, \sigma^2, y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\Sigma_\beta\right)^{-1}\left(\beta - \hat{\beta}\right)^2\right]$$

$$\beta | X, \sigma^2, y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X^2 + n_\beta\right)^{-1}\right)$$

$$f\left(X | \beta, \sigma^2, y\right) \propto \left|\Sigma_X\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\Sigma_X\right)^{-1}\left(X - \hat{X}\right)^2\right]$$

$$X | \beta, \sigma^2, y \sim N\left(\hat{X}, \Sigma_X = \sigma^2\left(\beta^2 + n_x\right)^{-1}\right)$$

$$f\left(\sigma^2 | \beta, X, y\right) \propto \left(\sigma^2\right)^{-(\alpha_* + 1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2 | \beta, X, y \sim IG(\alpha_*, \gamma_{**})$$

Gibbs Sampler:

$$\beta_{(0)}, X_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(0)}, \sigma^2_{(1)} \downarrow X_{(1)} \downarrow X_{(1)}, \sigma^2_{(1)} \downarrow \beta_{(1)}$$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

$$\alpha_* = \frac{2\alpha + 2}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y - X\beta)^2 + n_\beta(\beta - \beta_0)^2 + n_x(X - X_0)^2 + 2\gamma\right]$$

3. Model 5

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right)$$

X is Unobservable

Parameter	Dimensions
y	1×1
X	$1 \times p$
β	$p \times 1$

Data Likelihood:

$$f\left(y|X,\beta,\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$ *(normal)*

Priors:

$$f\left(\beta|\beta_0,\sigma^2,n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right]$$ *(normal)*

$$f\left(X|X_0,\sigma^2,n_x\right) \propto \left(\sigma^2\right)^{-\frac{p}{2}} \exp\left[-\frac{n_x}{2\sigma^2}(X - X_0)(X - X_0)'\right]$$ *(normal)*

$$f\left(\sigma^2|\alpha,\gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right]$$ *(inverse gamma)*

Posterior:

$$f\left(\beta,\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-\left(\frac{2p+2\alpha+1}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(X - X_0)(X - X_0)' + 2\gamma$$

3. Model 5
Marginals

$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2)$

X is Unobservable

Parameter	Dimensions
y	1×1
X	$1 \times p$
β	$p \times 1$

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_p)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + p}{2}}$$

$$\beta|X,y \sim t(\nu_\beta, \hat{\beta}, \mathrm{T}), E(\beta|X,y) = \hat{\beta} = (X'X + n_\beta I_p)^{-1}(X'y + n_\beta \beta_0),\ \ Var(\beta|X,y) = \frac{\nu_\beta}{\nu_\beta - 2}\mathrm{T}$$

$$f(X|\beta,y) \propto \left[1 + \frac{1}{\nu_x}(X - \hat{X}) \left[\frac{\nu_x(\beta\beta' + n_x I_p)}{\theta}\right](X - \hat{X})'\right]^{-\frac{\nu_x + p}{2}}$$

$$X|\beta,y \sim t(\nu_x, \hat{X}, \Delta), E(X|\beta,y) = \hat{X} = (y\beta' + n_x X_0)(\beta\beta' + n_x I_p)^{-1},\ \ Var(X|\beta,y) = \frac{\nu_x}{\nu_x - 2}\Delta$$

Gibbs Sampler:

$\beta_{(0)}$
↓
$X_{(1)}$
↓
$\beta_{(1)}$
↓
$X_{(2)}$

- Integrate out σ^2
- Can't integrate out X or β
- Can use Gibbs sampling to estimate X and β
- Mainly use marginals to check conditionals

$$\nu_\beta = p + 2\alpha + 1,\ \nu_x = p + 2\alpha + 1$$

$$\mathrm{T} = \frac{\phi}{\nu_\beta}(X'X + n_\beta I_p)^{-1},\ \Delta = \frac{\theta}{\nu_x}(\beta\beta' + n_x I_p)^{-1}$$

$$\phi = y'y - \hat{\beta}'(X'X + n_\beta I_p)\hat{\beta} + n_x(X - X_0)(X - X_0)' + n_\beta \beta_0'\beta_0 + 2\gamma$$

$$\theta = y'y - \hat{X}(\beta\beta' + n_x I_p)\hat{X}' + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x X_0 X_0' + 2\gamma$$

3. Model 5
Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2)$$

X is Unobservable

Parameter	Dimensions
y	1×1
X	$1 \times p$
β	$p \times 1$

Posterior Conditionals:

$$f(\beta | X, \sigma^2, y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta - \hat{\beta})'(\Sigma_\beta)^{-1}(\beta - \hat{\beta})\right]$$

$$\beta | X, \sigma^2, y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2 (X'X + n_\beta I_p)^{-1}\right)$$

$$f(X | \beta, \sigma^2, y) \propto |\Sigma_X|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(X - \hat{X})(\Sigma_X)^{-1}(X - \hat{X})'\right]$$

$$X | \beta, \sigma^2, y \sim N\left(\hat{X}, \Sigma_X = \sigma^2 (\beta\beta' + n_x I_p)^{-1}\right)$$

$$f(\sigma^2 | \beta, X, y) \propto (\sigma^2)^{-(\alpha_* + 1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2 | \beta, X, y \sim IG(\alpha_*, \gamma_{**})$$

Gibbs Sampler:

$\beta_{(0)}, X_{(0)}$
$\downarrow$
$\sigma^2_{(1)}$
$\downarrow$
$\beta_{(0)}, \sigma^2_{(1)}$
$\downarrow$
$X_{(1)}$
$\downarrow$
$X_{(1)}, \sigma^2_{(1)}$
$\downarrow$
$\beta_{(1)}$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

$$\alpha_* = \frac{2p + 2\alpha + 1}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(X - X_0)(X - X_0)' + 2\gamma\right]$$

