Symmetric Matrix PDFs
(Multivariate Gamma and Inverse Gamma)

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Outline

The Wishart PDF (multivariate gamma)

The Inverse Wishart PDF (multivariate inverse gamma)

Discussion

Homework

The Wishart PDF

In our first Statistics course we learned that if $x_1,x_2,\ldots,x_n$

are iid $N(\mu,\sigma^2)$ and we calculate

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ , then } y = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \text{ .}$$

That is, the PDF of y is

$$f(y|\nu) = \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\nu/2}\Gamma\left(\nu/2\right)}$$

where $\nu=n$-1 and $y>0$.

The Wishart PDF

$$y = \frac{(n-1)s^2}{\sigma^2}$$

If we perform the transformation of variable

$$s^2 = \frac{\sigma^2}{(n-1)} y$$

(s^2 is treated as one symbol)

$$f(s^2 \mid \alpha,\beta) = \frac{(s^2)^{\alpha-1} e^{-\frac{s^2}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

$$E(s^2 \mid \alpha,\beta) = \alpha\beta$$

$$\mathrm{var}(s^2 \mid \alpha,\beta) = \alpha\beta^2$$

Then $s^2 \sim \Gamma\left(\frac{\nu}{2}, \frac{2\sigma^2}{\nu}\right)$ where $\nu = n-1$

$$\alpha = \frac{\nu}{2} \qquad \beta = \frac{2\sigma^2}{\nu}$$

with $f(s^2 \mid \nu,\sigma^2) = \dfrac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$ and $s^2, \sigma^2 > 0$.

$$E(s^2 \mid \nu,\sigma^2) = \sigma^2 \qquad var(s^2 \mid \nu,\sigma^2) = \frac{2\sigma^4}{\nu}$$

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i.e. n=10, m=10^6

If we were to simulate a large number of samples m each of size n from $N(\mu,\sigma^2)$ then calculate $\bar{x}$ and s^2 for each

sample

$$
\begin{array}{ccccc}
x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & & x_1^{(m)} \\
x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots & x_2^{(m)} \\
\vdots & \vdots & \vdots & & \vdots \\
x_n^{(1)} & x_n^{(2)} & x_n^{(3)} & & x_n^{(m)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bar{x}_{(1)} & \bar{x}_{(2)} & \bar{x}_{(3)} & \cdots & \bar{x}_{(m)} \\
s_{(1)}^2 & s_{(2)}^2 & s_{(3)}^2 & \cdots & s_{(m)}^2
\end{array}
$$

If we made a histogram of $\bar{x}$'s we would see that they are $N(\mu,\sigma^2/n)$ and …

If we made a histogram of s^2's, then we would see that they are $\Gamma\left(\frac{\nu}{2},\frac{2\sigma^2}{\nu}\right)$.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

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We can write the gamma PDF

$$f(s^2 \mid \nu,\sigma^2) = \frac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma\left(\nu/2\right)(2\sigma^2/\nu)^{\nu/2}}$$

$$\nu = n-1$$

$$s^2,\sigma^2 > 0$$

as

$$f(s^2 \mid \nu,\sigma^2) = k\left|\frac{\sigma^2}{\nu}\right|^{-\frac{\nu}{2}} \left|s^2\right|^{\frac{\nu-1-1}{2}} e^{-\frac{1}{2}\left(\frac{\sigma^2}{\nu}\right)^{-1}s^2}$$

Note the eccentric way I wrote this.

$$k = \frac{1}{\Gamma\left(\nu/2\right)2^{\nu/2}}$$

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In bivariate (multivariate) statistics if

$$f(s^2|\nu,\sigma^2)=\frac{(s^2)^{\frac{\nu}{2}-1}e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$$

$\underset{2\times1}{x_1}, \underset{2\times1}{x_2},.., \underset{2\times1}{x_n}$ are iid $N(\mu,\Sigma)$ and we calculate the covariance matrix

$\underset{2\times2}{S}=\frac{1}{n-1}\sum_{i=1}^{n}\underbrace{(\underset{2\times1}{x_i}-\underset{2\times1}{\bar{x}})}_{2\times1}\underbrace{(\underset{2\times1}{x_i}-\underset{2\times1}{\bar{x}})'}_{1\times2}$, then the PDF of the covariance matrix

$\underset{2\times2}{S}$ has the (multivariate) generalization of the gamma distribution

$$f(\underset{2\times2}{S}|\underset{2\times2}{\Sigma},\nu)=k_W\,|\underset{2\times2}{\Sigma/\nu}|^{-\frac{\nu}{2}}\,|\underset{2\times2}{S}|^{\frac{\nu-\overset{p}{2}-1}{2}}\,e^{-\frac{1}{2}tr(\underset{2\times2}{\Sigma/\nu})^{-1}\underset{2\times2}{S}}\ .$$

$S,\Sigma>0$

$\nu=n-1$

$tr()$=trace

normalizing constant

Remember the eccentric way I wrote this.

$$f(s^2|\nu,\sigma^2)=k\left|\frac{\sigma^2}{\nu}\right|^{-\frac{\nu}{2}}|s^2|^{\frac{\nu-\overset{p}{1}-1}{2}}e^{-\frac{1}{2}\left(\frac{\sigma^2}{\nu}\right)^{-1}s^2}$$

The Wishart PDF

i.e. n=10, m=10^6

If we were to simulate a large number of samples m each of size n from $N(\mu,\Sigma)$ then calculate $\bar{x}$ and S for each.

sample

$$
\begin{array}{ccccc}
x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & & x_1^{(m)} \\
x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots & x_2^{(m)} \\
\vdots & \vdots & \vdots & & \vdots \\
x_n^{(1)} & x_n^{(2)} & x_n^{(3)} & & x_n^{(m)} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bar{x}_{(1)} & \bar{x}_{(2)} & \bar{x}_{(3)} & \cdots & \bar{x}_{(m)} \\
S_{(1)} & S_{(2)} & S_{(3)} & \cdots & S_{(m)}
\end{array}
$$

If we made histograms of $\bar{x}$'s we would see that they are $N(\mu,\Sigma/n)$ and …

If we made histograms of S's, then we would see that they are $W(\Sigma/\nu,\nu)$.

$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$$

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Multivariate version of gamma distribution.

