Bayesian Multivariate Multiple Linear Regression

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Univariate Simple Sampling

Univariate Simple Linear Regression

Univariate Multiple Linear Regression

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Univariate Simple Sampling

Model i^{th} Observation:

$$y_i = \underbrace{\beta_0}_{\mu} + \varepsilon_i$$

$$E(\varepsilon_i) = 0 \qquad \text{cov}(\varepsilon_i) = \sigma^2$$

or equivalently

$$i = 1, \ldots, n$$

$$\underset{1\times 1}{y_i} = \underset{1\times 1}{(1)} \underset{1\times 1}{\underbrace{\left(\beta_0\right)}_{\mu}} + \underset{1\times 1}{\varepsilon_i}$$

Univariate Simple Sampling

$$y_i = \beta_0 + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n \times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n \times 1}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} \left(\underset{\mu}{\beta_0} \right) + \underset{n \times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n \times 1}{y} = \underset{n \times 1}{X} \; \underset{1 \times 1}{\beta} + \underset{n \times 1}{\varepsilon}$$

Univariate Simple Sampling

$y_i = \beta_0 + \varepsilon_i$

The model

column of 1's

$$\underset{n\times 1}{y} = \underset{n\times 1}{X}\ \underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon)=0 \qquad \text{cov}(\varepsilon)=\sigma^2 I_n \qquad i=1,\ldots,n$$

μ

with coefficients and variance estimated as

$$\underset{1\times 1}{\hat{\beta}} = \underset{1\times 1}{(X'X)^{-1}}\ \underset{1\times n}{X'}\ \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

$\hat{\mu} = \bar{x}$

from likelihood function

Require $n\geq 1$, general rule $n\geq 10$

$$f(\underset{n\times 1}{y}\,|\,\beta,\sigma^2) = (2\pi)^{-n/2}\,|\sigma^2 I_{q+1}|^{-n/2}\exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

.

Univariate Simple Sampling

$$\underset{n\times 1}{y} = \underset{n\times 1}{X}\ \underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$$\underset{1\times 1}{\hat{\beta}} = (X'X)^{-1}X'\ y \quad \text{is } N(\underset{n\times 1}{E(\hat{\beta})} = \beta,\ \text{cov}(\hat{\beta}) = \sigma^2 \underset{1\times 1}{(X'X)}^{-1})$$

$n\geq 1$, general rule $n\geq 10$

$\hat{\mu} = \bar{x}$

and

$$\underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n-1}(y - \underset{1\times n}{X}\hat{\beta})'(y - \underset{n\times 1}{X}\hat{\beta}) \qquad \text{is gamma}(\nu,\sigma^2/\nu)!$$

$$f(\underset{n\times 1}{y}|\beta,\sigma^2) = (2\pi)^{-n/2}|\sigma^2 I_{q+1}|^{-n/2}\exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

Univariate Simple Linear Regression

Model i^{th} Observation:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$E(\varepsilon_i) = 0 \qquad \operatorname{cov}(\varepsilon_i) = \sigma^2$$

$$i = 1, \ldots, n$$

or equivalently

$$\underset{1\times 1}{y_i} = \underset{1\times 2}{(1, x_i)} \underset{2\times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}} + \underset{1\times 1}{\varepsilon_i}$$

Univariate Simple Linear Regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n\times 2}{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}} + \underset{n\times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n\times 1}{y} = \underset{n\times 2}{X} \; \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Univariate Simple Linear Regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

The model

$i = 1,\ldots,n$

$$\underset{n\times 1}{y} = \underset{n\times 2}{X}\ \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0 \qquad \text{cov}(\varepsilon) = \sigma^2 I_n \qquad i = 1,\ldots,n$$

with coefficients and variance estimated as

$$\underset{2\times 1}{\hat{\beta}} = \underset{2\times 2}{(X'X)^{-1}}\ \underset{2\times n}{X'}\ \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

from likelihood function

Require $n \geq 2$, general rule $n \geq 20$

$$f(\underset{n\times 1}{y} \mid \beta, \sigma^2) = (2\pi)^{-n/2} \left| \sigma^2 I_2 \right|^{-n/2} \exp\left[-\frac{1}{2} (\sigma^2 I_2)^{-1} (y - X\beta)'(y - X\beta) \right]$$

.

Univariate Simple Linear Regression

$$\underset{n\times 1}{y} = \underset{n\times 2}{X}\ \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$\underset{2\times 1}{\hat{\beta}} = (X'X)^{-1}X'\ y$ is $N(\underset{2\times 1}{E(\hat{\beta})} = \beta,\ \mathrm{cov}(\hat{\beta}) = \sigma^2 (\underset{2\times 2}{X'X})^{-1})$ $n\geq 2$, general rule $n>20$

and

$\underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n-2}(y - \underset{1\times n}{X\hat{\beta}})'(y - \underset{n\times 1}{X\hat{\beta}})$ is gamma(ν,σ^2/ν)!

$$f(\underset{n\times 1}{y} \mid \beta, \sigma^2) = (2\pi)^{-n/2} \left|\sigma^2 I_2\right|^{-n/2} \exp\left[-\frac{1}{2}(\sigma^2 I_2)^{-1}(y - X\beta)'(y - X\beta)\right]$$

Univariate Multiple Linear Regression

Model i^{th} Observation:

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

$$E(\varepsilon_i) = 0 \qquad \text{cov}(\varepsilon_i) = \sigma^2$$

or equivalently

$$i = 1, \ldots, n$$

$$\underset{1 \times 1}{y_i} = \underset{1 \times (q+1)}{(1, x_{i1}, \ldots, x_{iq})} \underset{(q+1) \times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{pmatrix}} + \underset{1 \times 1}{\varepsilon_i}$$

Univariate Multiple Linear Regression

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n \times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n \times (q+1)}{\begin{pmatrix} 1 & x_{11} & \cdots & x_{1q} \\ 1 & x_{21} & & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nq} \end{pmatrix}} \underset{(q+1) \times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{pmatrix}} + \underset{n \times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n \times 1}{y} = \underset{n \times (q+1)}{X} \underset{(q+1) \times 1}{\beta} + \underset{n \times 1}{\varepsilon}$$

Univariate Multiple Linear Regression

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

The model

$$i = 1, \ldots, n$$

$$\underset{n\times 1}{y} = \underset{n\times (q+1)}{X} \underset{(q+1)\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0 \qquad \mathrm{cov}(\varepsilon) = \sigma^2 I_n \qquad i = 1, \ldots, n$$

with coefficients and variance estimated as

$$\underset{(q+1)\times 1}{\hat{\beta}} = \underset{(q+1)\times (q+1)}{(X'X)^{-1}} \underset{(q+1)\times n}{X'} \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

from likelihood function

Require $n \geq (q+1)$, general rule $n \geq 10(q+1)$

$$f(\underset{n\times 1}{y} | \beta, \sigma^2) = (2\pi)^{-n/2} |\sigma^2 I_{q+1}|^{-n/2} \exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta) \right]$$

.

