Bivariate Probability Density Functions Continued

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Outline

Bivariate Student-t PDF

Bivariate N-IG Conditional & Marginal PDFs
(Normal-Inverse Gamma)

Bivariate L-IG Conditional & Marginal PDFs
(Laplace-Inverse Gamma)

Discussion

Homework

Bivariate Student-t PDF

We know that if we have a univariate standard normal PDF

$$f(z)=\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$ where $-\infty<z<\infty$

and we transform to $x=az+\mu$, then

$$f(x\,|\,\mu,\sigma^2)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$ where $-\infty<x,\mu<\infty$ and $0<\sigma=a$.

$E(X)=\mu$, $var(X)=\sigma^2=aa$.

Bivariate Student-t PDF

Similarly, if we have a univariate (standard) Student-t PDF

$$f(t|\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t^2\right)^{(\nu+1)/2}}$$

where $-\infty<z<\infty$

and we transform to $s=\tau t+\delta$, then

$$f(s|\nu,\delta,\tau^2)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{(\tau^2)^{-1/2}}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}\left(\frac{s-\delta}{\tau}\right)^2\right)^{(\nu+1)/2}}$$

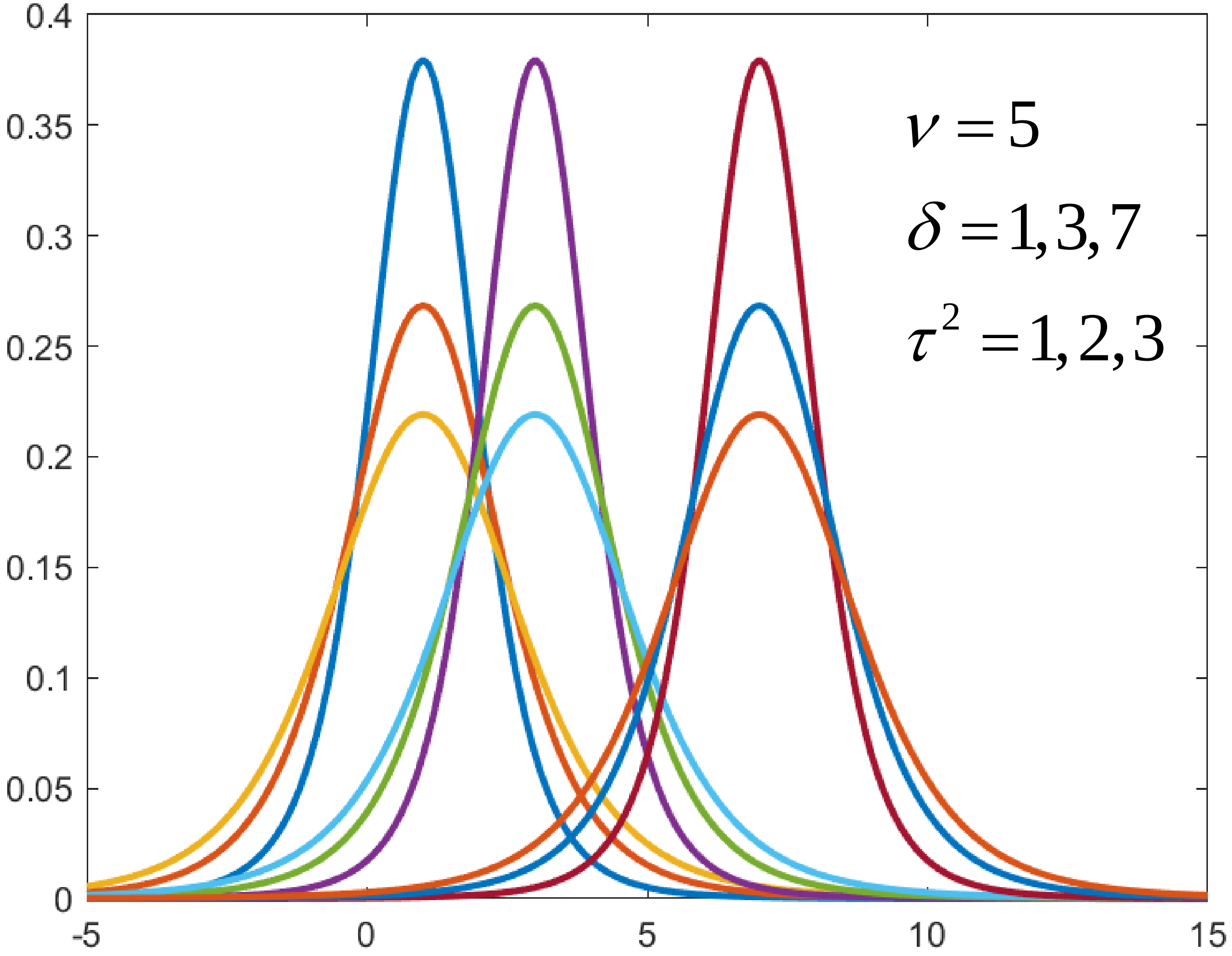
where $-\infty<s,\delta<\infty$ and $0<\tau$

$\nu=1,2,\ldots$

$E(S)=\delta$ for $\nu>1$, $var(S)=\nu\tau^2/(\nu-2)$ for $\nu>2$.

Bivariate Student-t PDF

As we can see, the location and scale of the Student-t PDF are evident from selection of varied parameters.



$$f(s\,|\,\nu,\delta,\tau^2)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{(\tau^2)^{-1/2}}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}\left(\frac{s-\delta}{\tau}\right)^2\right)^{(\nu+1)/2}}$$

$$\nu=1,2,\ldots$$

$$-\infty<s,\delta<\infty$$

$$0<\tau$$

Bivariate Student-t PDF

```
s=(-5:.1:15)';
nu=5; delta=[1,3,7]; tau2=[1,2,3]
figure;
for counti=1:length(nu)
    for countj=1:length(delta)
        for countk=1:length(tau2)
            mytpdf=gamma((nu(counti)+1)/2)/gamma(nu(counti)/2)...
                /sqrt(nu(counti)*pi*tau2(countk))...
                ./(1+((s-delta(countj)).^2)/tau2(countk)/nu(counti))...
                .^((nu(counti)+1)/2);
            plot(s,mytpdf,'LineWidth',2)
            ylim([0 .4]), xlim([-5 15])
            hold on
        end
    end
end
```

Bivariate Student-t PDF

We know that we can form a bivariate standard normal PDF

by multiplying two independent univariate standard normal PDFs.

With $z=(z_1,z_2)'$, we have

$$f(z)=(2\pi)^{-1/2}e^{-\frac{z_1^2}{2}}(2\pi)^{-1/2}e^{-\frac{z_2^2}{2}}$$

and using the theorem in the last lecture that if $\underset{2\times1}{x}=\underset{2\times2}{A}\underset{2\times1}{z}+\underset{2\times1}{\mu}$,

then $E(X|\mu,\Sigma)=\underset{2\times1}{\mu}$ and $cov(X|\mu,\Sigma)=\underset{2\times2}{\Sigma}$, $\underset{2\times2}{\Sigma}=\underset{2\times2}{A}\underset{2\times2}{A}'$.

Further $\underset{2\times1}{x}=(x_1,x_2)'$ has a bivariate normal PDF.