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3. Model 6

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_n\right)$$

X is Unobservable

Parameter	Dimensions
y	$n \times 1$
X	$n \times p$
β	$p \times 1$

Data Likelihood:

$$f\left(y|X, \beta, \sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$ *(normal)*

Priors:

$$f\left(\beta|\beta_0, \sigma^2, n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right]$$ *(normal)*

$$f\left(X|X_0, \sigma^2, n_x\right) \propto \left(\sigma^2\right)^{-\frac{np}{2}} \exp\left[-\frac{n_x}{2\sigma^2} tr\{(X - X_0)(X - X_0)'\}\right]$$ *(multivariate normal)*

$$f\left(\sigma^2|\alpha, \gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right]$$ *(inverse gamma)*

Posterior:

$$f\left(\beta, \sigma^2|X, y\right) \propto \left(\sigma^2\right)^{-\left(\frac{np+n+p+2\alpha}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{(X - X_0)(X - X_0)'\} + 2\gamma$$

tr = trace(matrix)
Sum of the diagonals

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3. Model 6

Marginals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

X is Unobservable

Parameter	Dimensions
y	$n \times 1$
X	$n \times p$
β	$p \times 1$

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_p)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + p}{2}}$$

$$\beta|X,y \sim t(\nu_\beta, \hat{\beta}, \mathrm{T}), E(\beta|X,y) = \hat{\beta} = (X'X + n_\beta I_p)^{-1}(X'y + n_\beta \beta_0),\ Var(\beta|X,y) = \frac{\nu_\beta}{\nu_\beta - 2}\mathrm{T}$$

$$f(X|\beta,y) \propto \left[1 + \frac{1}{\nu_x}(X - \hat{X}) \left[\frac{\nu_x(\beta\beta' + n_x I_p)}{\theta}\right](X - \hat{X})'\right]^{-\frac{\nu_x + np}{2}}$$

$$X|\beta,y \sim Mt(\nu_x, \hat{X}, \Delta),\ E(X|\beta,y) = \hat{X} = (y\beta' + n_x X_0)(\beta\beta' + n_x I_p)^{-1},\ Var(X|\beta,y) = \frac{\nu_x}{\nu_x - 2}\Delta$$

Gibbs Sampler:

$\beta_{(0)}$
↓
$X_{(1)}$
↓
$\beta_{(1)}$
↓
$X_{(2)}$

- Integrate out σ^2
- Can't integrate out X or β
- Can use Gibbs sampling to estimate X and β
- Mainly use marginals to check conditionals

$$\nu_\beta = np + p + 2\alpha,\ \nu_x = n + p + 2\alpha$$

$$\mathrm{T} = \frac{\phi}{\nu_\beta}(X'X + n_\beta I_p)^{-1},\ \Delta = \frac{\theta}{\nu_x}(\beta\beta' + n_x I_p)^{-1}$$

$$\phi = y'y - \hat{\beta}'(X'X + n_\beta I_p)\hat{\beta} + n_x tr\{(X - X_0)(X - X_0)'\} + n_\beta \beta_0'\beta_0 + 2\gamma$$

$$\theta = y'y - tr\{\hat{X}(\beta\beta' + n_x I_p)\hat{X}'\} + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{X_0 X_0'\} + 2\gamma$$

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3. Model 6

Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_n\right)$$

X is Unobservable

Parameter	Dimensions
y	$n \times 1$
X	$n \times p$
β	$p \times 1$

Posterior Conditionals:

$$f(\beta|X,\sigma^2,y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta-\hat{\beta})'(\Sigma_\beta)^{-1}(\beta-\hat{\beta})\right]$$

$$\beta|X,\sigma^2,y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_p\right)^{-1}\right)$$

$$f(X|\beta,\sigma^2,y) \propto |\Sigma_X|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}tr\{(X-\hat{X})(\Sigma_X)^{-1}(X-\hat{X})'\}\right]$$

$$X|\beta,\sigma^2,y \sim MN\left(\hat{X}, \Sigma_X = \sigma^2\left(\beta\beta' + n_x I_p\right)^{-1}\right)$$

$$f(\sigma^2|\beta,X,y) \propto (\sigma^2)^{-(\alpha_*+1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2|\beta,X,y \sim IG(\alpha_*, \gamma_{**})$$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

$$\alpha_* = \frac{np+n+p+2\alpha}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y-X\beta)'(y-X\beta) + n_\beta(\beta-\beta_0)'(\beta-\beta_0) + n_x tr\{(X-X_0)(X-X_0)'\} + 2\gamma\right]$$

Gibbs Sampler:

$$\beta_{(0)}, X_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(0)}, \sigma^2_{(1)} \downarrow X_{(1)} \downarrow X_{(1)}, \sigma^2_{(1)} \downarrow \beta_{(1)}$$

3. Models 4-6

Quick Review

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

X is Unobservable

Posterior: $f(\beta, \sigma^2 | X, y) \propto (\sigma^2)^{-\left(\frac{n+p+2\alpha}{2}+1\right)} \exp\left[-\frac{1}{2\sigma^2}\{(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{(X - X_0)(X - X_0)'\} + 2\gamma\}\right]$

Marginal for $\boldsymbol{\beta}$: $f(\beta | X, y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_p)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + p}{2}}$

Conditional for $\boldsymbol{\beta}$: $f(\beta | X, \sigma^2, y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\beta - \hat{\beta})'(X'X + n_\beta I_p)(\beta - \hat{\beta})\right]$