$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$$

A random $p \times p$ matrix variate $\underset{p\times p}{S}$ follows the Wishart

distribution with scale matrix $\underset{p\times p}{\Sigma}/\nu$ and ν *df* denoted $\underset{p\times p}{S} \sim W(\underset{p\times p}{\Sigma}/\nu,\nu)$

iff $f(\underset{p\times p}{S}|\underset{p\times p}{\Sigma},\nu) = k_W \left|\underset{p\times p}{\Sigma}/\nu\right|^{-\frac{\nu}{2}} \left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma}/\nu)^{-1}\underset{p\times p}{S}}$

where $k_W^{-1} = 2^{\frac{\nu p}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p}\Gamma\left(\frac{\nu+1-j}{2}\right)$

If p=1, $f(s^2|\nu,\sigma^2) = \dfrac{(s^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2\sigma^2}s^2}}{\Gamma(\nu/2)(2\sigma^2/\nu)^{\nu/2}}$ is

Gamma distribution by $\alpha=\nu/2$ and $\beta=2\sigma^2/\nu$.

The Wishart PDF

The Wishart matrix PDF is

$$f(\underset{p\times p}{S}|\Sigma,\nu)=k_W\left|\underset{p\times p}{\Sigma}/\nu\right|^{-\frac{\nu}{2}}\left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}}e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma}/\nu)^{-1}\underset{p\times p}{S}}$$

$$k_W^{-1}=2^{\frac{\nu p}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma\left(\frac{\nu+1-j}{2}\right)$$

The mean, variance, and covariance of its elements are

$$E(\underset{p\times p}{S}|\Sigma,\nu)=\underset{p\times p}{\Sigma}$$

$$\mathrm{var}(S_{ij}|\Sigma,\nu)=(\Sigma_{ij}^2+\Sigma_{ii}\Sigma_{jj})/\nu$$

$$\mathrm{cov}(S_{ij}S_{kl}|\Sigma,\nu)=(\Sigma_{ik}\Sigma_{jl}+\Sigma_{il}\Sigma_{jk})/\nu$$

If p=1

$$E(s|\sigma^2,\nu)=\sigma^2$$

$$var(s^2|\sigma^2,\nu)=\sigma^4/\nu$$

The Wishart PDF

Theorem:

If S is a $p \times p$ random matrix variable from $f(S|\Sigma,\nu)$, with

$$f(\underset{p\times p}{S}|\Sigma,\nu) = k_W \left|\underset{p\times p}{\Sigma}/\nu\right|^{-\frac{\nu}{2}} \left|\underset{p\times p}{S}\right|^{\frac{\nu-p-1}{2}} e^{-\frac{1}{2}tr(\underset{p\times p}{\Sigma}/\nu)^{-1}\underset{p\times p}{S}}$$

then if we form $\underset{r\times r}{Q} = \underset{r\times p}{A}\ \underset{p\times p}{S}\ \underset{p\times r}{A'}$ where dimensions match

and A full row rank (A: $r \times p$, $r \leq p$), then $\underset{r\times r}{Q} \sim W(\underset{r\times r}{\Delta} = A\,\Sigma\,A'/\nu,\nu)$

$$E(\underset{r\times r}{Q}|\Delta,\nu) = \underset{r\times r}{\Delta}$$

$$\mathrm{var}(Q_{ij}|\Delta,\nu) = (\Delta_{ij}^2 + \Delta_{ii}\Delta_{jj})/\nu$$

$$\mathrm{cov}(Q_{ij}Q_{kl}|\Delta,\nu) = (\Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk})/\nu$$

.

The Wishart PDF

$$\underset{2\times 1}{\mu} = \begin{pmatrix} 67 \\ 150 \end{pmatrix} \quad \underset{2\times 2}{\Sigma} = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix}$$

$$\nu = 9$$

Took 10^6 sets of n=10 variates $\underset{2\times 1}{x}$, subtracted mean $\underset{2\times 1}{\bar{x}}$

from each set, transpose multiplied each value, added the 10 values

in set and divided by 9 to form each $\underset{2\times 2}{S}$. The $\underset{2\times 2}{S}$'s are now $W(\underset{2\times 2}{\Sigma}/\nu,\nu=n\text{-}1)$.

$$E(\underset{p\times p}{S} \mid \Sigma,\nu) = \underset{p\times p}{\Sigma}$$

$$\text{var}(S_{ij} \mid \Sigma,\nu) = (\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj}) / \nu$$

$$\text{cov}(S_{ij}S_{kl} \mid \Sigma,\nu) = (\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) / \nu$$

$$\underset{p\times p}{S} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)$$

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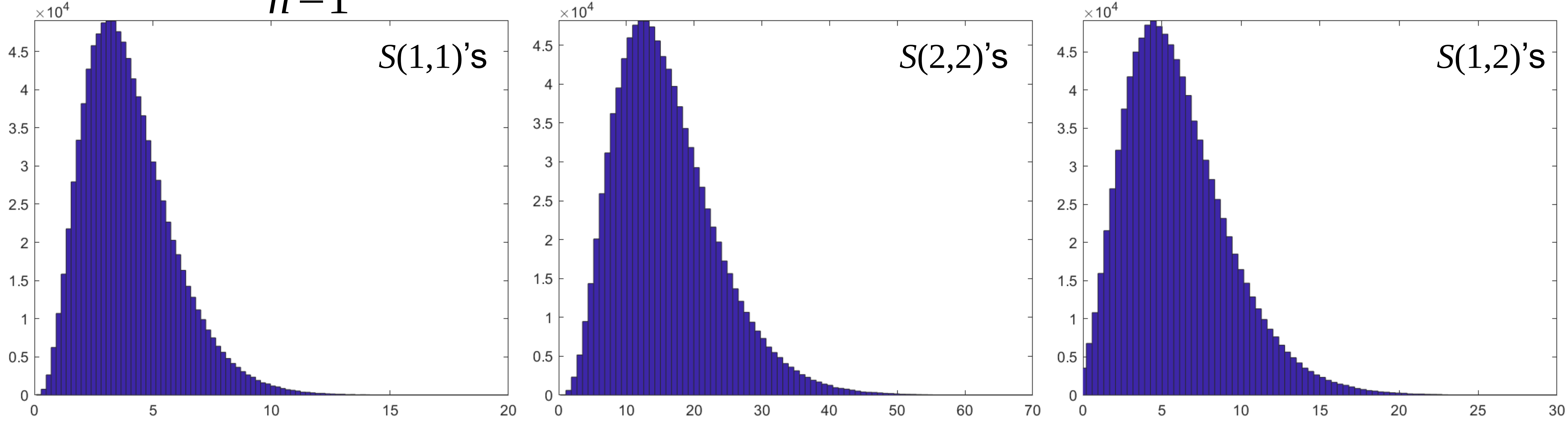
$$\underset{2\times 1}{\mu}=\begin{pmatrix}67\\150\end{pmatrix}\quad \underset{2\times 2}{\Sigma}=\begin{pmatrix}4 & 6\\ 6 & 16\end{pmatrix}$$

$$\nu=9$$

The S's, $S=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})$ are now $W(\Sigma/\nu,\nu=n\text{-}1)$.