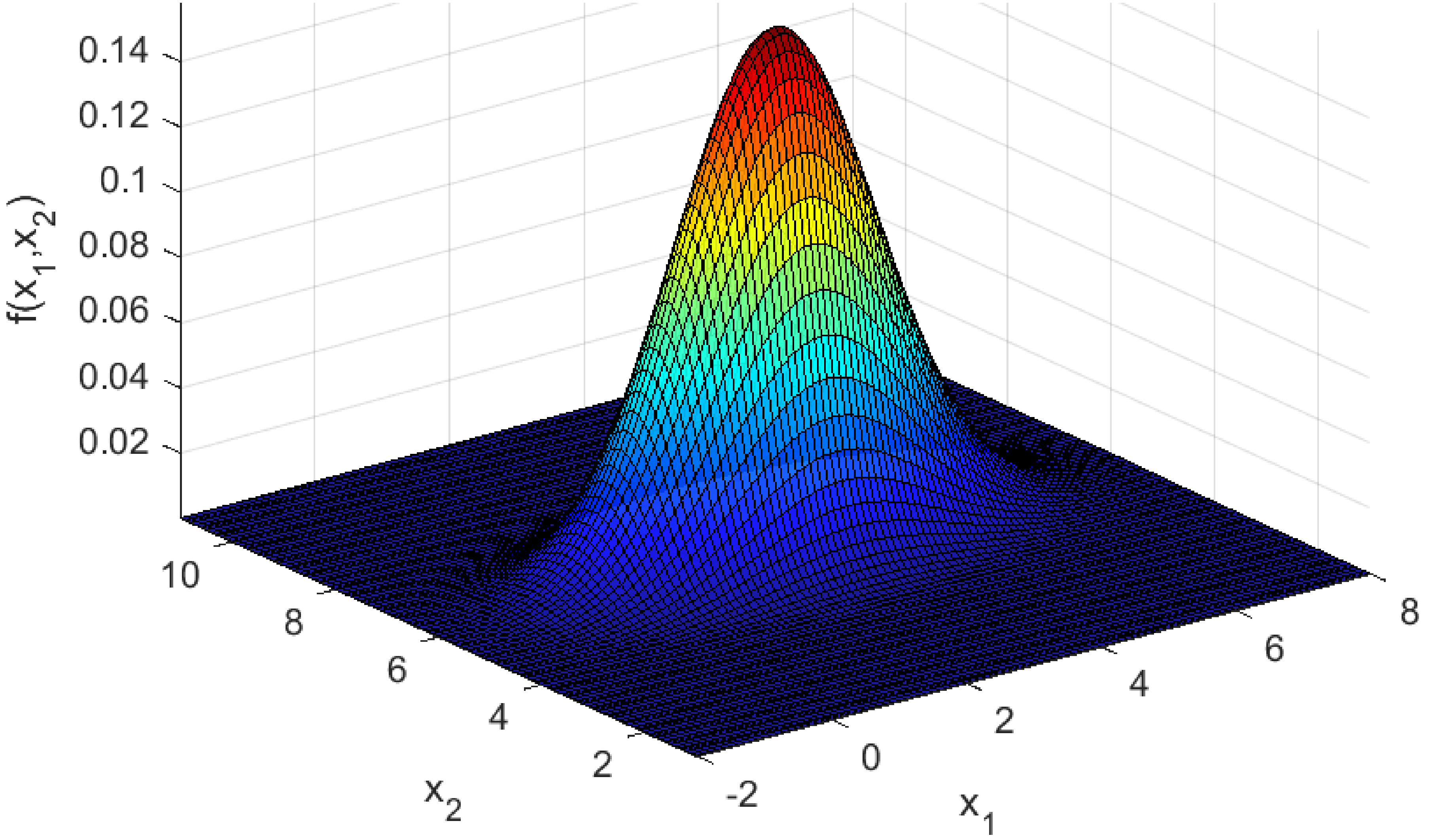
That is, $\underset{2\times1}{x}\sim N(\underset{2\times1}{\mu},\underset{2\times2}{\Sigma})$ or $f(x\,|\,\mu,\Sigma)=(2\pi)^{-2/2}\,|\Sigma|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$.

Bivariate Student-t PDF

That is, $\underset{2\times 1}{x} \sim N(\underset{2\times 1}{\mu},\underset{2\times 2}{\Sigma})$.

$$\mu = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$



$$f(x|\mu,\Sigma) = (2\pi)^{-2/2}\,|\Sigma|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

$$x,\mu \in \mathbb{R}^2$$
$$0 < \Sigma$$

Bivariate Student-t PDF

We can form a bivariate Student-t PDF with two independent univariate Student-t PDFs.

With t=(t_1,t_2)', we have

$$f(t\,|\,\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t_1^2\right)^{(\nu+1)/2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t_2^2\right)^{(\nu+1)/2}}$$

and using the transformation theorem that if $\underset{2\times1}{s}=\underset{2\times2}{A}\,\underset{2\times1}{t}+\underset{2\times1}{\delta}$,

then $E(S|v,\delta,\mathrm{T})=\underset{2\times1}{\delta}$ and $cov(S|\delta,\mathrm{T})=\dfrac{\nu}{\nu-2}\mathrm{T}=\Sigma$, $\underset{2\times2}{\Sigma}=\underset{2\times2}{A}\,\underset{2\times2}{A'}$,

and $\underset{2\times1}{s}$=(s_1,s_2)' has a bivariate Student-t PDF.

$p=2$

That is, $\underset{2\times1}{s}\sim t(v,\underset{2\times1}{\delta},\underset{2\times2}{\mathrm{T}})$.

$$f(s\,|\,\nu,\delta,\mathrm{T})=\frac{\Gamma(\frac{\nu+p}{2})|\mathrm{T}|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})}[1+(s-\delta)'\mathrm{T}^{-1}(s-\delta)/\nu]^{-(\nu+p)/2}$$

Bivariate Student-t PDF

Theorem: Marginal

If s is a 2-D (or p-D) random variable from $f(s|\delta,T)$ Student-t, with

$$E(s|\nu,\delta,\mathrm{T}) = \underset{p\times 1}{\delta} \text{ and } cov(s|\,\nu,\delta,\mathrm{T}) = \frac{\nu}{\nu-2}\underset{p\times p}{\mathrm{T}}$$ think of p=2

and partition $\underset{p\times 1}{s} = \begin{pmatrix} s_A \\ s_B \end{pmatrix} \begin{matrix} \}\, p_A\times 1 \\ \}\, p_B\times 1 \end{matrix}$, $\underset{p\times 1}{\delta} = \begin{pmatrix} \delta_A \\ \delta_B \end{pmatrix} \begin{matrix} \}\, p_A\times 1 \\ \}\, p_B\times 1 \end{matrix}$, $\underset{p\times p}{T} = \begin{pmatrix} T_{AA} & T_{AB} \\ \underbrace{T_{BA}}_{p_A\times 1} & \underbrace{T_{BB}}_{p_B\times 1} \end{pmatrix} \begin{matrix} \}\, p_A\times 1 \\ \}\, p_B\times 1 \end{matrix}$

where p_A+p_B=p, then the marginal PDFs of $\underset{p_A\times 1}{s_A}$ and $\underset{p_B\times 1}{s_B}$ are

$s_A \sim t(\nu,\delta_A,\mathrm{T}_{AA})$ and $s_B \sim t(\nu,\delta_B,\mathrm{T}_{BB})$!