Univariate Multiple Linear Regression

$$\underset{n\times 1}{y} = \underset{n\times 2}{X} \; \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$\underset{(q+1)\times 1}{\hat{\beta}} = (X'X)^{-1}X'y$ is $N(\underset{(q+1)\times 1}{E(\hat{\beta})} = \beta, \ \mathrm{cov}(\hat{\beta}) = \sigma^2 \underset{(q+1)\times(q+1)}{(X'X)^{-1}})$ $\quad$ $n \geq q+1$, general rule $n > 10(q+1)$

and

$\underset{1\times 1}{\hat{\sigma}^2} = \dfrac{1}{n-q-1}(\underset{1\times n}{y - X\hat{\beta}})'(\underset{n\times 1}{y - X\hat{\beta}})$ is gamma(ν,σ^2/ν)!

$$f(\underset{n\times 1}{y} \mid \beta, \sigma^2) = (2\pi)^{-n/2} \left| \sigma^2 I_{q+1} \right|^{-n/2} \exp\left[-\frac{1}{2} (\sigma^2 I_{q+1})^{-1} (y - X\beta)'(y - X\beta) \right]$$

Matrix Normal and Matrix-T PDFs

Separable used for vectors over time or row-col correlation in images..

Let $\underset{np\times 1}{y}$ be have a multivariate normal PDF, $\underset{np\times 1}{y} \sim N(\underset{np\times 1}{\mu}, \underset{np\times np}{\Omega})$

$$f(\underset{np\times 1}{y} | \underset{np\times 1}{\mu}, \underset{np\times np}{\Omega}) = (2\pi)^{-np/2} |\Omega|^{-1/2} \exp\left[-\frac{1}{2}(y-\mu)'\Omega^{-1}(y-\mu)\right] \qquad y,\mu \in \mathbb{R}^{np} \qquad \Omega > 0$$

and that Ω has a separable or Kronecker product structure $\underset{np\times np}{\Omega}=\underset{n\times n}{\Phi}\otimes\underset{p\times p}{\Sigma}$.
We utilize mathematical results that

$$|\underset{np\times np}{\Omega}| = |\underset{n\times n}{\Phi} \otimes \underset{p\times p}{\Sigma}| = |\underset{n\times n}{\Phi}|^{p} |\underset{p\times p}{\Sigma}|^{n} \quad \text{and} \quad (y-\mu)'(\Phi\otimes\Sigma)^{-1}(y-\mu) = tr\Sigma^{-1}(Y-M)'\Phi^{-1}(Y-M)$$

where $\underset{np\times 1}{y}=vec(\underset{n\times p}{Y})$ and $\underset{np\times 1}{\mu}=vec(\underset{n\times p}{M})$ to arrive at a reformulated PDF.

$$Y = (y_1,\ldots,y_n)$$
$$M = (\mu_1,\ldots,\mu_n)$$

Matrix Normal and Matrix-T PDFs

Separable used for vectors over time or row-col correlation in images..

The new Matrix Normal PDF for $\underset{n\times p}{Y}$ is

$$f(\underset{n\times p}{Y}|\underset{n\times p}{M},\ \underset{n\times n}{\Phi},\ \underset{p\times p}{\Sigma})=(2\pi)^{-np/2}|\Phi|^{-p/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-M)'\Phi^{-1}(Y-M)\right]$$

and we write $\underset{n\times p}{Y}\sim N(\underset{n\times p}{M},\underset{n\times n}{\Phi},\underset{p\times p}{\Sigma})$ aka $\underset{np\times 1}{y}\sim N(\underset{np\times 1}{\mu},\underset{np\times np}{\Phi\otimes\Sigma})$.

n=1→multivariate
n,p=1→univariate

There are various mean and covariance notations,

$E(\underset{n\times p}{Y})=\underset{n\times p}{M}$ or $E(\underset{np\times 1}{\text{vec}(Y)})=\underset{np\times 1}{\mu}$ and $cov(\underset{n\times p}{Y})=\underset{np\times np}{\Phi\otimes\Sigma}$ or $cov(\underset{np\times 1}{y})=\underset{np\times np}{\Phi\otimes\Sigma}$.

$$Y=(y_1,\ldots,y_n)$$
$$M=(\mu_1,\ldots,\mu_n)$$

Matrix Normal and Matrix-T PDFs

There is also a CLT result that MT becomes MN.

the Matrix-T distribution can be arrived at via a transformation of variable from a matrix normal to a standard matrix normal and a Wishart PDF.

$$f(\underset{n \times p}{Y} | \nu, M, \Sigma, \Phi) = k_T \frac{|\Phi|^{-\frac{p}{2}} |\Sigma|^{-\frac{n}{2}}}{\left| I_n + \frac{1}{\nu}\Phi^{-1}(Y-M)'\Sigma^{-1}(Y-M) \right|^{\frac{\nu+p}{2}}}$$

$$k_T = \frac{\prod_{j=1}^{n} \Gamma\left(\frac{\nu+p+1-j}{2}\right)}{(\nu\pi)^{\frac{np}{2}} \prod_{j=1}^{n} \Gamma\left(\frac{\nu+1-j}{2}\right)}$$

and we write $\underset{n \times p}{Y} \sim T(\nu, \underset{n \times p}{M}, \underset{n \times n}{\Phi}, \underset{p \times p}{\Sigma})$ aka $\underset{np \times 1}{y} \sim T(\underset{np \times 1}{\mu}, \underset{np \times np}{\Phi \otimes \Sigma})$.

n=1→multivariate
n,p=1→univariate

There are various mean and covariance notations,

$E(\underset{n \times p}{Y}) = \underset{n \times p}{M}$ or $E(\underset{np \times 1}{\text{vec}(Y)}) = \underset{np \times 1}{\mu}$ and $cov(\underset{n \times p}{Y}) = \frac{\nu}{\nu-2} \underset{np \times np}{\Phi \otimes \Sigma}$ or $cov(\underset{np \times 1}{y}) = \frac{\nu}{\nu-2} \underset{np \times np}{\Phi \otimes \Sigma}$.