Marginal for $\boldsymbol{X}$: $f(X | \beta, y) \propto \left[1 + \frac{1}{\nu_x}(X - \hat{X}) \left[\frac{\nu_x(\beta\beta' + n_x I_p)}{\theta}\right](X - \hat{X})'\right]^{-\frac{\nu_x + np}{2}}$

Conditional for $\boldsymbol{X}$: $f(X | \beta, \sigma^2, y) \propto |\Sigma_X|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2} tr\{(X - \hat{X})(\beta\beta' + n_x I_p)(X - \hat{X})'\}\right]$

	Model 4 $\boldsymbol{n = 1, p = 1}$	Model 5 $\boldsymbol{n = 1, p = 3}$	Model 6 $\boldsymbol{n = 4, p = 3}$
α_*	$\frac{2\alpha + 1 + 1 + 1}{2}$	$\frac{2\alpha + p + p + 1}{2}$	$\frac{2\alpha + p + np + n}{2}$
ν_β	$2\alpha + 1 + 1$	$2\alpha + p + 1$	$2\alpha + np + n$
β Marginal exponent	$\frac{\nu_\beta + 1}{2}$	$\frac{\nu_\beta + p}{2}$	$\frac{\nu_\beta + p}{2}$
ν_x	$2\alpha + 1 + 1$	$2\alpha + p + 1$	$2\alpha + p + n$
X Marginal exponent	$\frac{\nu_x + 1}{2}$	$\frac{\nu_x + p}{2}$	$\frac{\nu_x + np}{2}$

4. Complex-Valued Statistics

Basic Overview of Complex Number

Complex Model: $y_c = X_c\beta_c + \varepsilon_c$

$y_c = y_R + iy_I$ $X_c = X_R + iX_I$ $\beta_c = \beta_R + i\beta_I$ $\varepsilon_c = \varepsilon_R + i\varepsilon_I$

Real-valued Isomorphism:

$$y = \begin{bmatrix} y_R \\ y_I \end{bmatrix} \qquad X = \begin{bmatrix} X_R & -X_I \\ X_I & X_R \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_R \\ \beta_I \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_R \\ \varepsilon_I \end{bmatrix}$$

Model: $y = X\beta + \varepsilon$

$$\begin{bmatrix} y_R \\ y_I \end{bmatrix} = \begin{bmatrix} X_R & -X_I \\ X_I & X_R \end{bmatrix} \begin{bmatrix} \beta_R \\ \beta_I \end{bmatrix} + \begin{bmatrix} \varepsilon_R \\ \varepsilon_I \end{bmatrix}$$

Example:

$$\begin{bmatrix} \beta_{1c} \\ \beta_{2c} \\ \beta_{3c} \end{bmatrix} = \begin{bmatrix} \beta_{1R} + i\beta_{1I} \\ \beta_{2R} + i\beta_{2I} \\ \beta_{3R} + i\beta_{3I} \end{bmatrix} \rightarrow \begin{bmatrix} \beta_{1R} \\ \beta_{2R} \\ \beta_{3R} \\ \beta_{1I} \\ \beta_{2I} \\ \beta_{3I} \end{bmatrix} \longrightarrow \begin{bmatrix} \beta_{1c} \\ \beta_{2c} \\ \beta_{3c} \end{bmatrix} = \begin{bmatrix} 5 + 7i \\ 3 + 4i \\ 9 + 10i \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 3 \\ 9 \\ 7 \\ 4 \\ 10 \end{bmatrix}$$

5. Models 7-9
Complex-Valued

$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$

X is Unobservable

Complex-valued parameters

Model 7

$n = 1$

$p = 1$

$$y_{1c} = x_{1c}\beta_{1c} + \varepsilon_{1c}$$

Model 8

$n = 1$

$p = 3$

$$y_{1c} = \begin{bmatrix} x_{1c} & x_{2c} & x_{3c} \end{bmatrix} \begin{bmatrix} \beta_{1c} \\ \beta_{2c} \\ \beta_{3c} \end{bmatrix} + \varepsilon_{1c}$$

Model 9

$n = 4$

$p = 3$

$$\begin{bmatrix} y_{1c} \\ y_{2c} \\ y_{3c} \\ y_{4c} \end{bmatrix} = \begin{bmatrix} x_{11c} & x_{12c} & x_{13c} \\ x_{21c} & x_{22c} & x_{23c} \\ x_{31c} & x_{32c} & x_{33c} \\ x_{41c} & x_{42c} & x_{43c} \end{bmatrix} \begin{bmatrix} \beta_{1c} \\ \beta_{2c} \\ \beta_{3c} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1c} \\ \varepsilon_{2c} \\ \varepsilon_{3c} \\ \varepsilon_{4c} \end{bmatrix}$$

	Model 7	**Model 8**	**Model 9**
y	1×1	1×1	$\mathrm{n} \times 1$
X	1×1 (observed)	$1 \times p$ (observed)	$\mathrm{n} \times p$ (observed)
β	1×1	$p \times 1$	$p \times 1$

- Known Parameters: y
- Parameters to estimate:
  - $X,\ \beta,\ \sigma^2$
- Marginals not “Friendly”
- Need to use numerical methods to find expectations of the marginals