Bivariate Student-t PDF

Theorem: Conditional

If s is a 2-D (or p-D) random variable from $f(s|\delta,T)$ Student-t, with

$E(s|\nu,\delta,\mathrm{T}) = \underset{p\times 1}{\delta}$ and $cov(s|\nu,\delta,\mathrm{T}) = \frac{\nu}{\nu-2}\underset{p\times p}{\mathrm{T}}$ think of p=2

and partition $\underset{p\times 1}{s} = \begin{pmatrix} s_A \\ s_B \end{pmatrix} \begin{matrix} \left.\right\} p_A\times 1 \\ \left.\right\} p_B\times 1 \end{matrix}$, $\underset{p\times 1}{\delta} = \begin{pmatrix} \delta_A \\ \delta_B \end{pmatrix} \begin{matrix} \left.\right\} p_A\times 1 \\ \left.\right\} p_B\times 1 \end{matrix}$, $\underset{p\times p}{T} = \begin{pmatrix} T_{AA} & T_{AB} \\ \underbrace{T_{BA}}_{p_A\times 1} & \underbrace{T_{BB}}_{p_B\times 1} \end{pmatrix} \begin{matrix} \left.\right\} p_A\times 1 \\ \left.\right\} p_B\times 1 \end{matrix}$

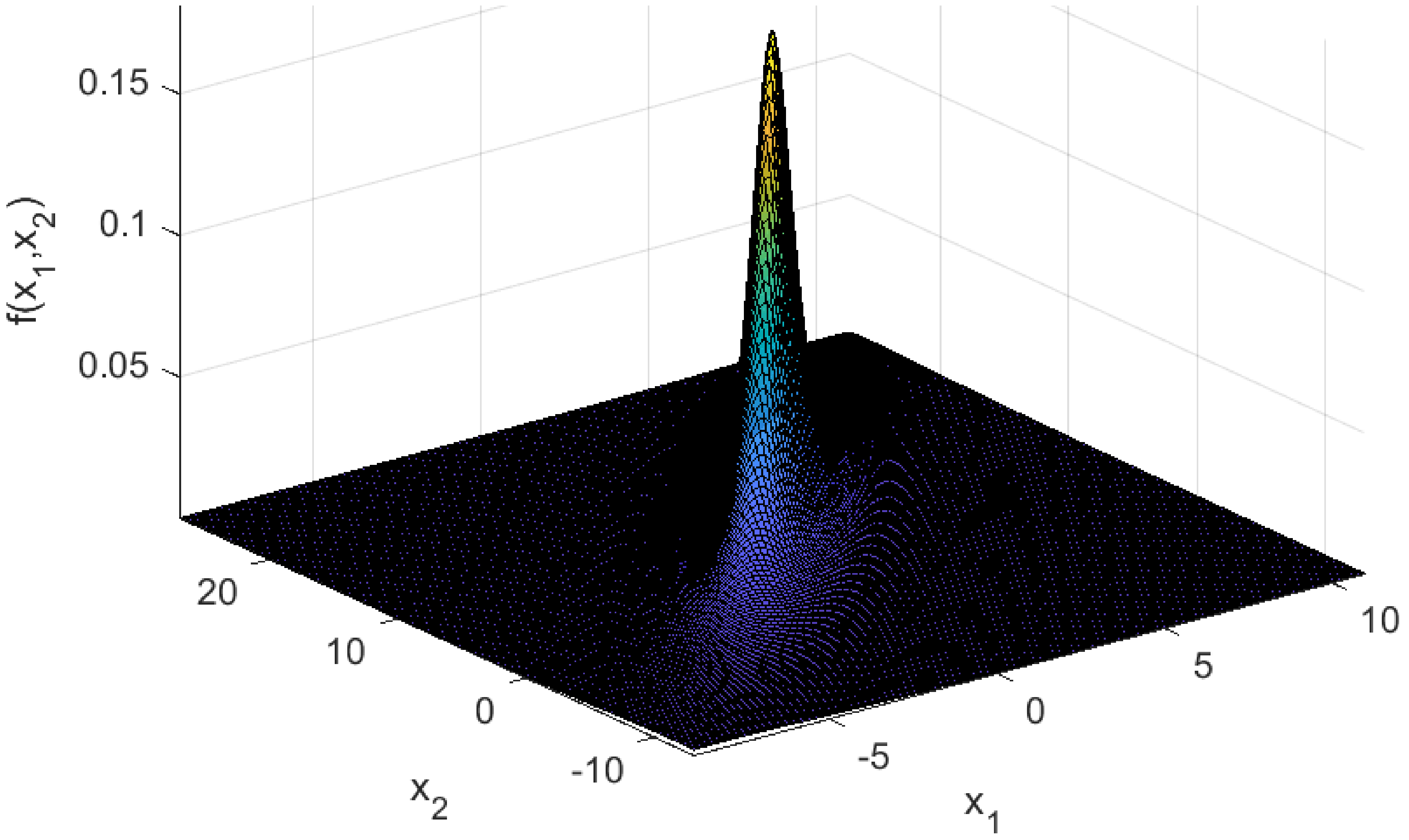
where $p_A+p_B=p$, then the conditional PDFs of $s_B|\underset{p_A\times 1}{s_A}$ is

$s_B|s_A \sim t(\underbrace{\nu+p_A}, \underbrace{\delta_B+T_{BA}T_{AA}^{-1}(s_A-\delta_A)}, \underbrace{(\nu+(s_A-\delta_A)'T_{AA}^{-1}(s_A-\delta_A)))\Lambda_{BB}/(\nu+p_A)})$

where $\Lambda_{BB} = T_{BB}-T_{BA}T_{AA}^{-1}T_{AB}$.

Bivariate Student-t PDF

Theoretical PDF.



$$\delta = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\mathrm{T} = \begin{pmatrix} 1 & 1.8 \\ 1.8 & 4 \end{pmatrix}$$

$$\nu = 5$$

$$f(s \mid \nu, \delta, \mathrm{T}) = \frac{\Gamma(\frac{\nu+p}{2}) \left|\mathrm{T}\right|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})} \frac{1}{[1+\frac{1}{\nu}(s-\delta)'\mathrm{T}^{-1}(s-\delta)]^{(\nu+p)/2}}$$