$$E(S\,|\,\Sigma,\nu)=\Sigma\quad=\begin{pmatrix}4 & 6\\ 6 & 16\end{pmatrix}\qquad \mathrm{var}(S_{ij}\,|\,\Sigma,\nu)=(\Sigma_{ij}^2+\Sigma_{ii}\Sigma_{jj})/\nu=\begin{pmatrix}3.56 & 11.11\\ 11.11 & 56.89\end{pmatrix}$$

$$\Sigma=AA'$$

$$\mathrm{cov}(S_{ij}S_{kl}\,|\,\Sigma,\nu)=(\Sigma_{ik}\Sigma_{jl}+\Sigma_{il}\Sigma_{jk})/\nu=\underset{11,22}{5.33},\underset{11,12}{8.00},\underset{22,12}{21.33}\leftarrow ij,kl$$

$$A=\begin{pmatrix}2 & 0\\ 3 & \sqrt{7}\end{pmatrix}$$

The Wishart PDF

```
rng('default')
% define dimensions
p=2;     % dimension of vectors
n=10;    % sample size
m=10^6;  % repeated samples
nu=n-1;

% specify the mean vector
mu=[67;150]

% specify the covariance matrix
Sigma=[2^2,2*4*.75;2*4*.75,4^2]

% Cholesky factorize
A=chol(Sigma)';
A=[2,0;3,sqrt(7)]
```

```
% Wishart Distribution %%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Generate via Cholesky
% generate the simulated observations
XX=A*randn(p,n*m)+repmat(mu,[1,n*m]);

X=zeros(p,n,m);
for j=1:m
    X(:,:,j)=XX(:,(j-1)*n+1:n*j);
end
clear j
figure;
hist(squeeze(S(1,1,:)),100)
xlabel('var(height)'), axis tight, xlim([0,20])
figure;
hist(squeeze(S(2,2,:)),100)
xlabel('var(weight)'), axis tight, xlim([0,70])
figure;
hist(squeeze(S(1,2,:)),100)
xlabel('cov(height,weight)'), axis tight,
xlim([0,30])
```

The Wishart PDF

```
% compute the mean in each sample
meanX=squeeze(mean(X,2));
mean(meanX,2)

figure;
hist(meanX(1,:),100)
xlabel('mean(height)'), axis tight
figure;
hist(meanX(2,:),100)
xlabel('mean(weight)'), axis tight

% covariance matrix for each sample
S=zeros(p,p,m);
for j=1:m
    S(:,:,j)=cov(squeeze(X(:,:,j)'),0);
end
clear j
```

```
% covariance
[Sigma,mean(S,3)]
% true variances
i=1; j=1;
vS11=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
i=1; j=2;
vS12=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
i=2; j=2;
vS22=(Sigma(i,j)^2+Sigma(i,i)*Sigma(j,j))/nu;
V=[vS11,vS12;vS12,vS22];
[V,var(S,1,3)]
```

The Wishart PDF

```
figure;
hist(squeeze(S(1,1,:)),100)
xlabel('var(height)'), axis tight, xlim([0,20])
figure;
hist(squeeze(S(2,2,:)),100)
xlabel('var(weight)'), axis tight, xlim([0,70])
figure;
hist(squeeze(S(1,2,:)),100)
xlabel('cov(height,weight)'), axis tight, xlim([0,30])

% true covariances
i=1; j=1; k=1; l=2;
cS1112=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=1; k=2; l=2;
cS1122=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=2; k=2; l=2;
cS1222=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
[cS1112,cS1122,cS1222]
% sample covariances
covS1112=cov(squeeze(S(1,1,:)),squeeze(S(1,2,:)));
covS1122=cov(squeeze(S(1,1,:)),squeeze(S(2,2,:)));
covS1222=cov(squeeze(S(1,2,:)),squeeze(S(2,2,:)));
[covS1112(1,2),covS1122(1,2),covS1222(1,2)]
```

```
% make histograms of the SS elements
figure;
hist(squeeze(S(1,1,:)),100)
xlabel('var(height)'), axis tight
figure;
hist(squeeze(S(2,2,:)),100)
xlabel('var(weight)'), axis tight
figure;
hist(squeeze(S(1,2,:)),100)
xlabel('cov(height,weight)'), axis tight
```

The Wishart PDF

We can generate random matrix variate observations directly from the Wishart PDF via the Matlab function wishrnd(Sigma/nu,nu)

```
% define dimensions
p=2;     % dimension of vectors
n=10;    % sample size
m=10^6; % repeated samples
nu=n-1; % degrees of freedom
% specify the covariance matrix
Sigma=[2^2,2*4*.75;2*4*.75,4^2]
```

```
% use matlab function %%%%%%%%%%%%%%%%%%%%%%%
Wmat=zeros(p,p,m); D=chol(Sigma/nu); Hmat=zeros(p,p,m);
for count=1:m
    Wmat(:,:,count) = wishrnd(Sigma,nu,D);
    Hmat(:,:,count) = inv(squeeze(Wmat(:,:,count))); % use later
end
```

and compare sample to theoretical values!