5. Models 7-9

Real-Valued Representation

$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$

X is Unobservable

Real-valued parameters

Model 7 $n = 1$ $p = 1$	$\begin{bmatrix} y_{1R} \\ y_{1I} \end{bmatrix} = \begin{bmatrix} x_{1R} & -x_{1I} \\ x_{1I} & x_{1R} \end{bmatrix} \begin{bmatrix} \beta_{1R} \\ \beta_{1I} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1R} \\ \varepsilon_{1I} \end{bmatrix}$	$y: 2 \times 1$ $X: 2 \times 2$ $\beta: 2 \times 1$
Model 8 $n = 1$ $p = 3$	$\begin{bmatrix} y_{1R} \\ y_{1I} \end{bmatrix} = \begin{bmatrix} x_{1R} & x_{2R} & x_{3R} & -x_{1I} & -x_{2I} & -x_{3I} \\ x_{1I} & x_{2I} & x_{3I} & x_{1R} & x_{2R} & x_{3R} \end{bmatrix} \begin{bmatrix} \beta_{1R} \\ \beta_{2R} \\ \beta_{3R} \\ \beta_{1I} \\ \beta_{2I} \\ \beta_{3I} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1R} \\ \varepsilon_{1I} \end{bmatrix}$	$y: 2 \times 1$ $X: 2 \times 2p$ $\beta: 2p \times 1$
Model 9 $n = 4$ $p = 3$	$\begin{bmatrix} y_{1R} \\ y_{2R} \\ y_{3R} \\ y_{4R} \\ y_{1I} \\ y_{2I} \\ y_{3I} \\ y_{4I} \end{bmatrix} = \begin{bmatrix} x_{11R} & x_{12R} & x_{13R} & -x_{11I} & -x_{12I} & -x_{13I} \\ x_{21R} & x_{22R} & x_{23R} & -x_{21I} & -x_{22I} & -x_{23I} \\ x_{31R} & x_{32R} & x_{33R} & -x_{31I} & -x_{32I} & -x_{33I} \\ x_{41R} & x_{42R} & x_{43R} & -x_{41I} & -x_{42I} & -x_{43I} \\ x_{11I} & x_{12I} & x_{13I} & x_{11R} & x_{12R} & x_{13R} \\ x_{21I} & x_{22I} & x_{23I} & x_{21R} & x_{22R} & x_{23R} \\ x_{31I} & x_{32I} & x_{33I} & x_{31R} & x_{32R} & x_{33R} \\ x_{41I} & x_{42I} & x_{43I} & x_{41R} & x_{42R} & x_{43R} \end{bmatrix} \begin{bmatrix} \beta_{1R} \\ \beta_{2R} \\ \beta_{3R} \\ \beta_{1I} \\ \beta_{2I} \\ \beta_{3I} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1R} \\ \varepsilon_{2R} \\ \varepsilon_{3R} \\ \varepsilon_{4R} \\ \varepsilon_{1I} \\ \varepsilon_{2I} \\ \varepsilon_{3I} \\ \varepsilon_{4I} \end{bmatrix}$	$y: 2n \times 1$ $X: 2n \times 2p$ $\beta: 2p \times 1$

5. Model 7

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_2\right)$$

X is Unobservable

Parameter	Dimensions
y	2×1
X	2×2
x	1×2
β	2×1

	Dim
y_R, y_I	1×1
x_R, x_I	1×1
β_R, β_I	1×1

$$\underset{2\times1}{y} = \begin{bmatrix} y_R \\ y_I \end{bmatrix} \qquad \underset{2\times2}{X} = \begin{bmatrix} x_R & -x_I \\ x_I & x_R \end{bmatrix}$$

$$\underset{1\times2}{x} = \begin{bmatrix} x_R & x_I \end{bmatrix} \qquad \underset{2\times1}{\beta} = \begin{bmatrix} \beta_R \\ \beta_I \end{bmatrix}$$

Data Likelihood:

$$f\left(y|X,\beta,\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{2}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right] \qquad \textit{(normal)}$$

Priors:

$$f\left(\beta|\beta_0,\sigma^2,n_\beta\right) \propto \left(\sigma^2\right)^{-\frac{2}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right] \qquad \textit{(normal)}$$

$$f\left(x|x_0,\sigma^2,n_x\right) \propto \left(\sigma^2\right)^{-\frac{2}{2}} \exp\left[-\frac{n_x}{2\sigma^2}(x - x_0)(x - x_0)'\right] \qquad \textit{(multivariate normal)}$$

$$f\left(\sigma^2|\alpha,\gamma\right) \propto \left(\sigma^2\right)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right] \qquad \textit{(inverse gamma)}$$

Posterior:

$$f\left(\beta,\sigma^2|X,y\right) \propto \left(\sigma^2\right)^{-\left(\frac{2\alpha+6}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(x - x_0)(x - x_0)' + 2\gamma$$

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5. Model 7
Marginals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_2\right)$$

X is Unobservable

Parameter	Dimensions
y	2×1
X	2×2
x	1×2
β	2×1

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_2)}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + 2}{2}}$$

$$\beta|X,y \sim t\left(\nu_\beta, \hat{\beta}, \mathrm{T}\right), E(\beta|X,y) = \hat{\beta} = \left(X'X + n_\beta I_2\right)^{-1}\left(X'y + n_\beta \beta_0\right),\ Var(\beta|X,y) = \frac{\nu_\beta}{\nu_\beta - 2}\mathrm{T}$$

$$f(x|\beta,y) \propto \left[1 + \frac{1}{\nu_x}(x - \hat{x}) \left[\frac{\nu_x(CC' + n_x I_2)}{\theta}\right](x - \hat{x})'\right]^{-\frac{\nu_x + 2}{2}}$$

$$x|\beta,y \sim t(\nu_x, \hat{x}, \Delta),\ E(x|\beta,y) = \hat{x} = (YC' + n_x x_0)(CC' + n_x I_2)^{-1}, Var(x|\beta,y) = \frac{\nu_x}{\nu_x - 2}\Delta$$