toggle forward

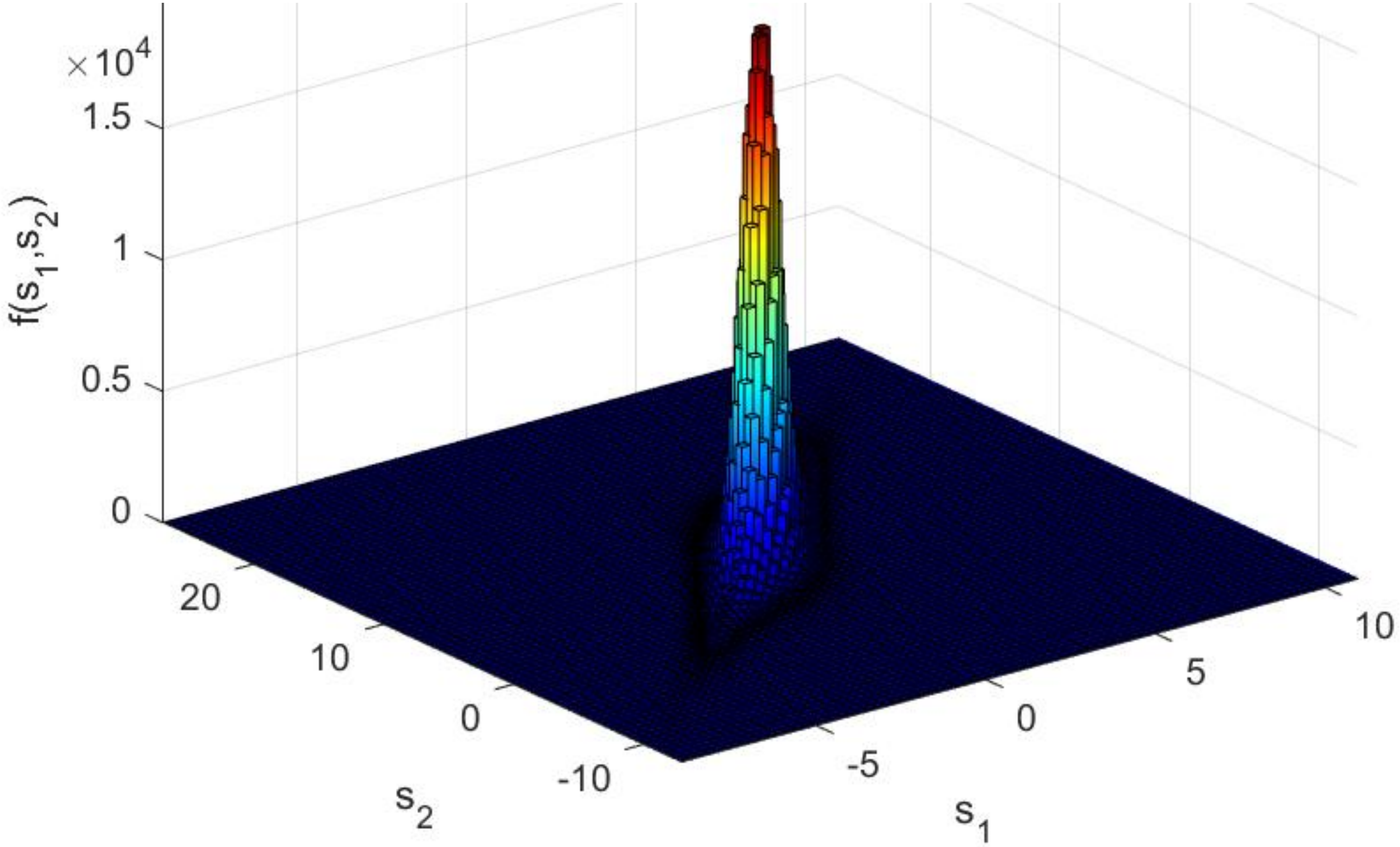
Bivariate Student-t PDF

Histogram PDF .

$$\delta = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\mathrm{T} = \begin{pmatrix} 1 & 1.8 \\ 1.8 & 4 \end{pmatrix}$$

$$\nu = 5$$



$$f(s\,|\,\nu,\delta,\mathrm{T}) = \frac{\Gamma(\frac{\nu+p}{2})|\mathrm{T}|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})} \frac{1}{[1+\frac{1}{\nu}(s-\delta)'\mathrm{T}^{-1}(s-\delta)]^{(\nu+p)/2}}$$

toggle backward

Bivariate Student-t PDF

```
p=2; nu=5; delta=[1;7]; sig11=1; sig22=4; rho=.9;
T=[sig11,sqrt(sig11*sig22)*rho;sqrt(sig11*sig22)*rho,sig22];
figure;
x=(delta(1,1)-10:.2:delta(1,1)+10);,y=(delta(2,1)-20:.2:delta(2,1)+20);
lenx=length(x); leny=length(y);
kt=gamma((nu+p)/2)/gamma((nu)/2)/(nu*pi)^(p/2);
for i=1:lenx
    for j=1:leny
      X(j,i)=x(i);
      Y(j,i)=y(j);
      Z(j,i)=(1+([x(i);y(j)]-delta)'*inv(T)*([x(i);y(j)]-delta)/nu)...
    ^((-nu+p)/2);
    end
end
Z=kt/sqrt(det(T))*Z;
surf(X,Y,Z,'FaceAlpha',0.9,'FaceColor','interp')
xlabel('x_1'),ylabel('x_2'),zlabel('f(x_1,x_2)')
axis tight
```

Bivariate Student-t PDF

```
rng default;
p=2; nu=5; delta=[1;7];
sig11=1; sig22=4; rho=.9;
T=[sig11,sqrt(sig11*sig22)*rho;sqrt(sig11*sig22)*rho,sig22];
n=10^6;
Sigma=nu*T/(nu-2); A=chol(T)';
% S = mvtrnd(T,nu,n); % generates mean=0, var=nu/(nu-2), rho corr
S = (A*trnd(nu,[2,n])+repmat(delta,[1,n]))';
figure;
hist3(S,[200,200])
colormap(jet)
xlim([delta(1,1)-10 delta(1,1)+10]),ylim([delta(2,1)-20 delta(2,1)+20])
xlabel('x_1'), ylabel('x_2'),zlabel('f(x_1,x_2)’)
```

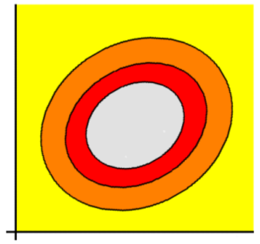
Bivariate Student-t PDF

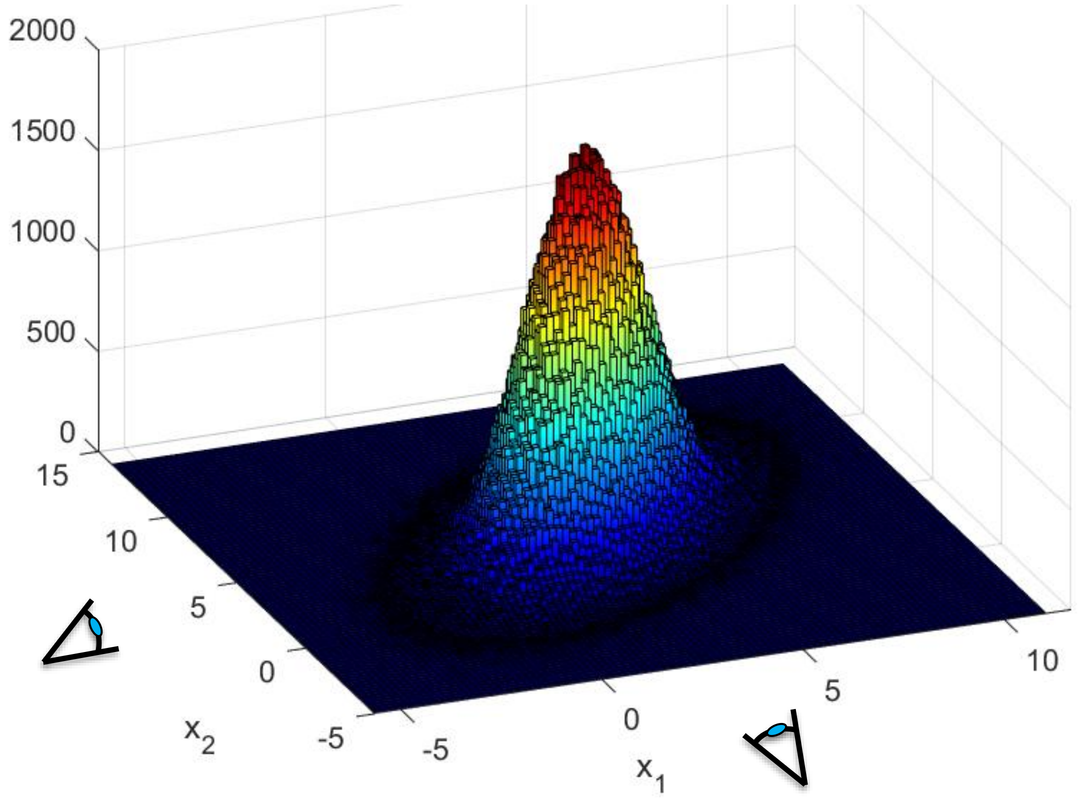
$$f(s|\nu,\delta,\mathrm{T})=\frac{\Gamma(\frac{\nu+p}{2})|\mathrm{T}|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})}\frac{1}{[1+\frac{1}{\nu}(s-\delta)'\mathrm{T}^{-1}(s-\delta)]^{(\nu+p)/2}}$$

$$\underset{2\times 1}{\delta}=\begin{pmatrix}1\\7\end{pmatrix}\qquad \underset{2\times 2}{\mathrm{T}}=\begin{pmatrix}1 & 1.8\\1.8 & 4\end{pmatrix}$$