$$Y = (y_1, \ldots, y_n)$$
$$M = (\mu_1, \ldots, \mu_n)$$

Multivariate Multiple Linear Regression

Model i^{th} Observation:

$$y_{ij} = \beta_{0j} + \beta_{1j} x_{i1} + \ldots + \beta_{qj} x_{iq} + \varepsilon_{ij}$$

$$E(\varepsilon_i) = 0 \qquad \operatorname{cov}(\varepsilon_i) = \Sigma$$

or equivalently

$$\begin{aligned} i &= 1, \ldots, n \\ j &= 1, \ldots, p \end{aligned}$$

$$\underset{1\times p}{(y_{i1}, \ldots, y_{ip})} = \underset{1\times(q+1)}{(1, x_{i1}, \ldots, x_{iq})} \underset{(q+1)\times p}{\begin{pmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0p} \\ \beta_{11} & \beta_{12} & & \beta_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix}} + \underset{1\times p}{(\varepsilon_{i1}, \ldots, \varepsilon_{ip})}$$

$$\varepsilon_i = \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{ip} \end{pmatrix}$$

Multivariate Multiple Linear Regression

$$(y_{i1},..,y_{ip}) = (1, x_{i1},..., x_{iq})\begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \beta_{21} & \beta_{22} & & \beta_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix} + (\varepsilon_{i1} + ... + \varepsilon_{ip})$$

$$i = 1,...,n$$

Model n Observations:

$$\underset{n\times p}{\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{np} \end{pmatrix}} = \underset{n\times(q+1)}{\begin{pmatrix} 1 & x_{11} & \cdots & x_{1q} \\ 1 & x_{21} & & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nq} \end{pmatrix}} \underset{(q+1)\times p}{\begin{pmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0p} \\ \beta_{11} & \beta_{12} & & \beta_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix}} + \underset{n\times p}{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1p} \\ \varepsilon_{21} & \varepsilon_{22} & & \varepsilon_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} & \cdots & \varepsilon_{np} \end{pmatrix}}$$

$$i = 1,...,n$$

more compactly

$$\underset{n\times p}{Y} = \underset{n\times(q+1)}{X}\ \underset{(q+1)\times p}{B} + \underset{n\times p}{E}$$

$$\underset{(q+1)\times p}{B} = (\underbrace{\beta_1,...,\beta_p}_{\text{cols}}) = (\underbrace{b_1,...,b_n}_{\text{rows}})'$$

Multivariate Multiple Linear Regression

$$(y_{i1},..,y_{ip}) = (1,x_{i1},...,x_{iq})\begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \beta_{21} & \beta_{22} & & \beta_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix} + (\varepsilon_{i1}+...+\varepsilon_{ip})$$

$i=1,...,n$

The model

$$\underset{n\times p}{Y} = \underset{n\times (q+1)}{X} \underset{(q+1)\times p}{B} + \underset{n\times 1}{E} \qquad E(vec(E))=0 \qquad \text{cov}(vec(E)) = I_n \otimes \Sigma \qquad i=1,...,n$$

Each row of E has $\text{cov}(\varepsilon_i)=\Sigma$.

with coefficients covariance estimated as

$$\underset{(q+1)\times p}{\hat{B}} = \underset{(q+1)\times(q+1)}{(X'X)^{-1}} \underset{(q+1)\times n}{X'} \underset{n\times p}{Y} \qquad \underset{p\times p}{\hat{\Sigma}} = \frac{1}{n}(Y-X\hat{B})'(Y-X\hat{B})$$

Require $n\geq(q+1)p$, general rule $n\geq 10p(q+1)$

from likelihood function

$$f(\underset{n\times p}{Y}|B,\Sigma) = (2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

.

$tr(\cdot)$ is trace, sum of diagonal elements

Multivariate Multiple Linear Regression

$$\underset{n\times p}{Y} = \underset{n\times(q+1)}{X}\ \underset{(q+1)\times 1}{B} + \underset{n\times p}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$\underset{(q+1)\times p}{\hat{B}} = (X'X)^{-1}X'Y$ is $N\Big(E(\hat{B}) = B,\ \mathrm{cov}(vec(\hat{B})) = \big(\underset{(q+1)\times(q+1)}{X'X}\big)^{-1} \otimes \underset{p\times p}{\Sigma}\Big)$

$n \geq (q+1)$, general rule $n > 10p(q+1)$

and

$$\mathrm{cov}(\underset{\text{cols}}{\hat{\beta}_j}) = \sigma_j^2 (X'X)^{-1} \qquad \mathrm{cov}(\underset{\text{rows}}{\hat{b}_i}) = w_{ii}\Sigma \qquad W = (X'X)^{-1}$$

$\underset{p\times p}{\hat{\Sigma}} = \dfrac{1}{n-q-1}(\underset{p\times n}{Y - X\hat{B}})'(\underset{n\times p}{Y - X\hat{B}})$ is Wishart(ν,Σ/ν)!

$$\underset{p(q+1)\times p(q+1)}{\mathrm{cov}(vec(\hat{B}))} = \begin{bmatrix} w_{11}\Sigma & w_{12}\Sigma & & \\ w_{21}\Sigma & w_{22}\Sigma & & \\ & & \ddots & \\ & & & w_{(q+1),(q+1)}\Sigma \end{bmatrix}$$

$$f(\underset{n\times p}{Y} \mid B,\Sigma) = (2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y - XB)'(Y - XB)\right]$$

Multivariate Multiple Linear Regression

Example:

It is believed that the heights of sons (y_s) and daughters (y_d) are linearly dependent upon the heights of their fathers (x_f) and mothers (x_m).

$$(y_{is}, y_{id}) = (1, x_{if}, x_{im}) \begin{pmatrix} \beta_{0s} & \beta_{0d} \\ \beta_{1s} & \beta_{1d} \\ \beta_{2s} & \beta_{2d} \end{pmatrix} + (\varepsilon_{is}, \varepsilon_{id}) \qquad (\varepsilon_{is}, \varepsilon_{id})' \sim N(0, \Sigma) \qquad i = 1, \ldots, n$$

Assume that the true parameter values are:

$$\underset{(q+1)\times p}{B} = \begin{pmatrix} 14.1 & 10.8 \\ 0.41 & 0.39 \\ 0.43 & 0.43 \end{pmatrix} \qquad \underset{p \times p}{\Sigma} = \begin{pmatrix} 2.250 & 1.125 \\ 1.125 & 2.250 \end{pmatrix}$$

could also imagine grandparents heights in model as x's

http://chance.amstat.org/2013/09/1-pagano/

Multivariate Multiple Linear Regression

Example:

Selected

$x_f = \{66,67,68,69,70,71,72,73,74,75\}$,

$x_m = \{60,61,62,63,64,65,66,67,68,69\}$

combinations (n=100), multiplied each with B

then added correlated normal noise with Σ.