Gibbs Sampler:

$$\beta_{(0)} \downarrow x_{(1)} \downarrow \beta_{(1)} \downarrow x_{(2)}$$

$$\underset{2\times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\underset{1\times 2}{Y} = [y_R \quad y_I]$$

$$\nu_\beta = 2\alpha + 4,\ \nu_x = 2\alpha + 4$$

$$\mathrm{T} = \frac{\phi}{\nu_\beta}\left(X'X + n_\beta I_2\right)^{-1},\ \Delta = \frac{\theta}{\nu_x}(CC' + n_x I_2)^{-1}$$

$$\phi = y'y - \hat{\beta}'\left(X'X + n_\beta I_2\right)\hat{\beta} + n_x(x - x_0)(x - x_0)' + n_\beta \beta_0'\beta_0 + 2\gamma$$

$$\theta = y'y - \hat{x}(CC' + n_x I_2)\hat{x}' + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x x_0 x_0' + 2\gamma$$

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5. Model 7
Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_2\right)$$

X is Unobservable

Parameter	Dimensions
y	2×1
X	2×2
x	1×2
β	2×1

Posterior Conditionals:

$$f\left(\beta | X, \sigma^2, y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\beta - \hat{\beta}\right)'\left(\Sigma_\beta\right)^{-1}\left(\beta - \hat{\beta}\right)\right]$$

$$\beta | X, \sigma^2, y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_2\right)^{-1}\right)$$

$$f\left(x | \beta, \sigma^2, y\right) \propto \left|\Sigma_X\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x - \hat{x})\left(\Sigma_X\right)^{-1}(x - \hat{x})'\right]$$

$$x | \beta, \sigma^2, y \sim N\left(\hat{x}, \Sigma_X = \sigma^2\left(CC' + n_x I_2\right)^{-1}\right)$$

$$f\left(\sigma^2 | \beta, X, y\right) \propto \left(\sigma^2\right)^{-(\alpha_* + 1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2 | \beta, X, y \sim IG(\alpha_*, \gamma_{**})$$

Gibbs Sampler:

$$\beta_{(0)}, X_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(0)}, \sigma^2_{(1)} \downarrow X_{(1)} \downarrow X_{(1)}, \sigma^2_{(1)} \downarrow \beta_{(1)}$$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

$$\underset{2\times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\alpha_* = \frac{2\alpha + 6}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(x - x_0)(x - x_0)' + 2\gamma\right]$$

5. Model 8

Likelihood, Priors, Posterior

$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_2)$

X is Unobservable

Parameter	Dimensions
y	2×1
X	$2 \times 2p$
x	$1 \times 2p$
β	$2p \times 1$

	Dim
y_R, y_I	1×1
x_R, x_I	$1 \times p$
β_R, β_I	$p \times 1$

$$\underset{2\times 1}{y} = \begin{bmatrix} y_R \\ y_I \end{bmatrix} \qquad \underset{2\times 2p}{X} = \begin{bmatrix} x_R & -x_I \\ x_I & x_R \end{bmatrix}$$

$$\underset{1\times 2p}{x} = \begin{bmatrix} x_R & x_I \end{bmatrix} \qquad \underset{2p\times 1}{\beta} = \begin{bmatrix} \beta_R \\ \beta_I \end{bmatrix}$$

Data Likelihood:

$$f(y|X,\beta,\sigma^2) \propto (\sigma^2)^{-\frac{2}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right] \qquad \textit{(normal)}$$

Priors:

$$f(\beta|\beta_0,\sigma^2,n_\beta) \propto (\sigma^2)^{-\frac{2p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right] \qquad \textit{(normal)}$$

$$f(x|x_0,\sigma^2,n_x) \propto (\sigma^2)^{-\frac{2p}{2}} \exp\left[-\frac{n_x}{2\sigma^2}(x - x_0)(x - x_0)'\right] \qquad \textit{(multivariate normal)}$$

$$f(\sigma^2|\alpha,\gamma) \propto (\sigma^2)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right] \qquad \textit{(inverse gamma)}$$

Posterior:

$$f(\beta,\sigma^2|X,y) \propto (\sigma^2)^{-\left(\frac{4p+2\alpha+2}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(x - x_0)(x - x_0)' + 2\gamma$$

5. Model 8
Marginals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_2\right)$$

X is Unobservable

Parameter	Dimensions
y	2×1
X	$2 \times 2p$
x	$1 \times 2p$
β	$2p \times 1$

Gibbs Sampler:
$\beta_{(0)}$
$\downarrow$
$x_{(1)}$
$\downarrow$
$\beta_{(1)}$
$\downarrow$
$x_{(2)}$

Posterior Marginals:

$$f(\beta|X, y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_{2p})}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + 2p}{2}}$$

$$\beta|X, y \sim t\left(\nu_\beta, \hat{\beta}, \mathrm{T}\right), E(\beta|X, y) = \hat{\beta} = \left(X'X + n_\beta I_{2p}\right)^{-1}\left(X'y + n_\beta \beta_0\right), Var(\beta|X, y) = \frac{\nu_\beta}{\nu_\beta - 2}\mathrm{T}$$

$$f(x|\beta, y) \propto \left[1 + \frac{1}{\nu_x}(x - \hat{x}) \left[\frac{\nu_x(CC' + n_x I_{2p})}{\theta}\right](x - \hat{x})'\right]^{-\frac{\nu_x + 2p}{2}}$$

$$x|\beta, y \sim t(\nu_x, \hat{x}, \Delta),\ \ E(x|\beta, y) = \hat{x} = (YC' + n_x x_0)\left(CC' + n_x I_{2p}\right)^{-1},\ \ Var(x|\beta, y) = \frac{\nu_x}{\nu_x - 2}\Delta$$

$$\underset{2p\times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\underset{1\times 2}{Y} = \begin{bmatrix} y_R & y_I \end{bmatrix}$$

$$\nu_\beta = 2p + 2\alpha + 2,\ \nu_x = 2p + 2\alpha + 2$$

$$\mathrm{T} = \frac{\phi}{\nu_\beta}\left(X'X + n_\beta I_{2p}\right)^{-1},\ \Delta = \frac{\theta}{\nu_x}\left(CC' + n_x I_{2p}\right)^{-1}$$

$$\phi = y'y - \hat{\beta}'\left(X'X + n_\beta I_{2p}\right)\hat{\beta} + n_x(x - x_0)(x - x_0)' + n_\beta \beta_0{}'\beta_0 + 2\gamma$$

$$\theta = y'y - \hat{x}\left(CC' + n_x I_{2p}\right)\hat{x}' + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x x_0 x_0{}' + 2\gamma$$