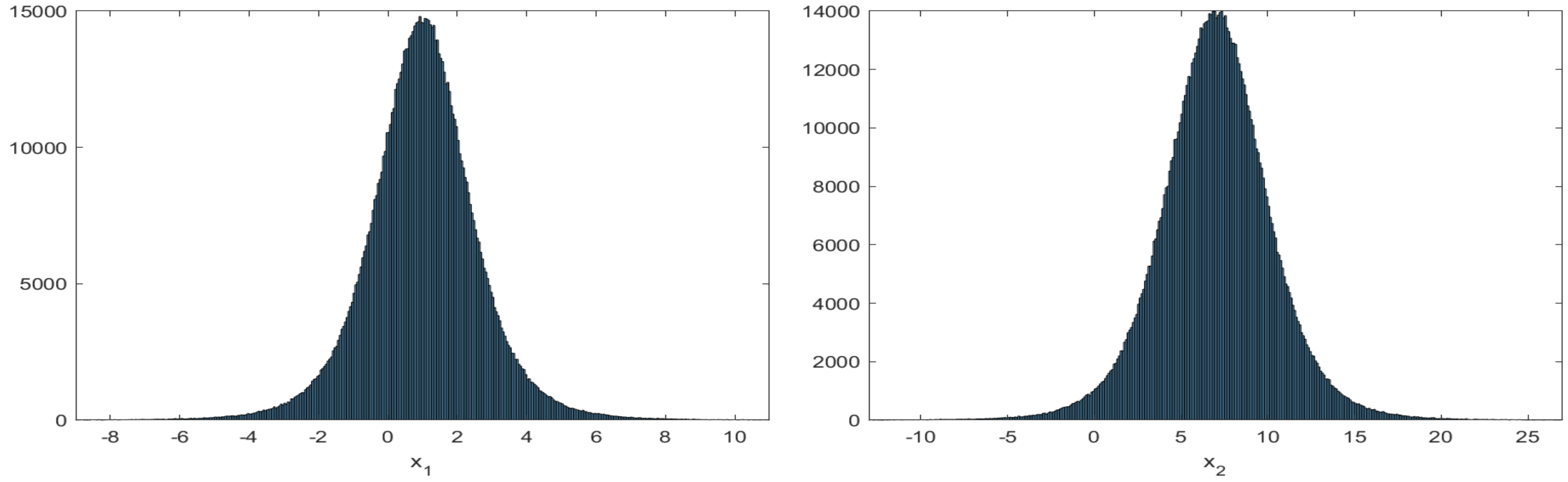
$$\nu=5$$

cross
section →





marginals are Student-t



Bivariate Student-t PDF

```
figure;
histogram(S(:,1))
xlim([delta(1,1)-10 delta(1,1)+10])
xlabel('x_1')

figure;
histogram(S(:,2))
xlim([delta(2,1)-20 delta(2,1)+20])
xlabel('x_2')
```

Bivariate Student-t PDF

From the bivariate Student-t PDF,

$$f(s|\nu,\delta,\mathrm{T})=\frac{\Gamma(\frac{\nu+p}{2})|\mathrm{T}|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})}\frac{1}{[1+\frac{1}{\nu}(s-\delta)'\mathrm{T}^{-1}(s-\delta)]^{(\nu+p)/2}}$$

The marginal Student-t PDFs of s_1 and s_2 can be shown to be

$$f(s_i|\nu,\delta,\tau^2)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{(T_{ii})^{-1/2}}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}\left(\frac{s_i-\delta_i}{T_{ii}}\right)^2\right)^{(\nu+1)/2}}$$

$$i=1,2$$

$$-\infty<s_i<\infty$$
$$-\infty<\delta_i<\infty$$
$$T_{ii}>0$$

with marginal mean δ_i and variance $vT_{ii}/(v\text{-}2)$.

The conditionals are also Student-t.

Bivariate N-IG Conditional & Marginal PDFs

Since $f(x_1,x_2|\theta) = f(x_2|x_1,\theta)\ f(x_1|\theta)$, we can assemble a Bivariate distribution by specifying the conditional PDF of x_2 given x_1 and the marginal PDF of x_1.

This is useful when we have observations that come from a PDF with two parameters and wish to specify a bivariate prior PDF for the parameters.

Bivariate N-IG Conditional & Marginal PDFs

Later we will specify a bivariate PDF by specifying that

the conditional PDF of x_2 given x_1 is **normal**, and

the marginal PDF of x_1 is **inverse gamma**.

$$\underbrace{f(x_1,x_2\,|\,\theta)}_{\text{joint}} = \underbrace{f(x_2\,|\,x_1,\theta)}_{\text{conditional}}\underbrace{f(x_1\,|\,\theta)}_{\text{marginal}}$$

$$\underbrace{f(x_2\,|\,x_1,\mu)}_{\text{conditional}} = \frac{e^{-\frac{(x_2-\mu)^2}{2x_1}}}{\sqrt{2\pi x_1}} \qquad x_2,\mu\in\mathbb{R} \quad x_1\in\mathbb{R}^+$$

$$\underbrace{f(x_1\,|\,\alpha,\beta)}_{\text{marginal}} = \frac{\beta^\alpha}{\Gamma(\alpha)}x_1^{-\alpha-1}\,\mathrm{e}^{-\beta/x_1} \qquad \alpha,\beta\in\mathbb{R}^+$$

Bivariate N-IG Conditional & Marginal PDFs

Later we will specify a bivariate PDF by specifying that

the conditional PDF of x_2 given x_1 is **normal**, and

the marginal PDF of x_1 is **inverse gamma**.

$$\underbrace{f(x_1,x_2|\theta)}_{\text{joint}} = \underbrace{f(x_2|x_1,\theta)}_{\text{conditional}} \underbrace{f(x_1|\theta)}_{\text{marginal}}$$

$$\underbrace{f(x_1,x_2|\theta)}_{\text{joint}} = \underbrace{\frac{e^{-\frac{(x_2-\mu)^2}{2x_1}}}{\sqrt{2\pi x_1}}}_{\text{conditional}} \cdot \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} x_1^{-\alpha-1} \mathrm{e}^{-\beta/x_1}}_{\text{marginal}}$$