Repeated L=10,000 times.

$$B = \begin{pmatrix} 14.1 & 10.8 \\ 0.41 & 0.39 \\ 0.43 & 0.43 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2.250 & 1.125 \\ 1.125 & 2.250 \end{pmatrix}$$

$$A = \begin{pmatrix} 1.50 & 0 \\ 0.755 & 1.30 \end{pmatrix}$$

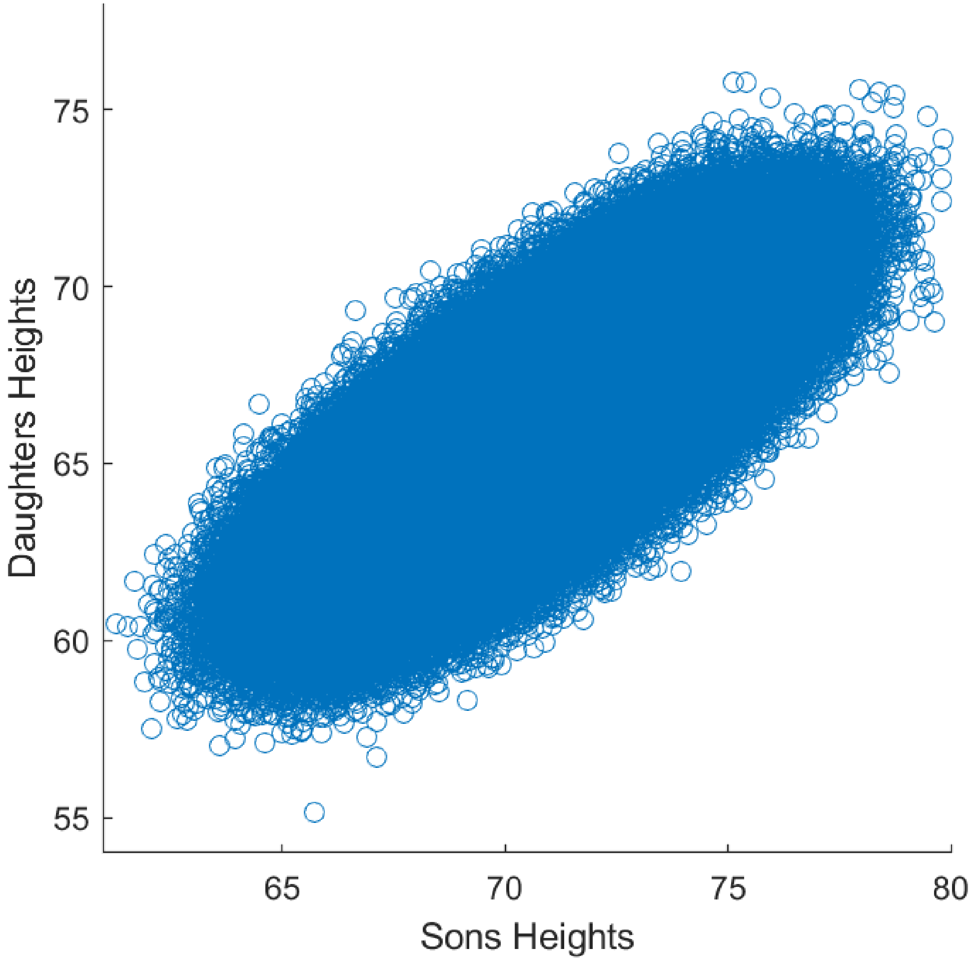
Multivariate Multiple Linear Regression

Example:

Estimates and plots from all 10^6.

$$\hat{B} = \begin{pmatrix} 14.0511 & 10.8487 \\ 0.4108 & 0.3899 \\ 0.4299 & 0.4294 \end{pmatrix}$$

$$\hat{\Sigma} = \begin{pmatrix} 2.2504 & 1.1207 \\ 1.1207 & 2.2465 \end{pmatrix}$$



$$B = \begin{pmatrix} 14.1 & 10.8 \\ 0.41 & 0.39 \\ 0.43 & 0.43 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2.250 & 1.125 \\ 1.125 & 2.250 \end{pmatrix}$$

$$A = \begin{pmatrix} 1.50 & 0 \\ 0.755 & 1.30 \end{pmatrix}$$

Multivariate Multiple Linear Regression

```
% Multivariate model for
% Y1=son height, Y2=daughter height
% X1 = father height, X2=mother height
clear all
close all
rng('default')

% define true parameter values
n=100;
B=[[14.1;.41;.43],[10.8;.39;.43]]
sigma1=1.5; sigma2=1.5; rho=.5;
Sigma=[sigma1^2,sigma1*sigma2*rho;...
       sigma1*sigma2*rho,sigma2^2]
A=chol(Sigma)'

% generate simulated data
hf=(66:66+9);,hm=(60:60+9);
[Xf,Xm]=meshgrid(hf,hm);
xf=reshape(Xf,n,1);,xm=reshape(Xm,n,1);
X=[ones(n,1),xf,xm]
```

```
replicate=10000;
X=repmat(X,replicate,1);
n=size(X,1);

E=(A*randn(2,n))';
Y=X*B+E;

% Estimate parameters
Bhat=inv(X'*X)*X'*Y;
SigmaHat=(Y-X*B)'*(Y-X*B)/(n-3);

[Bhat,B]
[SigmaHat,Sigma]

figure;
scatter(Y(:,1),Y(:,2))
axis square
xlabel('Sons Heights')
ylabel('Daughters Heights')
xlim([61,80]),ylim([54,78])
```

Bayesian Multivariate Multiple Linear Regression

The model

$$\underset{n\times p}{Y} = \underset{n\times (q+1)}{X}\ \underset{(q+1)\times p}{B} + \underset{n\times 1}{E} \qquad E(vec(E))=0 \qquad \mathrm{cov}(vec(E)) = I_n \otimes \Sigma \qquad i=1,\ldots,n$$