5. Model 8

Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_2\right)$$

X is Unobservable

Parameter	Dimensions
y	2×1
X	$2 \times 2p$
x	$1 \times 2p$
β	$2p \times 1$

Posterior Conditionals:

$$f\left(\beta | X, \sigma^2, y\right) \propto \left|\Sigma_\beta\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\beta - \hat{\beta}\right)'\left(\Sigma_\beta\right)^{-1}\left(\beta - \hat{\beta}\right)\right]$$

$$\beta | X, \sigma^2, y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_{2p}\right)^{-1}\right)$$

$$f\left(x | \beta, \sigma^2, y\right) \propto \left|\Sigma_X\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(x - \hat{x}\right)\left(\Sigma_X\right)^{-1}\left(x - \hat{x}\right)'\right]$$

$$x | \beta, \sigma^2, y \sim N\left(\hat{x}, \Sigma_X = \sigma^2\left(CC' + n_x I_{2p}\right)^{-1}\right)$$

$$f\left(\sigma^2 | \beta, X, y\right) \propto \left(\sigma^2\right)^{-(\alpha_* + 1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2 | \beta, X, y \sim IG(\alpha_*, \gamma_{**})$$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

Gibbs Sampler:

$$\beta_{(0)}, X_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(0)}, \sigma^2_{(1)} \downarrow X_{(1)} \downarrow X_{(1)}, \sigma^2_{(1)} \downarrow \beta_{(1)}$$

$$\underset{2p \times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\alpha_* = \frac{4p + 2\alpha + 2}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x(x - x_0)(x - x_0)' + 2\gamma\right]$$

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5. Model 9

Likelihood, Priors, Posterior

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_{2n}\right)$$

X is Unobservable

Parameter	Dimensions
y	$2n \times 1$
X	$2n \times 2p$
x	$n \times 2p$
β	$2p \times 1$

	Dim
y_R, y_I	$n \times 1$
x_R, x_I	$n \times p$
β_R, β_I	$p \times 1$

Data Likelihood:

$$f(y|X,\beta,\sigma^2) \propto (\sigma^2)^{-\frac{2n}{2}} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right] \quad \textit{(normal)}$$

Priors:

$$f(\beta|\beta_0,\sigma^2,n_\beta) \propto (\sigma^2)^{-\frac{2p}{2}} \exp\left[-\frac{n_\beta}{2\sigma^2}(\beta - \beta_0)'(\beta - \beta_0)\right] \quad \textit{(normal)}$$

$$f(x|x_0,\sigma^2,n_x) \propto (\sigma^2)^{-\frac{2np}{2}} \exp\left[-\frac{n_x}{2\sigma^2} tr\{(x - x_0)(x - x_0)'\}\right] \quad \textit{(multivariate normal)}$$

$$f(\sigma^2|\alpha,\gamma) \propto (\sigma^2)^{-(\alpha+1)} \exp\left[-\frac{\gamma}{\sigma^2}\right] \quad \textit{(inverse gamma)}$$

$$\underset{2n\times1}{y} = \begin{bmatrix} y_R \\ y_I \end{bmatrix} \quad \underset{2n\times2p}{X} = \begin{bmatrix} x_R & -x_I \\ x_I & x_R \end{bmatrix}$$

$$\underset{n\times2p}{x} = \begin{bmatrix} x_R & x_I \end{bmatrix} \quad \underset{2p\times1}{\beta} = \begin{bmatrix} \beta_R \\ \beta_I \end{bmatrix}$$

Posterior:

$$f(\beta,\sigma^2|X,y) \propto (\sigma^2)^{-\left(\frac{2np+2n+2p+2\alpha}{2}+1\right)} \exp\left[-\frac{h}{2\sigma^2}\right]$$

$$h = (y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{(x - x_0)(x - x_0)'\} + 2\gamma$$

tr = trace(matrix)
Sum of the diagonals

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5. Model 9
Marginals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_{2n}\right)$$

X is Unobservable

Parameter	Dimensions
y	$2n \times 1$
X	$2n \times 2p$
x	$n \times 2p$
β	$2p \times 1$

Posterior Marginals:

$$f(\beta|X,y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})' \left[\frac{\nu_\beta(X'X + n_\beta I_{2p})}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + 2p}{2}}$$

$$\beta|X,y \sim t(\nu_\beta, \hat{\beta}, \mathrm{T}),\ E(\beta|X,y) = \hat{\beta} = \left(X'X + n_\beta I_{2p}\right)^{-1}\left(X'y + n_\beta \beta_0\right),\ Var(\beta|X,y) = \frac{\nu_\beta}{\nu_\beta - 2}\mathrm{T}$$

$$f(x|\beta,y) \propto \left|I_n + \frac{1}{\nu_x}(x - \hat{x})\left[\frac{\nu_x(CC' + n_x I_{2p})}{\theta}\right](x - \hat{x})'\right|^{-\frac{\nu_x + 2np}{2}}$$

$$x|\beta,y \sim Mt(\nu_x, \hat{x}, \Delta),\quad E(x|\beta,y) = \hat{x} = (YC' + n_x x_0)\left(CC' + n_x I_{2p}\right)^{-1},\quad Var(x|\beta,y) = \frac{\nu_x}{\nu_x - 2}\Delta$$