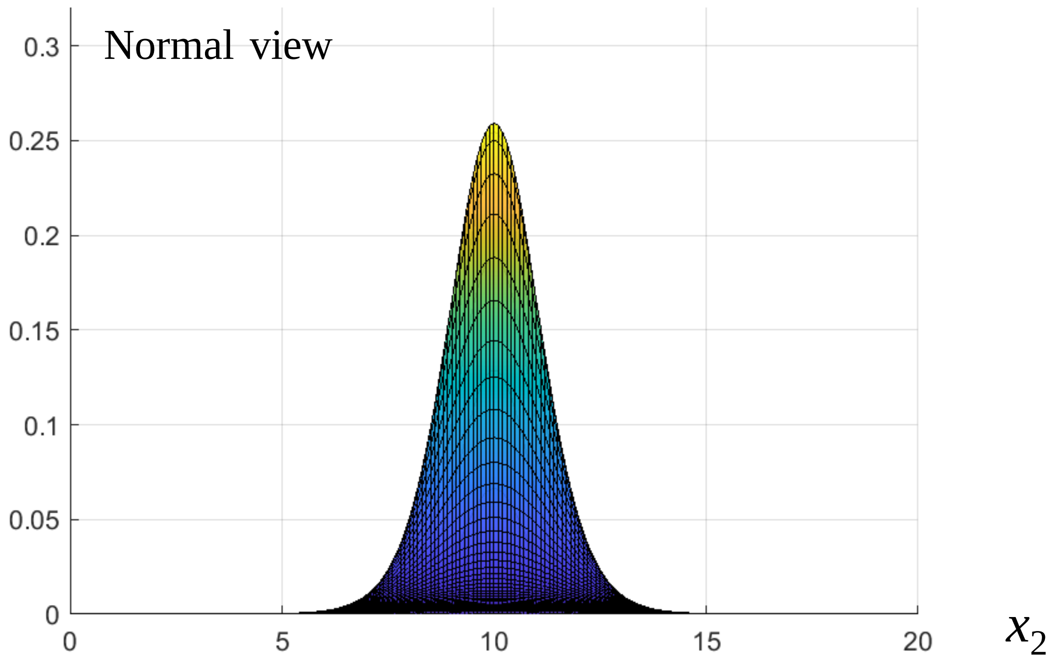
$$x_2, \mu \in \mathbb{R}$$

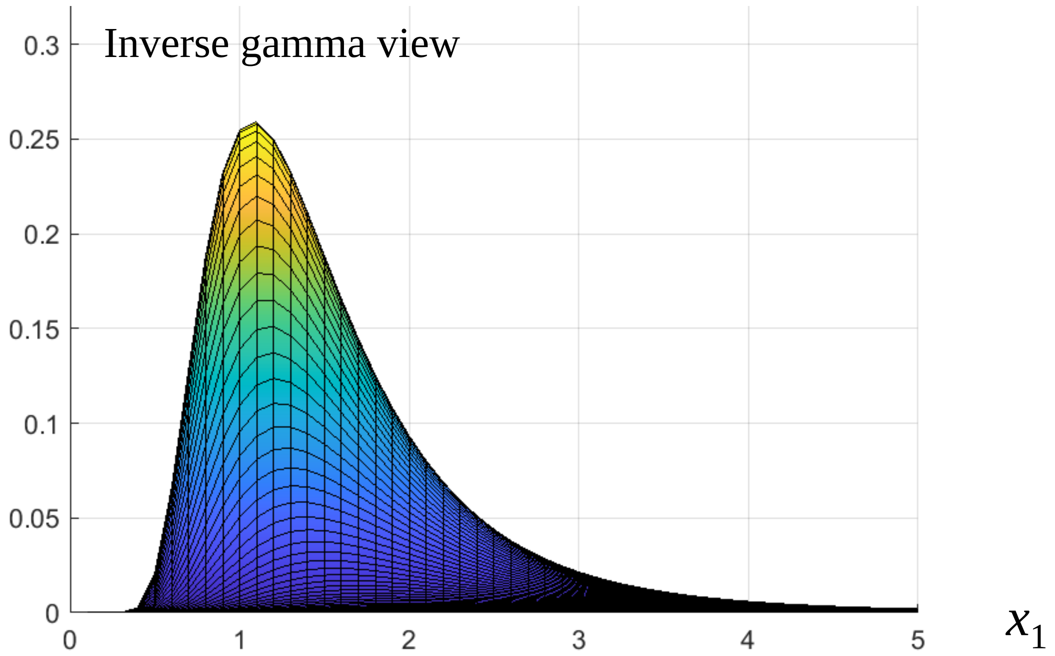
$$x_1 \in \mathbb{R}^+$$

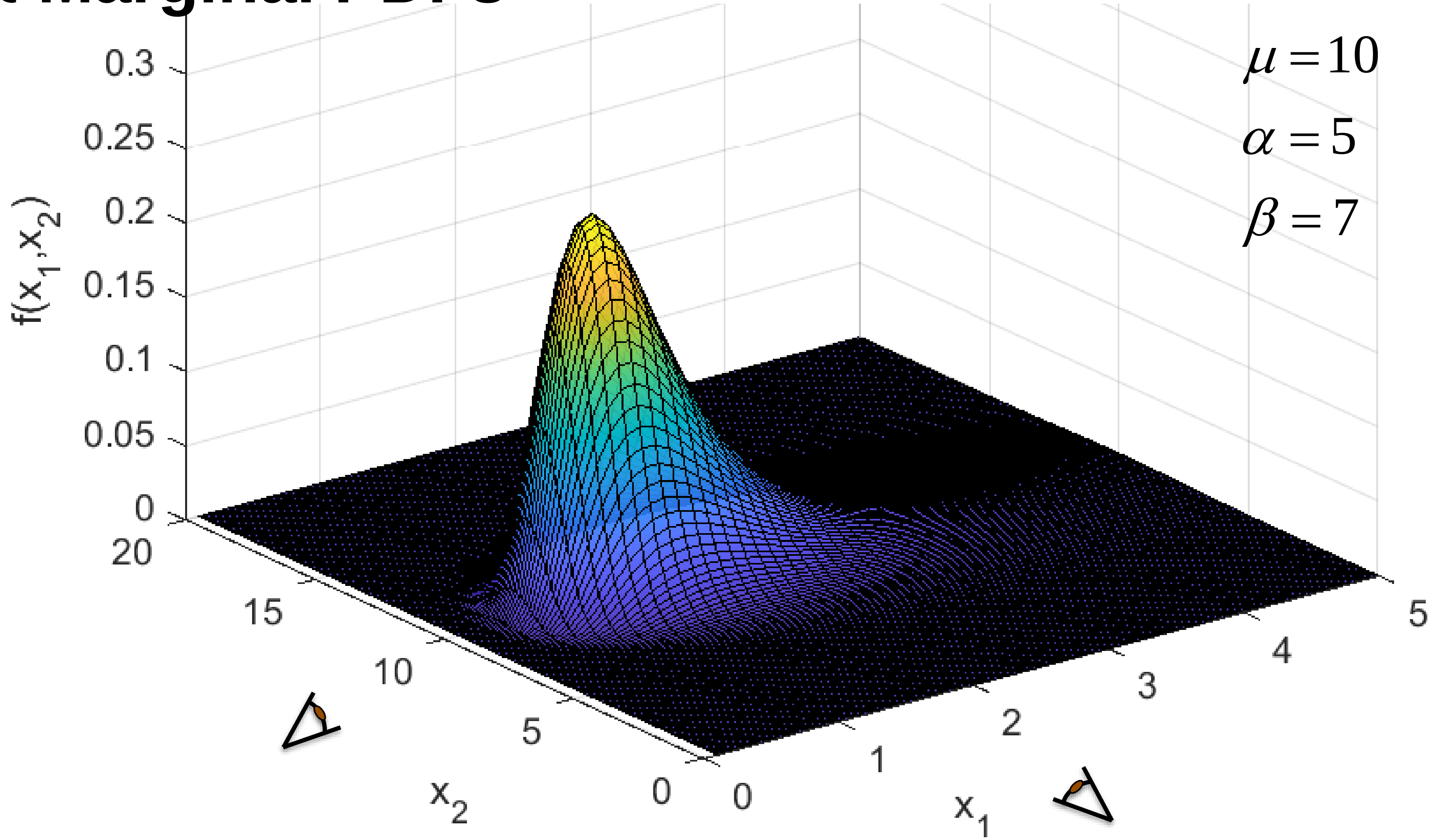
$$\underbrace{f(x_1,x_2|\theta)}_{\text{joint}} = \frac{\beta^\alpha}{\sqrt{2\pi}\,\Gamma(\alpha)} x_1^{-\alpha-3/2} e^{-\frac{(x_2-\mu)^2+2\beta}{2x_1}}$$

$$\alpha, \beta \in \mathbb{R}^+$$

Bivariate N-IG Conditional & Marginal PDFs







$$\mu = 10$$

$$\alpha = 5$$

$$\beta = 7$$

$$\underset{\text{joint}}{f(x_1, x_2 \mid \theta)} = \frac{\beta^{\alpha}}{\sqrt{2\pi}\Gamma(\alpha)} x_1^{-\alpha-3/2} e^{-\frac{(x_2-\mu)^2+2\beta}{2x_1}}$$

Bivariate N-IG Conditional & Marginal PDFs

```
mu=10; alpha=5; beta=7;
figure;
x=(0:.1:5);, y=(0:.1:20);
lenx=length(x); leny=length(y);
k=beta^alpha/(sqrt(2*pi)*gamma(alpha));
for i=1:lenx
    for j=1:leny
        X(j,i)=x(i);,
        Y(j,i)=y(j);
        Z(j,i)=x(i)^(-alpha-3/2)*exp(-((y(j)-mu)^2+2*beta)/2/x(i));
    end
end
Z=k*Z;
surf(X,Y,Z,'FaceAlpha',0.9,'FaceColor','interp')
xlabel('x_1'),ylabel('x_2'),zlabel('f(x_1,x_2)')
xlim([0,5]),ylim([0,20]), zlim([0,.35])
az=90; el=0; view(az,el) %normal view
az=0; el=0; view(az,el)  %inverse gamma view
```