The Wishart PDF

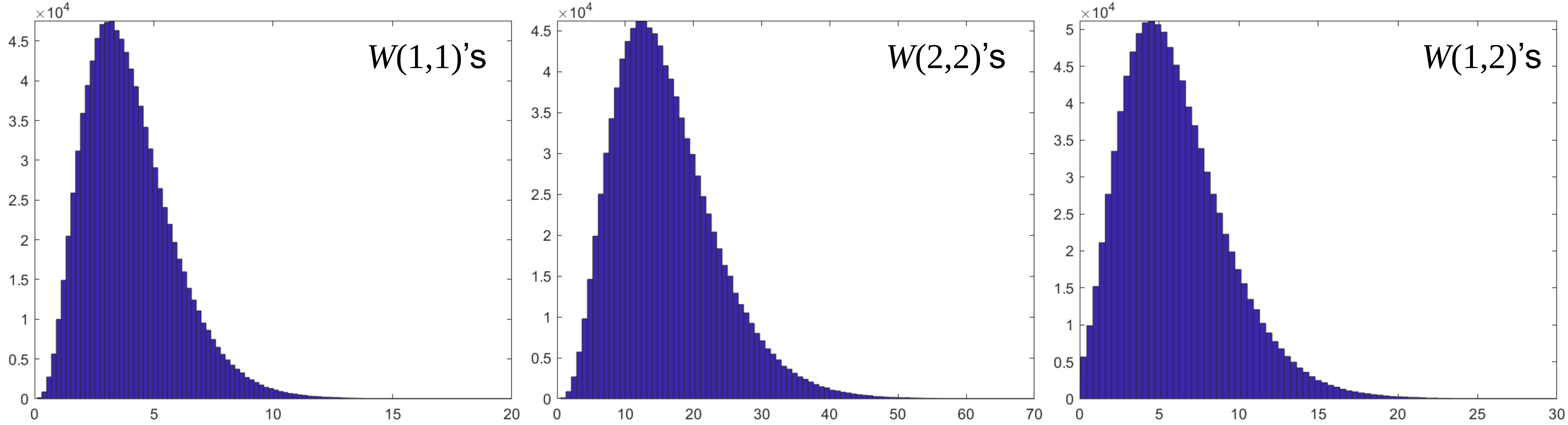
```
% mean and variances
[Sigma,mean(Wmat,3)]
[(Sigma(1,1)^2+Sigma(1,1)*Sigma(1,1)),
(Sigma(1,2)^2+Sigma(1,1)*Sigma(2,2)),...
(Sigma(2,2)^2+Sigma(2,2)*Sigma(2,2))]/nu
varWmat=var(Wmat,[],3);
[varWmat(1,1),varWmat(1,2),varWmat(2,2)]
% covariances
i=1; j=1; k=1; l=2;
cG1112=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=1; k=2; l=2;
cG1122=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
i=1; j=2; k=2; l=2;
cG1222=(Sigma(i,k)*Sigma(j,l)+Sigma(i,l)*Sigma(j,k))/nu;
[cG1112,cG1122,cG1222]
covG1112=cov(squeeze(Wmat(1,1,:)),squeeze(Wmat(1,2,:)));
covG1122=cov(squeeze(Wmat(1,1,:)),squeeze(Wmat(2,2,:)));
covG1222=cov(squeeze(Wmat(1,2,:)),squeeze(Wmat(2,2,:)));
[covG1112(1,2),covG1122(1,2),covG1222(1,2)]
```

The Wishart PDF

$$\mu=\begin{pmatrix}67\\150\end{pmatrix}\quad \Sigma=\begin{pmatrix}4&6\\6&16\end{pmatrix}$$

$$p=2\quad \nu=9$$

Matlab: Wmat=wishrnd(Sigma/nu,nu)



$$E(W)=\begin{pmatrix}4&6\\6&16\end{pmatrix}\quad \mathrm{var}(W)=\begin{pmatrix}3.5556&11.1111\\11.1111&56.8889\end{pmatrix}\quad \mathrm{cov}(W_{ij})=(5.3333,8.0000,21.3333)$$

$$\bar{W}=\begin{pmatrix}3.9978&5.9952\\5.9952&15.9916\end{pmatrix}\quad s_W^2=\begin{pmatrix}3.5426&11.0882\\11.0882&56.7960\end{pmatrix}\quad s_{W_{ij}}=(5.3152,7.9784,21.2961)$$

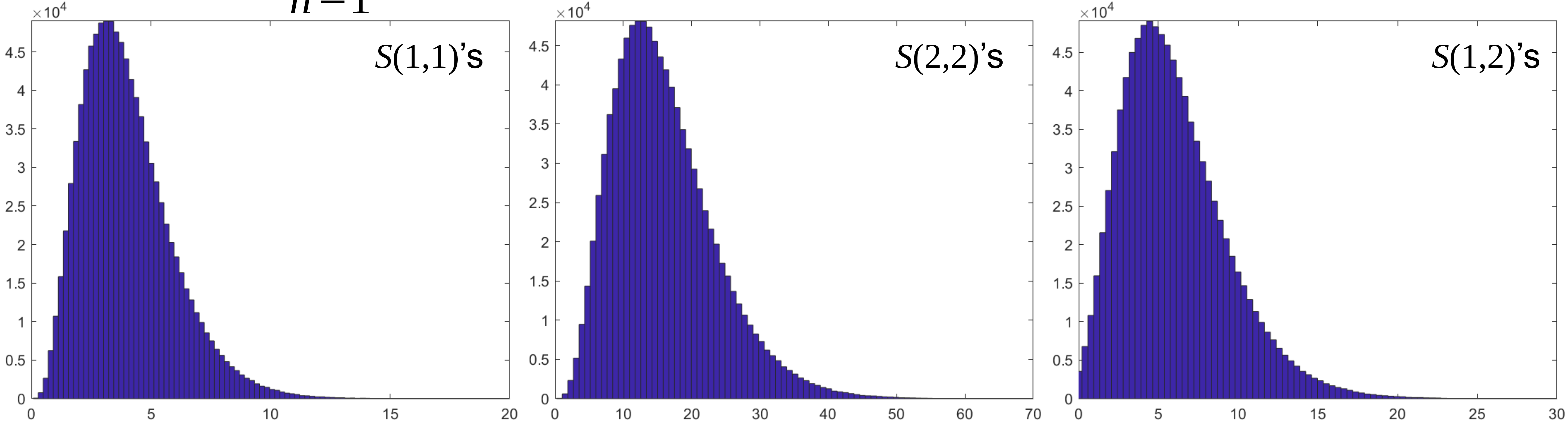
toggle forward

The Wishart PDF

$$\mu = \begin{pmatrix} 67 \\ 150 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix}$$