Each row of E has $\mathrm{cov}(\varepsilon_i)=\Sigma$.

with likelihood

$$\underset{n\times p}{f}(Y|B,\Sigma) = (2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

matrix normal

and priors

$$\underset{(q+1)\times p}{f}(B|\Sigma) = (2\pi)^{-p(q+1)/2}|\Sigma/n_0|^{-(q+1)/2}\exp\left[-\frac{n_0}{2}tr\Sigma^{-1}(B-B_0)'(B-B_0)\right]$$

matrix normal

$$\underset{p\times p}{f}(\Sigma) = k|H|^{\nu/2}|\Sigma|^{-(\nu+p+1)/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}H\right]$$

Inverse Wishart

Bayesian Multivariate Multiple Linear Regression

$$H_* = (Y - XB)'(Y - XB) + n_0(B - B_0)'(B - B_0) + H$$
$$\nu_* = \nu + n + q + 1$$

yields posterior

$$\underset{(q+1)\times p}{f(B,\Sigma|Y)} \propto |\Sigma|^{-(\nu_*+p+1)/2} \exp\left\{-\frac{1}{2} tr\Sigma^{-1} H_*\right\}$$

$$\hat{B} = (X'X + n_0 I)^{-1}(X'Y + n_0 B_0)$$

complete the square

$$\underset{(q+1)\times p}{f(B,\Sigma|Y)} \propto |\Sigma|^{-(\nu_*+p+1)/2} \exp\left\{-\frac{1}{2} tr\Sigma^{-1}\left[(B-\hat{B})'(X'X + n_0 I)(B-\hat{B}) + W\right]\right\}.$$

$$W = n_0 B_0' B_0 + H + Y'Y - \hat{B}'(X'X + n_0 I)\hat{B}$$

Integrate wrt to Σ to get Matrix Student-T PDF for B.

Integrate wrt to B to get Inverse Wishart PDF for Σ.

Bayesian Multivariate Multiple Linear Regression

$$f(Y|B,\Sigma)=(2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

$$f(B|\Sigma)=(2\pi)^{-p(q+1)/2}|\Sigma/n_0|^{-(q+1)/2}\exp\left[-\frac{n_0}{2}tr\Sigma^{-1}(B-B_0)'(B-B_0)\right]$$

$$f(\Sigma)=k|H|^{\nu/2}|\Sigma|^{-(\nu+p+1)/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}H\right]$$

The marginal PDF of B can be found

$$\underset{(q+1)\times p}{f(B|Y)}=\int_\Sigma f(B,\Sigma|Y)d\Sigma$$

$$\underset{(q+1)\times p}{f(B|Y)}\propto\int_\Sigma|\Sigma|^{-(\nu_*+p+1)/2}\exp\left\{-\frac{1}{2}tr\Sigma^{-1}\left[(B-\hat{B})'(X'X+n_0I)(B-\hat{B})+W\right]\right\}d\Sigma$$

$$\nu_*=\nu+n+q+1$$

$$\underset{(q+1)\times p}{f(B|Y)}\propto|H_*|^{-\nu_*/2}\underbrace{\int_\Sigma|H_*|^{\nu_*/2}|\Sigma|^{-(\nu_*+p+1)/2}e^{-\frac{1}{2}tr\Sigma^{-1}H_*}d\Sigma}_{=\text{constant}}$$

$$H_*=(B-\hat{B})'(X'X+n_0I)(B-\hat{B})+W$$

$$W=n_0B_0'B_0+H+Y'Y-\hat{B}'(X'X+n_0I)\hat{B}$$

Bayesian Multivariate Multiple Linear Regression

$$f(Y|B,\Sigma) = (2\pi)^{-np/2} |\Sigma|^{-n/2} \exp\left[-\frac{1}{2} tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

$$f(B|\Sigma) = (2\pi)^{-p(q+1)/2} |\Sigma / n_0|^{-(q+1)/2} \exp\left[-\frac{n_0}{2} tr\Sigma^{-1}(B-B_0)'(B-B_0)\right]$$

$$f(\Sigma) = k |H|^{\nu/2} |\Sigma|^{-(\nu+p+1)/2} \exp\left[-\frac{1}{2} tr\Sigma^{-1} H\right]$$

The marginal PDF of μ can be found

$$\underset{(q+1)\times p}{f(B|Y)} \propto |H_*|^{-\nu_*/2}$$

$$\underset{(q+1)\times p}{f(B|Y)} \propto \frac{1}{|W+(B-\hat{B})'(X'X+n_0 I)(B-\hat{B})|^{\nu_*/2}}$$

Note: If A $p\times q$ and B is $q\times p$, then

$$|I_p + AB| = |I_q + BA|$$

$$\nu_* = \nu + n + q + 1$$
$$\nu_\# = \nu + n - p + q + 1$$

$$\underset{(q+1)\times p}{f(B|Y)} \propto \frac{1}{\left|I_{q+1} + \frac{1}{\nu_\#}(X'X+n_0 I)(B-\hat{B})T^{-1}(B-\hat{B})'\right|^{(\nu_\#+p)/2}}$$

Matrix Student-T

$$\underset{(q+1)\times p}{E(B|Y)} = \hat{B}$$

$$T = \frac{W}{\nu_\#}$$

$$H_* = (B-\hat{B})'(X'X+n_0 I)(B-\hat{B}) + W$$

$$W = n_0 B_0' B_0 + H + Y'Y - \hat{B}'(X'X+n_0 I)\hat{B}$$

Bayesian Multivariate Multiple Linear Regression

$$f(Y|B,\Sigma)=(2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

$$f(B|\Sigma)=(2\pi)^{-p(q+1)/2}|\Sigma/n_0|^{-(q+1)/2}\exp\left[-\frac{n_0}{2}tr\Sigma^{-1}(B-B_0)'(B-B_0)\right]$$

$$f(\Sigma)=k|H|^{\nu/2}|\Sigma|^{-(\nu+p+1)/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}H\right]$$