Bivariate L-IG Conditional & Marginal PDFs

Later we will specify a bivariate PDF by specifying that

the conditional PDF of x_2 given x_1 is **Laplace**, and

the marginal PDF of x_1 is **inverse gamma**.

$$\underbrace{f(x_1,x_2|\theta)}_{\text{joint}} = \underbrace{f(x_2|x_1,\theta)}_{\text{conditional}} \underbrace{f(x_1|\theta)}_{\text{marginal}}$$

$$E(x_2|x_1,\mu)=\mu$$

$$\mathrm{var}(x_2|x_1,\mu)=2x_1^2$$

$$\underbrace{f(x_2|x_1,\mu)}_{\text{conditional}} = \frac{1}{2x_1}\mathrm{e}^{-\frac{|x_2-\mu|}{x_1}}$$

$$x_2,\mu\in\mathbb{R}$$

$$x_1\in\mathbb{R}^+$$

$$\underbrace{f(x_1|\alpha,\beta)}_{\text{marginal}} = \frac{\beta^\alpha}{\Gamma(\alpha)}x_1^{-\alpha-1}\mathrm{e}^{-\beta/x_1}$$

$$\alpha,\beta\in\mathbb{R}^+$$

Bivariate L-IG Conditional & Marginal PDFs

Later we will specify a bivariate PDF by specifying that

the conditional PDF of x_2 given x_1 is **Laplace**, and

the marginal PDF of x_1 is **inverse gamma**.

$$\underbrace{f(x_1,x_2\,|\,\theta)}_{\text{joint}} = \underbrace{f(x_2\,|\,x_1,\theta)}_{\text{conditional}}\underbrace{f(x_1\,|\,\theta)}_{\text{marginal}}$$

$$\underbrace{f(x_1,x_2\,|\,\theta)}_{\text{joint}} = \underbrace{\frac{1}{2x_1}\mathrm{e}^{-\frac{|x_2-\mu|}{x_1}}}_{\text{conditional}} \cdot \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}x_1^{-\alpha-1}\mathrm{e}^{-\beta/x_1}}_{\text{marginal}}$$