$$p = 2 \quad \nu = 9$$

From x's: $S = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})'(x_i - \bar{x})$



$$E(S) = \begin{pmatrix} 4 & 6 \\ 6 & 16 \end{pmatrix} \qquad \text{var}(S) = \begin{pmatrix} 3.5556 & 11.1111 \\ 11.1111 & 56.8889 \end{pmatrix} \qquad \text{cov}(S_{ij}) = (5.3333, 8.0000, 21.3333)$$

$$\bar{S} = \begin{pmatrix} 4.0010 & 6.0007 \\ 6.0007 & 15.9936 \end{pmatrix} \qquad s_W^2 = \begin{pmatrix} 3.5647 & 11.1342 \\ 11.1342 & 56.9372 \end{pmatrix} \qquad s_{S_{ij}} = (5.3477, 8.0200, 21.3676)$$

toggle backward

The Wishart PDF

```
% make histograms of the SS elements
figure;
hist(squeeze(Wmat(1,1,:)),100)
xlabel('var(height)'), axis tight, xlim([0,20])
figure;
hist(squeeze(Wmat(2,2,:)),100)
xlabel('var(weight)'), axis tight, xlim([0,70])
figure;
hist(squeeze(Wmat(1,2,:)),100)
xlabel('cov(height,weight)'),axis tight, xlim([0,30])
```

The Inverse Wishart PDF

A random variable h has a continuous

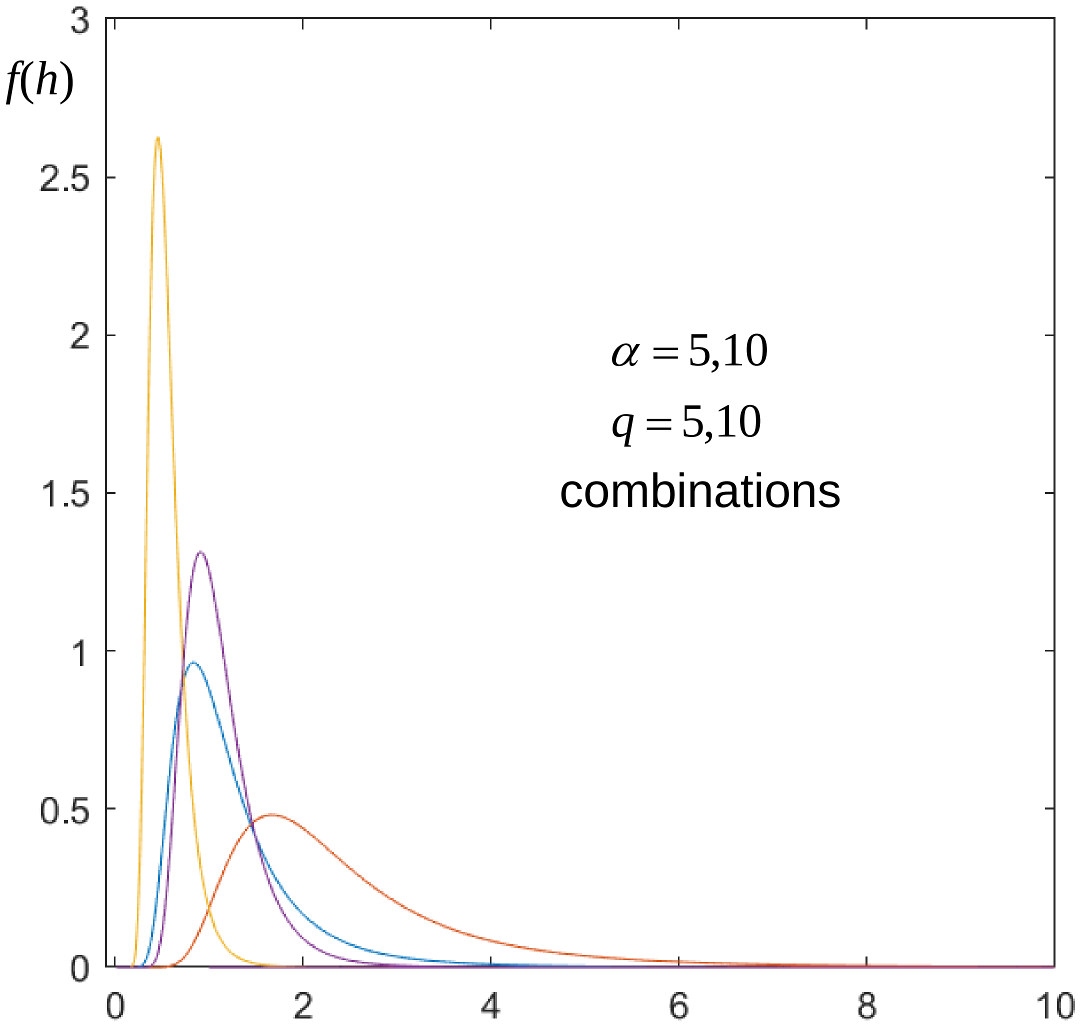
Inverse Gamma distribution, $h \sim \Gamma^{-1}(\alpha,q)$ if

$$f(h|\alpha,q)=\frac{q^{\alpha}}{\Gamma(\alpha)}h^{-\alpha-1}e^{-q/h} \quad ,$$

where $\alpha,q>0$ and $h>0$.

And can take on many shapes.

$\Gamma(a)=(a-1)!$ for integer a.



$$E(h|\alpha,q)=\frac{q}{\alpha-1}$$

$$\mathrm{var}(h|\alpha,q)=\frac{q^2}{(\alpha-1)^2(\alpha-2)}$$

The Inverse Wishart PDF

The inverse gamma PDF can be arrived at by starting with

$$f(s^2|\alpha,\beta)=\frac{(s^2)^{\alpha-1}e^{-\frac{s^2}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} \qquad \alpha=\frac{\nu}{2} \quad \beta=\frac{2\sigma^2}{\nu}$$

defining

$$h=s^{-2} \qquad J(s^2\to h)=h^{-2} \qquad q=1/\beta$$

and performing a transformation of variable resulting in

$$f(h|\alpha,q)=\frac{q^{\alpha}}{\Gamma(\alpha)}h^{-\alpha-1}e^{-qh^{-1}} \qquad \begin{array}{r} h>0 \\ q,\nu>0 \end{array}$$

for the univariate case.