The marginal PDF of Σ can be found

$$\underset{p\times p}{f(\Sigma|Y)}=\int_B f(B,\Sigma|Y)dB$$

$$\underset{p\times p}{f(\Sigma|Y)}\propto\int_B|\Sigma|^{-(\nu_*+p+1)/2}\exp\left\{-\frac{1}{2}tr\Sigma^{-1}\left[(B-\hat{B})'(X'X+n_0I)(B-\hat{B})+W\right]\right\}dB$$

$$\nu_*=\nu+n+q+1$$

$$\underset{p\times p}{f(\Sigma|Y)}\propto|\Sigma|^{-(\nu_*+p-q)/2}e^{-\frac{1}{2}tr\Sigma^{-1}W}\underbrace{\int_B|\Sigma|^{-(q+1)/2}\exp\left\{-\frac{1}{2}tr\Sigma^{-1}(B-\hat{B})'(X'X+n_0I)(B-\hat{B})\right\}dB}_{=\text{constant}}$$

$$W=n_0B_0'B_0+H+Y'Y-\hat{B}'(X'X+n_0I)\hat{B}$$

Bayesian Multivariate Multiple Linear Regression

The marginal PDF of Σ can be found to be inverse Wishart

$$f(\underset{p\times p}{\Sigma} | Y) = k |W|^{(\nu_{**})/2} |\Sigma|^{-(\nu_{**}+p+1)/2} e^{-\frac{1}{2} tr\Sigma^{-1}W}$$

with expectation, variances, and covariances

$$E(\underset{p\times p}{\Sigma} | Y) = \frac{W}{\nu_{**}-p-1} \quad ,$$

$$\mathrm{var}(\Sigma_{ii} | Y) = \frac{2W_{ii}^2}{(\nu_{**}-p-1)^2(\nu_{**}-p-3)}$$

$$\nu_{**} = \nu + n$$

$$\mathrm{var}(\Sigma_{ij} | Y) = \frac{(\nu_{**}-p+1)W_{ij}^2 + (\nu_{**}-p-1)W_{ii}W_{jj}}{(\nu_{**}-p)(\nu_{**}-p-1)^2(\nu_{**}-p-3)} \quad , \quad \mathrm{cov}(\Sigma_{ij}, \Sigma_{kl} | Y) = \frac{2W_{ij}W_{kl} + (\nu_{**}-p-1)(W_{ik}W_{jl} + W_{il}W_{kj})}{(\nu_{**}-p)(\nu_{**}-p-1)^2(\nu_{**}-p-3)} \; .$$

$$W = n_0 B_0' B_0 + H + Y'Y - \hat{B}'(X'X + n_0 I)\hat{B}$$

Discussion

We started with the univariate simple normal samples model and built up to the multivariate multiple linear regression model.

We formulated a Bayesian multivariate multiple regression model with assessed hyperparameters for conjugate priors and estimated the model parameters exactly *a posteriori*.

Discussion

Questions?

$$\underset{n\times p}{Y} = \underset{n\times(q+1)}{X}\ \underset{(q+1)\times p}{B} + \underset{n\times p}{E}$$

Non-Bayesian

$$\underset{(q+1)\times p}{\hat{B}_{MLE}} = (X'X)^{-1}X'\ Y$$

$$\underset{(q+1)p\ \times\ (q+1)p}{\mathrm{cov}(vec(\hat{B}))} = \left(X'X\right)^{-1} \otimes \hat{\Sigma}$$

$$\hat{\Sigma} = \frac{1}{n-q-1}(Y'Y - \hat{B}'X'X\hat{B})$$

Bayesian

$$\underset{(q+1)\times p}{\hat{B}} = (X'X + n_0 I)^{-1}[X'X\hat{B}_{MLE} + n_0 B_0]$$

$$\underset{(q+1)p\ \times\ (q+1)p}{\mathrm{cov}(vec(\hat{B}))} = \left(X'X + n_0 I\right)^{-1} \otimes \frac{\nu_\#}{\nu_\# - 2} T$$

$$T = [n_0 B_0' B_0 + H + Y'Y - \hat{B}'(X'X + n_0 I)\hat{B}] / \nu_\#$$

$$\nu_\# = \nu + n - p + q + 1$$

Homework 12

1***.Select observed known father x_{if} and mother x_{im} heights, multiply by coefficients B, add correlated error correlated ε_i with mean 0 and covariance Σ, repeat n=100 times.

$$\underset{(q+1)\times p}{\hat{B}} = (\underset{(q+1)\times (q+1)}{X'X})^{-1} \underset{(q+1)\times n}{X'} \underset{n\times p}{Y}$$

$$(y_{is}, y_{id}) = (1, x_{if}, x_{im}) \begin{pmatrix} \beta_{0s} & \beta_{0d} \\ \beta_{1s} & \beta_{1d} \\ \beta_{2s} & \beta_{2d} \end{pmatrix} + (\varepsilon_{is}, \varepsilon_{id}) \qquad i = 1,\ldots,n$$

$$\underset{p\times p}{\hat{\Sigma}} = \frac{(Y - X\hat{B})'(Y - X\hat{B})}{n - q - 1}$$

form Y and X, estimate $\hat{B}$ and $\hat{\Sigma}$.

Repeat a large number of times L so that you have $\hat{B}^{(1)},\ldots,\hat{B}^{(L)}$ and $\hat{\Sigma}^{(1)},\ldots,\hat{\Sigma}^{(L)}$.

Compute means, variances, make histograms, etc.

Compare to true values.

*** For enthusiastic students.

Homework 12

2***.Assess hyperparameters B_0, n_0, v, and H for the conjugate prior PDFs.

Compute Bayesian posterior estimates of data sets generated in #2.

try

$$B_0 = \begin{pmatrix} 14 & 11 \\ 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}$$

*** For enthusiastic students.

Univariate Simple Sampling

Slides Reordered

Model i^{th} Observation:

$$y_i = \overset{\mu}{\beta_0} + \varepsilon_i \qquad E(\varepsilon_i) = 0 \qquad \operatorname{cov}(\varepsilon_i) = \sigma^2$$

or equivalently $\qquad i = 1, \ldots, n$

$$\underset{1\times 1}{y_i} = \underset{1\times 1}{(1)} \underset{1\times 1}{\left(\overset{\mu}{\beta_0}\right)} + \underset{1\times 1}{\varepsilon_i}$$

Univariate Simple Linear Regression

Model i^{th} Observation:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad E(\varepsilon_i) = 0 \qquad \operatorname{cov}(\varepsilon_i) = \sigma^2$$

or equivalently $\qquad i = 1, \ldots, n$

$$\underset{1\times 1}{y_i} = \underset{1\times 2}{(1, x_i)} \underset{2\times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}} + \underset{1\times 1}{\varepsilon_i}$$