Gibbs Sampler:

$$\beta_{(0)} \downarrow x_{(1)} \downarrow \beta_{(1)} \downarrow x_{(2)}$$

$$\underset{2p\times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\underset{n\times 2}{Y} = \begin{bmatrix} y_R & y_I \end{bmatrix}$$

$$\nu_\beta = 2np + 2p + 2\alpha,\ \nu_x = 2n + 2p + 2\alpha$$

$$\mathrm{T} = \frac{\phi}{\nu_\beta}\left(X'X + n_\beta I_{2p}\right)^{-1},\ \Delta = \frac{\theta}{\nu_x}\left(CC' + n_x I_{2p}\right)^{-1}$$

$$\phi = y'y - \hat{\beta}'\left(X'X + n_\beta I_{2p}\right)\hat{\beta} + n_x tr\{(x - x_0)(x - x_0)'\} + n_\beta \beta_0{}'\beta_0 + 2\gamma$$

$$\theta = y'y - tr\{\hat{x}\left(CC' + n_x I_{2p}\right)\hat{x}'\} + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{x_0 x_0{}'\} + 2\gamma$$

5. Model 9
Conditionals

$$y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I_{2n}\right)$$

X is Unobservable

Parameter	Dimensions
y	$2n \times 1$
X	$2n \times 2p$
x	$n \times 2p$
β	$2p \times 1$

Posterior Conditionals:

$$f(\beta|X,\sigma^2,y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta-\hat{\beta})'(\Sigma_\beta)^{-1}(\beta-\hat{\beta})\right]$$

$$\beta|X,\sigma^2,y \sim N\left(\hat{\beta}, \Sigma_\beta = \sigma^2\left(X'X + n_\beta I_{2p}\right)^{-1}\right)$$

$$f(x|\beta,\sigma^2,y) \propto |\Sigma_X|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x-\hat{x})(\Sigma_X)^{-1}(x-\hat{x})'\right]$$

$$x|\beta,\sigma^2,y \sim N\left(\hat{x}, \Sigma_X = \sigma^2\left(CC' + n_x I_{2p}\right)^{-1}\right)$$

$$f(\sigma^2|\beta,X,y) \propto (\sigma^2)^{-(\alpha_*+1)} \exp\left[-\frac{\gamma_{**}}{\sigma^2}\right]$$

$$\sigma^2|\beta,X,y \sim IG(\alpha_*, \gamma_{**})$$

Gibbs Sampler:

$$\beta_{(0)}, X_{(0)} \downarrow \sigma^2_{(1)} \downarrow \beta_{(0)}, \sigma^2_{(1)} \downarrow X_{(1)} \downarrow X_{(1)}, \sigma^2_{(1)} \downarrow \beta_{(1)}$$

- Bayesian approach required
- Need MCMC technique to estimate β, X, σ^2

$$\underset{2p\times 2}{C} = \begin{bmatrix} \beta_R & \beta_I \\ -\beta_I & \beta_R \end{bmatrix}$$

$$\alpha_* = \frac{2np + 2n + 2p + 2\alpha}{2}$$

$$\gamma_{**} = \frac{1}{2}\left[(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{(x - x_0)(x - x_0)'\} + 2\gamma\right]$$

5. Models 7-9

Quick Review

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

X is Unobservable

Posterior: $f(\beta, \sigma^2 | X, y) \propto (\sigma^2)^{-\left(\frac{2np+2n+2p+2\alpha}{2}+1\right)} \exp\left[-\frac{1}{2\sigma^2}\{(y - X\beta)'(y - X\beta) + n_\beta(\beta - \beta_0)'(\beta - \beta_0) + n_x tr\{(x - x_0)(x - x_0)'\} + 2\gamma\}\right]$

Marginal for $\boldsymbol{\beta}$: $f(\beta | X, y) \propto \left[1 + \frac{1}{\nu_\beta}(\beta - \hat{\beta})'\left[\frac{\nu_\beta(X'X + n_\beta I_{2p})}{\phi}\right](\beta - \hat{\beta})\right]^{-\frac{\nu_\beta + 2p}{2}}$

Conditional for $\boldsymbol{\beta}$: $f(\beta | X, \sigma^2, y) \propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\beta - \hat{\beta})'(X'X + n_\beta I_{2p})(\beta - \hat{\beta})\right]$

Marginal for $\boldsymbol{X}$: $f(X | \beta, y) \propto \left[1 + \frac{1}{\nu_x}(X - \hat{X})\left[\frac{\nu_x(\beta\beta' + n_x I_{2p})}{\theta}\right](X - \hat{X})'\right]^{-\frac{\nu_x + 2np}{2}}$

Conditional for $\boldsymbol{X}$: $f(X | \beta, \sigma^2, y) \propto |\Sigma_X|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2} tr\{(x - \hat{x})(\beta\beta' + n_x I_{2p})(x - \hat{x})'\}\right]$