The Inverse Wishart PDF

$$f(h|\nu,q)=\frac{q^{\nu/2}}{\Gamma(\nu/2)}h^{-\nu/2+1}e^{-qh^{-1}}$$

And upon generalizing to higher dimensions, we obtain the inverse Wishart PDF.

$$f(\underset{p\times p}{H}|Q,\nu)=k_{IW}Q^{\nu/2}|\underset{p\times p}{H}|^{-(\nu+p+1)/2}\ e^{-\frac{1}{2}trQ_{p\times p}H^{-1}_{p\times p}}$$

$$H,Q>0$$
$$\nu>0$$

$$k_{IW}^{-1}=2^{\nu p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left(\tfrac{\nu-1-j}{2}\right)$$

$$E(\underset{p\times p}{H}|\underset{p\times p}{Q},\nu)=\frac{Q}{\nu-p-1}$$

$$\mathrm{var}(H_{ii}|Q,\nu)=\frac{2Q_{ii}^2}{(\nu-p-1)^2(\nu-p-3)}$$

$$\mathrm{var}(H_{ij}|Q,\nu)=\frac{(\nu-p+1)Q_{ij}^2+(\nu-p-1)Q_{ii}Q_{jj}}{(\nu-p)(\nu-p-1)^2(\nu-p-3)}$$

$$\mathrm{cov}(H_{ij}|Q,\nu)=\frac{2Q_{ij}Q_{kl}+(\nu-p-1)(Q_{ik}Q_{jl}+Q_{il}Q_{kj})}{(\nu-p)(\nu-p-1)^2(\nu-p-3)}$$

The Inverse Wishart PDF

The inverse Wishart PDF can be arrived at by starting with

$$f(S\,|\,V,\nu)=k_W\,|V|^{-\frac{\nu}{2}}\,|S|^{\frac{\nu-p-1}{2}}\,e^{-\frac{1}{2}trV^{-1}S}$$

defining

$$\underset{p\times p}{H}=\underset{p\times p}{S}^{-1} \qquad \underset{p\times p}{Q}=\underset{p\times p}{V}^{-1}=\nu\Sigma^{-1} \qquad J(S\to H)=H^{-(p+1)}$$

and performing a transformation of variable resulting in

$$f(\underset{p\times p}{H}\,|\,\underset{p\times p}{Q},\nu)=k_{IW}\,|\underset{p\times p}{Q}|^{\nu/2}\,|\underset{p\times p}{H}|^{-(\nu+p+1)/2}\,e^{-\frac{1}{2}tr(\underset{p\times p}{Q}\,\underset{p\times p}{H}^{-1})} \qquad \begin{matrix} H,Q>0 \\ \nu>0 \end{matrix}$$

$$k_{IW}^{-1}=2^{\nu p/2}\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left(\tfrac{\nu-p-j}{2}\right)$$

for the multivariate case.

$$f(h\,|\,\nu,q)=\frac{q^{\nu/2}}{\Gamma(\nu/2)}h^{-\nu/2-1}e^{-\frac{1}{2}qh^{-1}}$$

The Inverse Wishart PDF

We can invert each of the Wishart distributed matrix observations, $H=S^{-1}$, then the H has an inverse Wishart PDF **or** we can generate random matrix variate observations from the inverse Wishart PDF via the Matlab function iwishrnd(Sigma/nu,nu)

```
% Matlab function
IWmat=zeros(p,p,m); Q=inv(Sigma/nu); DI=chol(inv(Q));
for count=1:m
    IWmat(:,:,count) = iwishrnd(Q,nu,DI);
end
```

and compare sample to theoretical values

The Inverse Wishart PDF

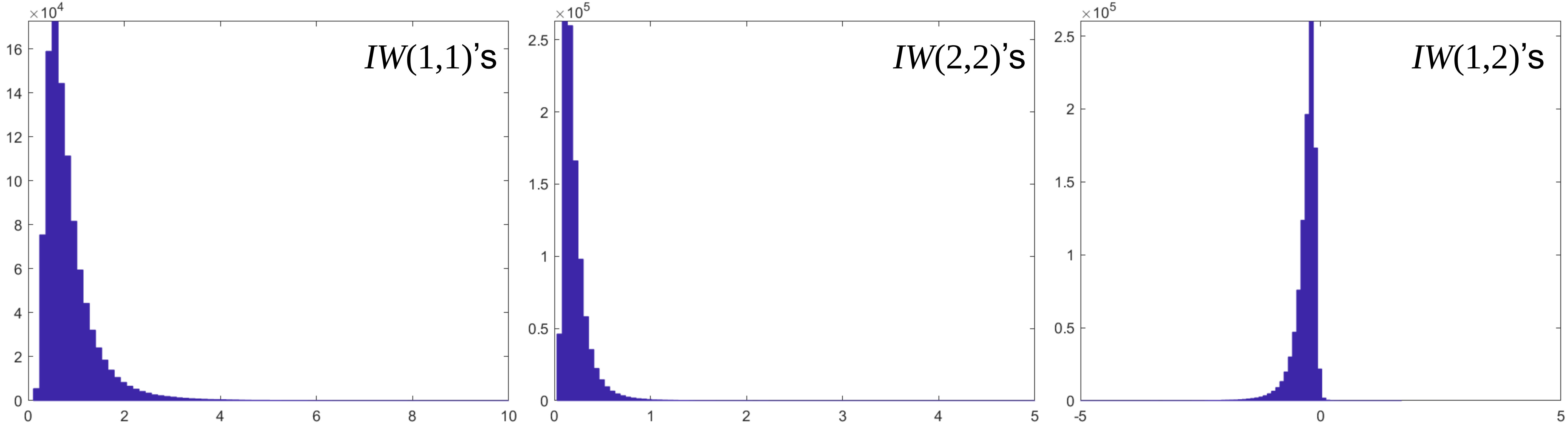
```
% Inverse Wishart Distribution %%%%%%%%%%%%%%%%%%%%%%%%%%%
Q=inv(Sigma/nu);
% means and variances
[Q/(nu-p-1),mean(Hmat,3)]
[2*Q(1,1)^2/((nu-p-1)^2*(nu-p-3)),...
    ((nu-p+1)*Q(1,2)^2+(nu-p-1)*Q(1,1)*Q(2,2))/((nu-p)*(nu-p-1)^2*(nu-p-3)),...
    2*Q(2,2)^2/((nu-p-1)^2*(nu-p-3))]
varHmat=var(Hmat,[],3);
[varHmat(1,1),varHmat(1,2),varHmat(2,2)]
% covariances
i=1;,j=1;,k=1;,l=2;
cH1112=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
i=1;,j=1;,k=2;,l=2;
cH1122=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
i=1;,j=2;,k=2;,l=2;
cH1222=(2*Q(i,j)*Q(k,l)+(nu-p-1)*(Q(i,k)*Q(j,l)+Q(i,l)*Q(k,j)))/((nu-p)*(nu-p-1)^2*(nu-p-3));
[cH1112,cH1122,cH1222]
covH1112=cov(squeeze(Hmat(1,1,:)),squeeze(Hmat(1,2,:)));
covH1122=cov(squeeze(Hmat(1,1,:)),squeeze(Hmat(2,2,:)));
covH1222=cov(squeeze(Hmat(1,2,:)),squeeze(Hmat(2,2,:)));
[covH1112(1,2),covH1122(1,2),covH1222(1,2)]
```