Univariate Multiple Linear Regression

Model i^{th} Observation:

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

$$E(\varepsilon_i) = 0 \qquad \operatorname{cov}(\varepsilon_i) = \sigma^2$$

$$i = 1, \ldots, n$$

or equivalently

$$\underset{1\times 1}{y_i} = \underset{1\times (q+1)}{(1, x_{i1}, \ldots, x_{iq})} \underset{(q+1)\times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{pmatrix}} + \underset{1\times 1}{\varepsilon_i}$$

Multivariate Multiple Linear Regression

Model i^{th} Observation:

$$y_{ij} = \beta_{0j} + \beta_{1j} x_{i1} + \ldots + \beta_{qj} x_{iq} + \varepsilon_{ij}$$

$$E(\varepsilon_i) = 0 \qquad \text{cov}(\varepsilon_i) = \Sigma$$

or equivalently

$$i = 1, \ldots, n$$
$$j = 1, \ldots, p$$

$$\underset{1 \times p}{(y_{i1}, .., y_{ip})} = \underset{1 \times (q+1)}{(1, x_{i1}, \ldots, x_{iq})} \underset{(q+1) \times p}{\begin{pmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0p} \\ \beta_{11} & \beta_{12} & & \beta_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix}} + \underset{1 \times p}{(\varepsilon_{i1}, \ldots, \varepsilon_{ip})}$$

$$\varepsilon_i = \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{ip} \end{pmatrix}$$

Univariate Simple Sampling

$$y_i = \beta_0 + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n\times 1}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} \left(\underset{\mu}{\beta_0} \right) + \underset{n\times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n\times 1}{y} = \underset{n\times 1}{X} \; \underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Univariate Simple Linear Regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n\times 2}{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}} + \underset{n\times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n\times 1}{y} = \underset{n\times 2}{X} \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Univariate Multiple Linear Regression

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

Model n Observations:

$$i = 1, \ldots, n$$

$$\underset{n \times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \underset{n \times (q+1)}{\begin{pmatrix} 1 & x_{11} & \cdots & x_{1q} \\ 1 & x_{21} & & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nq} \end{pmatrix}} \underset{(q+1) \times 1}{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{pmatrix}} + \underset{n \times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}$$

more compactly

$$\underset{n \times 1}{y} = \underset{n \times (q+1)}{X} \underset{(q+1) \times 1}{\beta} + \underset{n \times 1}{\varepsilon}$$

Multivariate Multiple Linear Regression

$$(y_{i1},..,y_{ip}) = (1, x_{i1},...,x_{iq}) \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \beta_{21} & \beta_{22} & & \beta_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix} + (\varepsilon_{i1} + ... + \varepsilon_{ip})$$

$$i = 1,...,n$$

Model n Observations:

$$\underset{n \times p}{\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{np} \end{pmatrix}} = \underset{n \times (q+1)}{\begin{pmatrix} 1 & x_{11} & \cdots & x_{1q} \\ 1 & x_{21} & & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nq} \end{pmatrix}} \underset{(q+1) \times p}{\begin{pmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0p} \\ \beta_{11} & \beta_{12} & & \beta_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{pmatrix}} + \underset{n \times p}{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1p} \\ \varepsilon_{21} & \varepsilon_{22} & & \varepsilon_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} & \cdots & \varepsilon_{np} \end{pmatrix}}$$

$$i = 1,...,n$$

more compactly

$$\underset{n \times p}{Y} = \underset{n \times (q+1)}{X} \; \underset{(q+1) \times p}{B} + \underset{n \times p}{E}$$

$$\underset{(q+1) \times p}{B} = (\underbrace{\beta_1,...,\beta_p}_{\text{cols}}) = (\underbrace{b_1,...,b_n}_{\text{rows}})'$$

Univariate Simple Sampling

$$y_i = \beta_0 + \varepsilon_i$$

The model

$$\underset{n\times 1}{y} = \underset{n\times 1}{X} \underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon)=0 \qquad \text{cov}(\varepsilon)=\sigma^2 I_n \qquad i=1,\ldots,n$$

μ

with coefficients and variance estimated as

$$\underset{1\times 1}{\hat{\beta}} = \underset{1\times 1}{(X'X)^{-1}} \underset{1\times n}{X'} \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

$\hat{\mu} = \bar{x}$

Require $n\geq 1$, general rule $n\geq 10$

from likelihood function

$$f(\underset{n\times 1}{y}|\beta,\sigma^2) = (2\pi)^{-n/2}|\sigma^2 I_{q+1}|^{-n/2}\exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

.

Univariate Simple Linear Regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$i = 1, \ldots, n$$

The model

$$\underset{n\times 1}{y} = \underset{n\times 2}{X} \; \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0 \qquad \operatorname{cov}(\varepsilon) = \sigma^2 I_n \qquad i = 1, \ldots, n$$

with coefficients and variance estimated as

$$\underset{2\times 1}{\hat{\beta}} = \underset{2\times 2}{(X'X)^{-1}} \; \underset{2\times n}{X'} \; \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

from likelihood function

Require $n \geq 2$, general rule $n \geq 20$

$$f(\underset{n\times 1}{y} \mid \beta, \sigma^2) = (2\pi)^{-n/2} \left| \sigma^2 I_2 \right|^{-n/2} \exp\left[-\frac{1}{2} (\sigma^2 I_2)^{-1} (y - X\beta)'(y - X\beta) \right]$$

.

Univariate Multiple Linear Regression

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_q x_{iq} + \varepsilon_i$$

The model

$i = 1, \ldots, n$

$$\underset{n\times 1}{y} = \underset{n\times (q+1)}{X}\ \underset{(q+1)\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0 \qquad \mathrm{cov}(\varepsilon) = \sigma^2 I_n \qquad i = 1, \ldots, n$$

with coefficients and variance estimated as

$$\underset{(q+1)\times 1}{\hat{\beta}} = \underset{(q+1)\times(q+1)}{(X'X)^{-1}}\ \underset{(q+1)\times n}{X'}\ \underset{n\times 1}{y} \qquad \underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta})$$

Require $n \geq (q+1)$, general rule $n \geq 10(q+1)$

from likelihood function

$$f(\underset{n\times 1}{y} | \beta, \sigma^2) = (2\pi)^{-n/2} |\sigma^2 I_{q+1}|^{-n/2} \exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

.