	Model 7 $n = 1, p = 1$	Model 8 $n = 1, p = 3$	Model 9 $n = 4, p = 3$
α_*	$\frac{2\alpha + 2 + 2 + 2}{2}$	$\frac{2\alpha + 2p + 2p + 2}{2}$	$\frac{2\alpha + 2p + 2np + 2n}{2}$
ν_β	$2\alpha + 2 + 2$	$2\alpha + 2p + 2$	$2\alpha + 2np + 2n$
β Marginal exponent	$\frac{\nu_\beta + 2}{2}$	$\frac{\nu_\beta + 2p}{2}$	$\frac{\nu_\beta + 2p}{2}$
ν_x	$2\alpha + 2 + 2$	$2\alpha + 2p + 2$	$2\alpha + 2p + 2n$
X Marginal exponent	$\frac{\nu_x + 2}{2}$	$\frac{\nu_x + 2p}{2}$	$\frac{\nu_x + 2np}{2}$

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$$y = X\beta + \varepsilon,\ \varepsilon \sim \mathrm{N}\left(0, \sigma^2 I_n\right)$$

6. Discussion

Models 1-9

Models 1,4,7

$n = 1$

$p = 1$

$$y_1 = x_1\beta_1 + \varepsilon_1$$

Models 2,5,8

$n = 1$

$p = 3$

$$y_1 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \varepsilon_1$$

Models 3,6,9

$n = 4$

$p = 3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

	Real (1-3)			Real (4-6)			Complex (7-9)		
y	1×1	1×1	$\mathrm{n} \times 1$	1×1	1×1	$\mathrm{n} \times 1$	1×1	1×1	$\mathrm{n} \times 1$
X	1×1 (observed)	$1 \times p$ (observed)	$\mathrm{n} \times p$ (observed)	1×1 (unobserved)	$1 \times p$ (unobserved)	$\mathrm{n} \times p$ (unobserved)	1×1 (unobserved)	$1 \times p$ (unobserved)	$\mathrm{n} \times p$ (unobserved)
β	1×1	$p \times 1$	$p \times 1$	1×1	$p \times 1$	$p \times 1$	1×1	$p \times 1$	$p \times 1$

$$y = X\beta + \varepsilon,\ \varepsilon \sim \mathrm{N}\left(0, \sigma^2 I_n\right)$$

6. Discussion

Model Buildup

Model 1 Real Numbers $n = 1, p = 1$ X is observed $y: 1 \times 1$ $X: 1 \times 1$ $\beta: 1 \times 1$	**Model 2** Real Numbers $n = 1, p = 3$ X is observed $y: 1 \times 1$ $X: 1 \times p$ $\beta: p \times 1$	**Model 3** Real Numbers $n = 4, p = 3$ X is observed $y: n \times 1$ $X: n \times p$ $\beta: p \times 1$
Model 6 Real Numbers $n = 4, p = 3$ X is unobserved $y: n \times 1$ $X: n \times p$ $\beta: p \times 1$	**Model 5** Real Numbers $n = 1, p = 3$ X is unobserved $y: 1 \times 1$ $X: 1 \times p$ $\beta: p \times 1$	**Model 4** Real Numbers $n = 1, p = 1$ X is unobserved $y: 1 \times 1$ $X: 1 \times 1$ $\beta: 1 \times 1$
Model 7 Complex Numbers $n = 1, p = 1$ X is unobserved $y: 2 \times 1$ $X: 2 \times 2$ $\beta: 2 \times 1$	**Model 8** Complex Numbers $n = 1, p = 3$ X is unobserved $y: 2 \times 1$ $X: 2 \times 2p$ $\beta: 2p \times 1$	**Model 9** Complex Numbers $n = 4, p = 3$ X is unobserved $y: 2n \times 1$ $X: 2n \times 2p$ $\beta: 2p \times 1$

Why go through this process?

- Easier to buildup up from a **simple real-valued** linear regression instead of going right to **complex-valued** linear regression with an unobserved design matrix
  - Understanding the **dimensions**
  - Going from X being **observed** to **unobserved**
  - Going from **real** numbers to **complex** numbers
- Helps understand the **prior knowledge** of the unobservable complex-valued design matrices
- Easier to set up parameter estimation techniques
  - Maximum A Posteriori (**MAP**) estimate using the Iterated Conditional Modes (**ICM**) algorithm
  - MCMC Gibbs Sampling

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Thank You!

Questions?

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Homework

1. Suppose we have a **complex-valued** linear regression with an observable design matrix X. For $n > 1$ and $p > 1$, find:

   Hint: Start with Model 9 and do not place a prior on X

   a) Data Likelihood: $f\left(y|\beta, X, \sigma^2\right)$
   b) Prior Distributions: $f\left(\beta|\beta_0, \sigma^2, n_\beta\right)$, $\quad f\left(\sigma^2|\alpha, \gamma\right)$
   c) Posterior Distribution: $f\left(\beta, \sigma^2|X, y\right)$
   d) Posterior Marginals: $f(\beta\ |X, y)$, $\quad f\left(\sigma^2|X, y\right)$
   e) Posterior Conditionals: $f\left(\beta\ |X, \sigma^2, y\right)$, $f\left(\sigma^2|\beta, X, y\right)$
2. Suppose we have a **real-valued** linear regression with an unobservable design matrix X. For $n > 1$ and $p = 1$, find:
   a) Data Likelihood: $f\left(y|\beta, X, \sigma^2\right)$
   b) Prior Distributions: $f\left(\beta|\beta_0, \sigma^2, n_\beta\right)$, $\quad f\left(X|X_0, \sigma^2, n_x\right)$, $\quad f\left(\sigma^2|\alpha, \gamma\right)$
   c) Posterior Distribution: $f\left(\beta, X, \sigma^2|y\right)$
   d) Posterior Marginals: $f(\beta\ |X, y)$, $\quad f(X|\beta, y)$
   e) Posterior Conditionals: $f\left(\beta\ |X, \sigma^2, y\right)$, $f\left(X\ |\beta, \sigma^2, y\right)$, $\quad f\left(\sigma^2|\beta, X, y\right)$