The Wishart PDF

$$E(H \mid Q,\nu) = \frac{Q}{\nu - p - 1} \qquad Q = \begin{pmatrix} 5.1429 & -1.9286 \\ -1.9286 & 1.2857 \end{pmatrix}$$

Matlab: IWmat = iwishrnd(Q,nu,DI);

$$p = 2 \quad \nu = 9$$



$$E(IW) = \begin{pmatrix} 0.8571 & -0.3214 \\ -0.3214 & 0.2143 \end{pmatrix} \qquad \mathrm{var}(IW) = \begin{pmatrix} 0.3673 & 0.0689 \\ 0.0689 & 0.0230 \end{pmatrix} \qquad \mathrm{cov}(IW_{ij}) = (-0.1378, 0.0574, -0.0344)$$

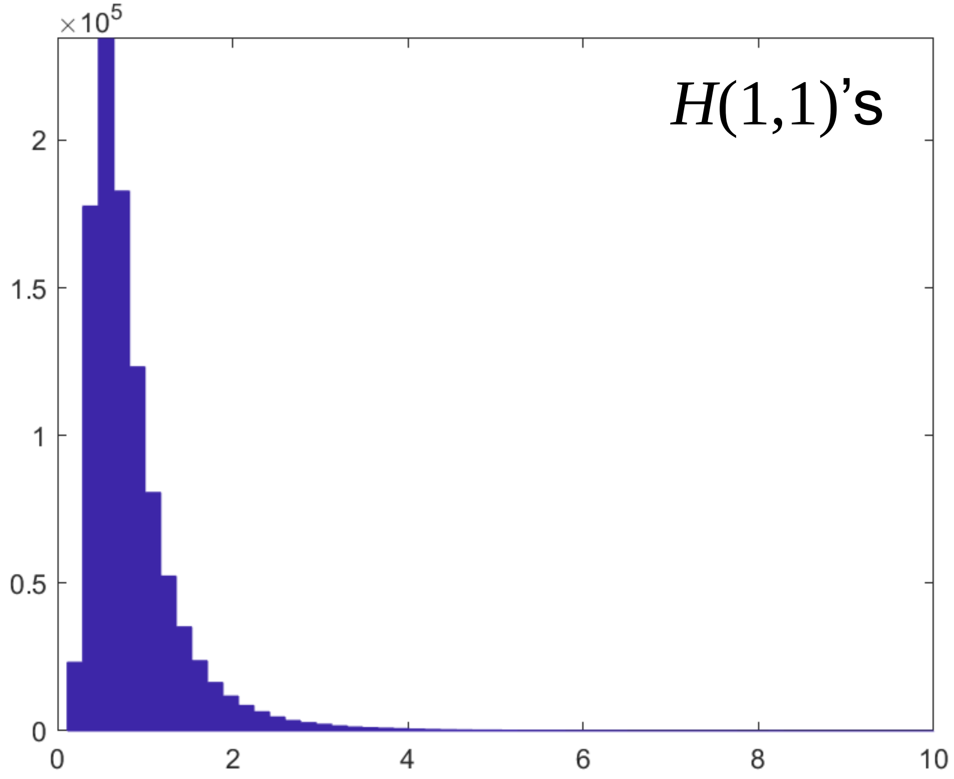
$$\overline{IW} = \begin{pmatrix} 0.8568 & -0.3216 \\ -0.3216 & 0.2144 \end{pmatrix} \qquad s_{IW}^2 = \begin{pmatrix} 0.3723 & 0.0721 \\ 0.0721 & 0.0245 \end{pmatrix} \qquad s_{IW_{ij}} = (-0.1400, 0.0583, -0.0348)$$