Multivariate Multiple Linear Regression

$$(y_{i1},..,y_{ip})=(1,x_{i1},...,x_{iq})\begin{pmatrix}\beta_{11} & \beta_{12} & \cdots & \beta_{1p}\\ \beta_{21} & \beta_{22} & & \beta_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp}\end{pmatrix}+(\varepsilon_{i1}+...+\varepsilon_{ip})$$

$$i=1,...,n$$

The model

$$\underset{n\times p}{Y}=\underset{n\times (q+1)}{X}\ \underset{(q+1)\times p}{B}+\underset{n\times 1}{E} \qquad E(vec(E))=0 \qquad \text{cov}(vec(E))=I_n\otimes\Sigma \qquad i=1,...,n$$

Each row of E has $\text{cov}(\varepsilon_i)=\Sigma$.

with coefficients covariance estimated as

$$\underset{(q+1)\times p}{\hat{B}}=\underset{(q+1)\times(q+1)}{(X'X)^{-1}}\ \underset{(q+1)\times n}{X'}\ \underset{n\times p}{Y} \qquad \underset{p\times p}{\hat{\Sigma}}=\frac{1}{n}(Y-X\hat{B})'(Y-X\hat{B})$$

Require $n\geq(q+1)p$, general rule $n\geq 10p(q+1)$

from likelihood function

$$f(\underset{n\times p}{Y}\,|\,B,\Sigma)=(2\pi)^{-np/2}\,|\Sigma|^{-n/2}\exp\left[-\frac{1}{2}tr\Sigma^{-1}(Y-XB)'(Y-XB)\right]$$

.

$tr(\cdot)$ is trace, sum of diagonal elements

Univariate Simple Sampling

$$\underset{n\times 1}{y} = \underset{n\times 1}{X}\ \underset{1\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$\underset{1\times 1}{\hat{\beta}} = (X'X)^{-1}X'\ y$ is $N(\underset{n\times 1}{E(\hat{\beta})} = \beta,\ \mathrm{cov}(\hat{\beta}) = \sigma^2 \underset{1\times 1}{(X'X)^{-1}})$ $n\geq 1$, general rule $n>10$

$\hat{\mu} = \bar{x}$

and

$\underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n-1}(y - X\hat{\beta})'(y - X\hat{\beta})$ is gamma(ν,σ^2/ν)!

$$f(\underset{n\times 1}{y} | \beta, \sigma^2) = (2\pi)^{-n/2} |\sigma^2 I_{q+1}|^{-n/2} \exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

Univariate Simple Linear Regression

$$\underset{n\times 1}{y} = \underset{n\times 2}{X}\ \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$$\underset{2\times 1}{\hat{\beta}} = (X'X)^{-1}X'\ y \quad \text{is } N(\underset{2\times 1}{E(\hat{\beta})} = \beta,\ \mathrm{cov}(\hat{\beta}) = \sigma^2 (\underset{2\times 2}{X'X})^{-1})$$

$n \geq 2$, general rule $n > 20$

and

$$\underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n-2}\underset{1\times n}{(y - X\hat{\beta})'}\underset{n\times 1}{(y - X\hat{\beta})} \qquad \text{is gamma}(\nu, \sigma^2/\nu)!$$

$$f(\underset{n\times 1}{y} \mid \beta, \sigma^2) = (2\pi)^{-n/2} |\sigma^2 I_2|^{-n/2} \exp\left[-\frac{1}{2}(\sigma^2 I_2)^{-1}(y - X\beta)'(y - X\beta)\right]$$

Univariate Multiple Linear Regression

$$\underset{n\times 1}{y} = \underset{n\times 2}{X}\ \underset{2\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$$\underset{(q+1)\times 1}{\hat{\beta}} = (X'X)^{-1}X'\ y \quad \text{is } N(\underset{(q+1)\times 1}{E(\hat{\beta}) = \beta}, \ \text{cov}(\hat{\beta}) = \sigma^2 \underset{(q+1)\times(q+1)}{(X'X)^{-1}})$$

$n \geq 2$, general rule $n > 20$

and

$$\underset{1\times 1}{\hat{\sigma}^2} = \frac{1}{n-q-1}\underset{1\times n}{(y - X\hat{\beta})'}\underset{n\times 1}{(y - X\hat{\beta})} \text{ is gamma}(\nu,\sigma^2/\nu)!$$

$$f(\underset{n\times 1}{y}|\beta,\sigma^2) = (2\pi)^{-n/2}|\sigma^2 I_{q+1}|^{-n/2}\exp\left[-\frac{1}{2}(\sigma^2 I_{q+1})^{-1}(y - X\beta)'(y - X\beta)\right]$$

Multivariate Multiple Linear Regression

$$\underset{n\times p}{Y} = \underset{n\times(q+1)}{X}\ \underset{(q+1)\times 1}{B} + \underset{n\times p}{\varepsilon}$$

Using the distributional assumption that if ε is normally distributed,

$\underset{(q+1)\times p}{\hat{B}} = (X'X)^{-1}X'\,Y$ is $N\left(E(\hat{B}) = B,\ \mathrm{cov}(vec(\hat{B})) = \left(\underset{(q+1)\times(q+1)}{X'X}\right)^{-1} \otimes \underset{p\times p}{\Sigma}\right)$ $\quad n \geq (q+1)$, general rule $n > 10p(q+1)$

and $\quad \underset{\text{cols}}{\mathrm{cov}(\hat{\beta}_j)} = \sigma_j^2 \left(X'X\right)^{-1} \quad \underset{\text{rows}}{\mathrm{cov}(\hat{b}_i)} = w_{ii}\Sigma \quad W = \left(X'X\right)^{-1}$

$\underset{p\times p}{\hat{\Sigma}} = \dfrac{1}{n-q-1}(Y - \underset{p\times n}{X\hat{B}})'(Y - \underset{n\times p}{X\hat{B}})$ is Wishart(ν,Σ/ν)!

$$\mathrm{cov}(vec(\hat{B})) = \begin{bmatrix} w_{11}\Sigma & w_{12}\Sigma & & \\ w_{21}\Sigma & w_{22}\Sigma & & \\ & & \ddots & \\ & & & w_{(q+1),(q+1)}\Sigma \end{bmatrix}$$

$$f(\underset{n\times p}{Y} \mid B,\Sigma) = (2\pi)^{-np/2}\,|\Sigma|^{-n/2} \exp\left[-\frac{1}{2} tr\Sigma^{-1}(Y - XB)'(Y - XB)\right]$$