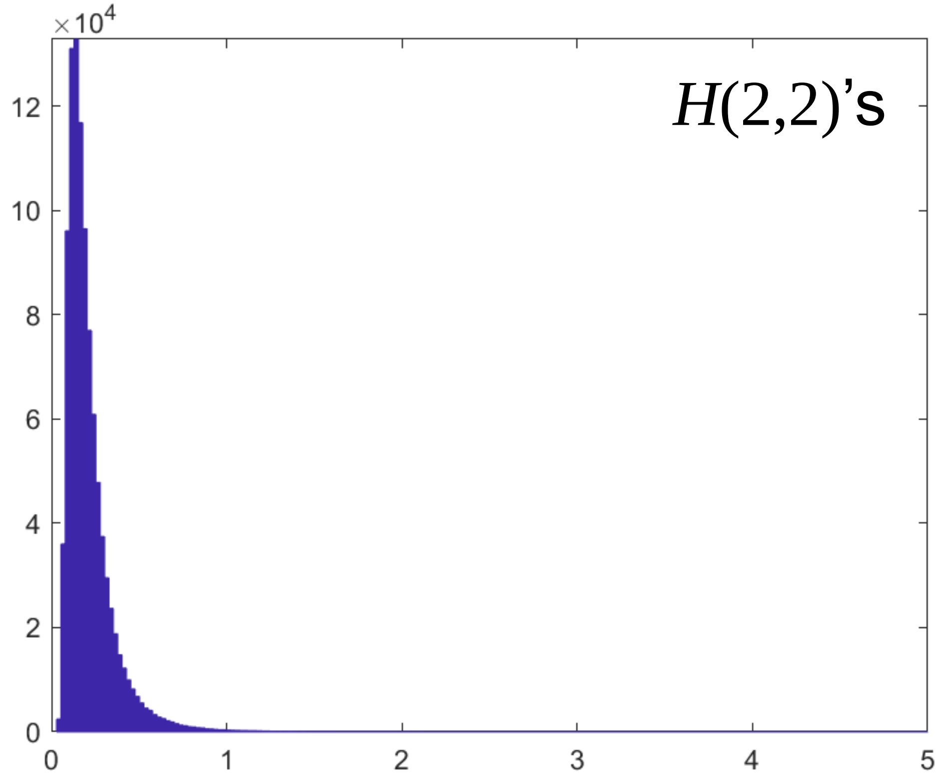
The Wishart PDF

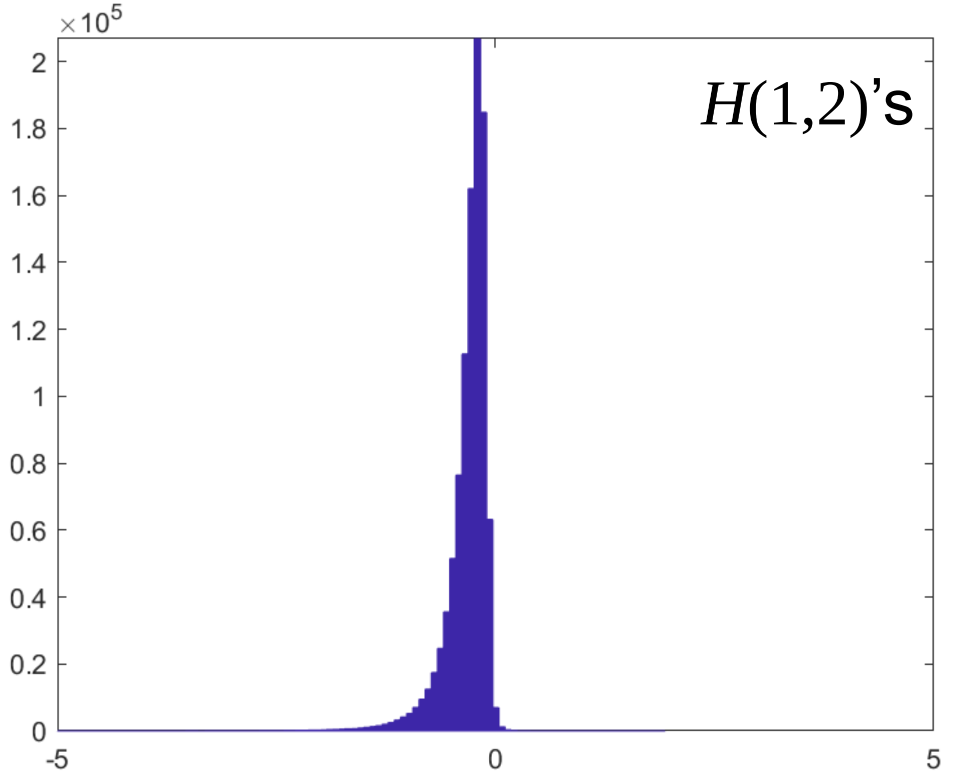
$$E(H \mid Q, \nu) = \frac{Q}{\nu - p - 1} \qquad Q = \begin{pmatrix} 5.1429 & -1.9286 \\ -1.9286 & 1.2857 \end{pmatrix}$$

From x's: $H = S^{-1}$

$$p = 2 \quad \nu = 9$$







$$E(H) = \begin{pmatrix} 0.8571 & -0.3214 \\ -0.3214 & 0.2143 \end{pmatrix} \qquad \text{var}(H) = \begin{pmatrix} 0.3673 & 0.0689 \\ 0.0689 & 0.0230 \end{pmatrix} \qquad \text{cov}(H_{ij}) = (-0.1378, 0.0574, -0.0344)$$

$$\bar{H} = \begin{pmatrix} 0.8568 & -0.3216 \\ -0.3216 & 0.2144 \end{pmatrix} \qquad s_H^2 = \begin{pmatrix} 0.3723 & 0.0721 \\ 0.0721 & 0.0245 \end{pmatrix} \qquad s_{H_{ij}} = (-0.1400, 0.0583, -0.0348)$$

The Inverse Wishart PDF

```
% make histograms of the SS elements
figure;
hist(squeeze(IWmat(1,1,:)),500),axis tight,xlim([0,10])
figure;
hist(squeeze(IWmat(2,2,:)),500),axis tight,xlim([0,5])
figure;
hist(squeeze(IWmat(1,2,:)),500),axis tight,xlim([-5,5])
```

Discussion

Questions?

Homework 6

1. Assume Marquette Undergrads heights have
μ_h=70 in and σ_h=3 in while their weights have
μ_w=160 lbs and σ_w=4 lbs with ρ=.75.
   a) Generate 10^7 h/w 2×1 vectors using the Cholesky, $\Sigma = AA'$.
   b) Divide the 10^7 random vectors into samples (sets) of 10.
   c) Calculate means, variances, and covariance from each 10.
   d) Calculate the mean of: means, variances, and covariance.
   e) Calculate the variance of: means, variances, and covariance.
   f) Calculate the covariance between: means, variances & covariances.
   g) Make histograms of everything. Do your results match theory?

Homework 6

2*.With pencil and paper sketch out the transformation from the Wishart RV S to the inverse Wishart RV H where H=S^{-1}, use $J(S\rightarrow H)=|H|^{-(p+1)}$.

This is a similar transformation as $h= (s^2)^{-1}$ using $J(s^2\rightarrow h)=h^{-2}$ for gamma to inverse gamma. (Do this one first.)

* For students in MSSC 5790.

Homework 6

3**.Compute the inverse of each of your S matrices from 1 to form H matrices.

Make histograms of 11, 22, and 12 elements of H. Calculate means, variances, and covariances of the elements of H. Compare to the theoretical values.

** For students that have had 6010 and 6020.