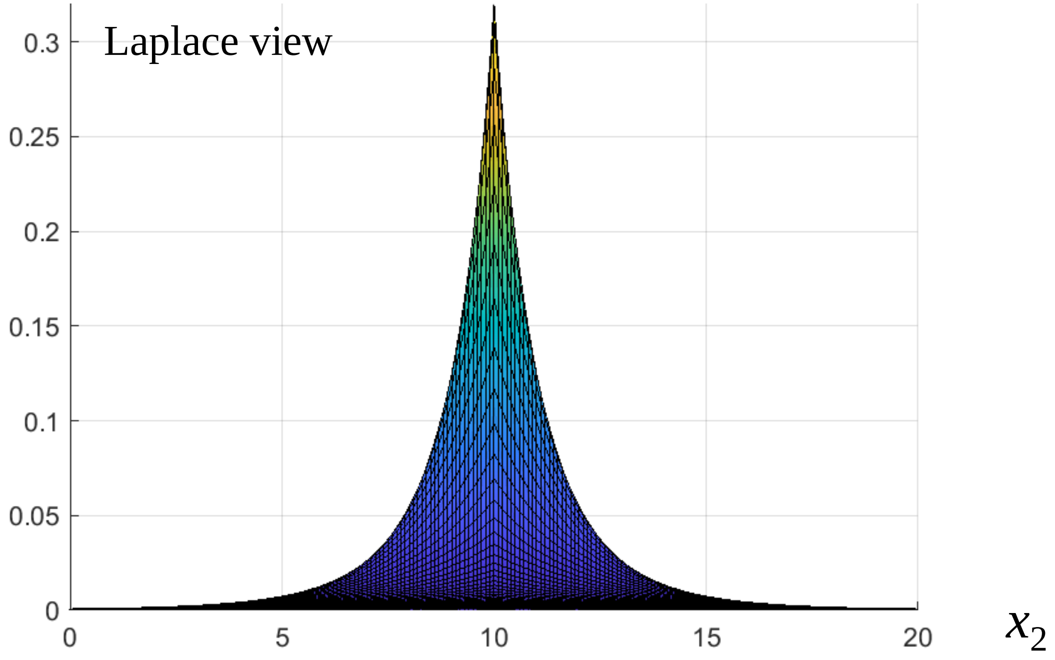
$$x_2,\mu \in \mathbb{R}$$

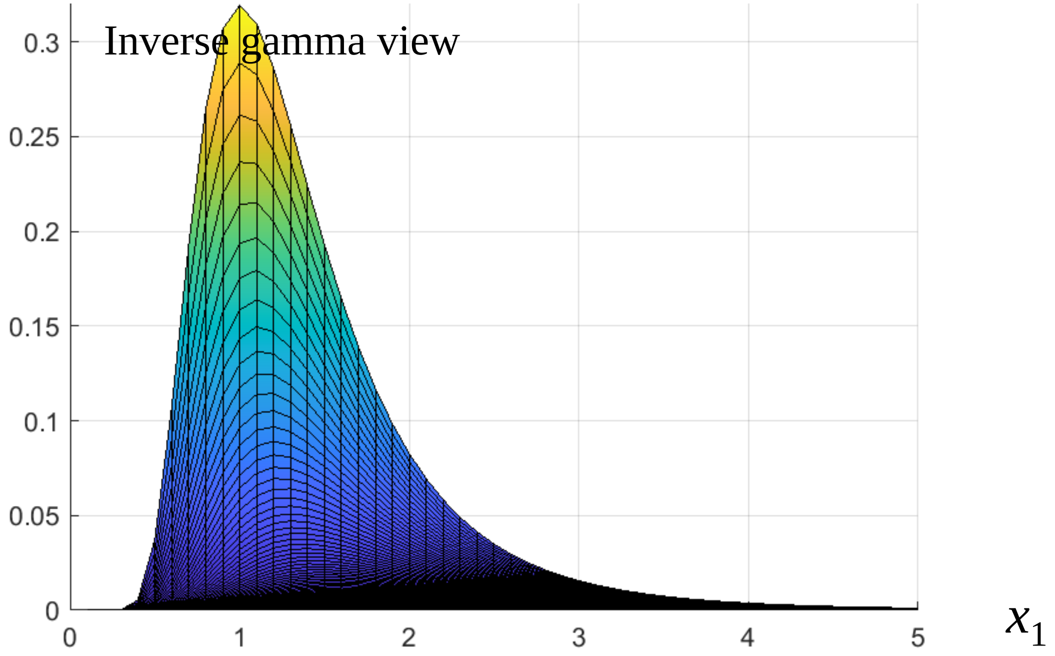
$$x_1 \in \mathbb{R}^+$$

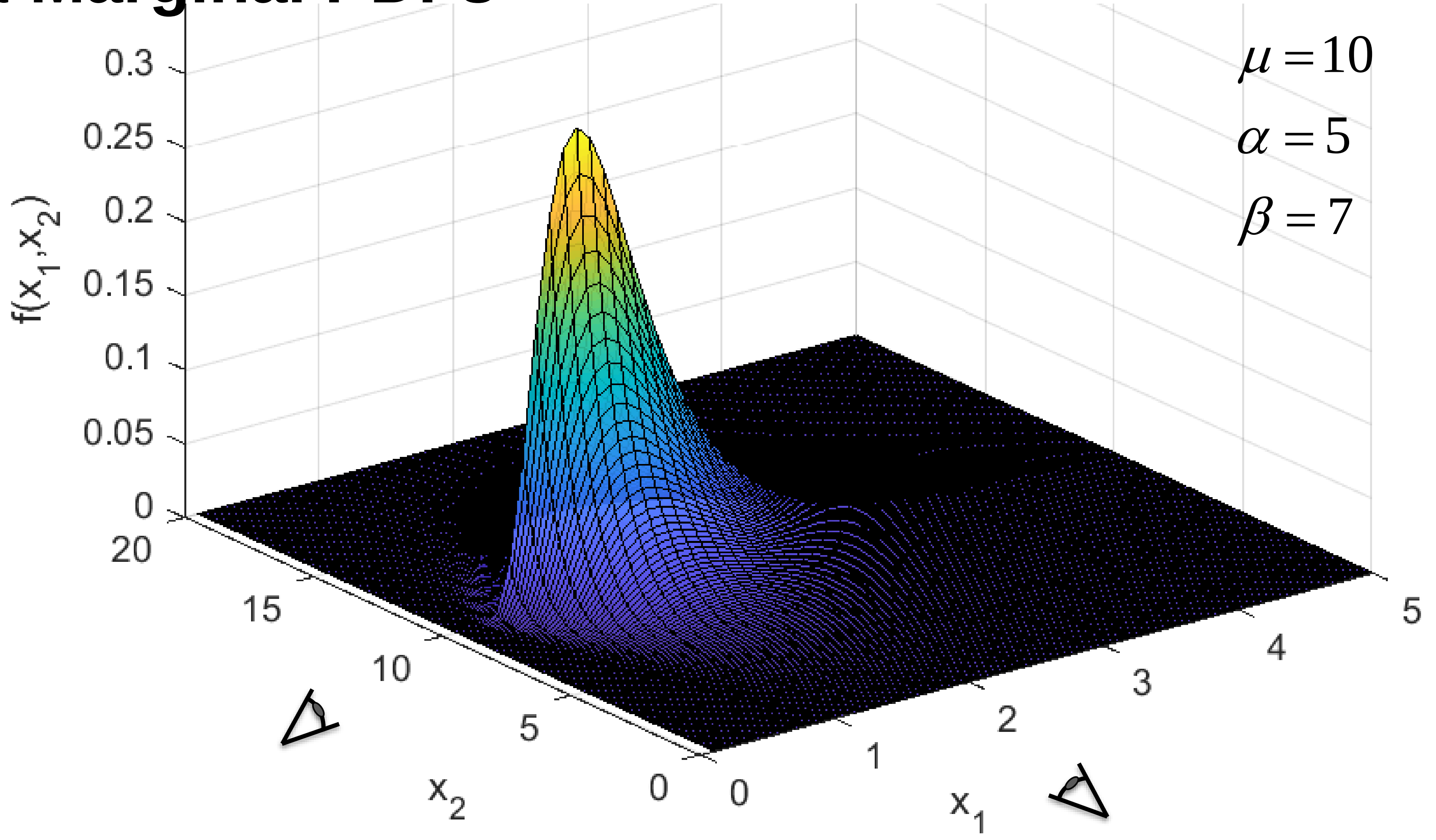
$$\underbrace{f(x_1,x_2\,|\,\theta)}_{\text{joint}} = \frac{\beta^\alpha}{2\Gamma(\alpha)}x_1^{-\alpha-2}e^{-\frac{|x_2-\mu|+\beta}{x_1}}$$

$$\alpha,\beta \in \mathbb{R}^+$$

Bivariate L-IG Conditional & Marginal PDFs







$\mu = 10$

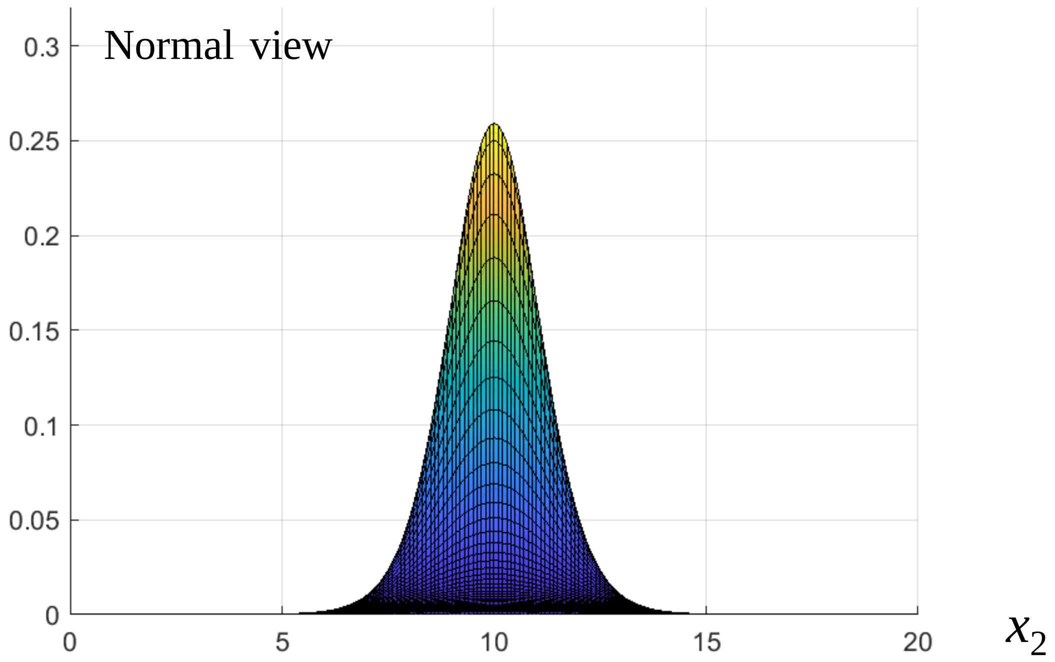
$\alpha = 5$

$\beta = 7$

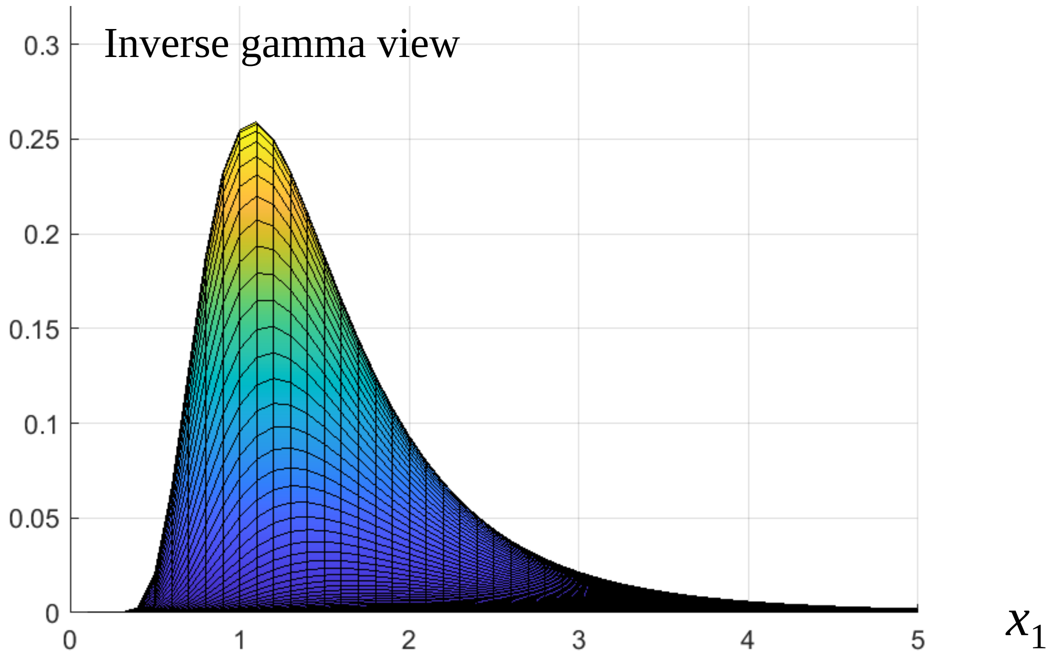
$$\underset{\text{joint}}{f(x_1, x_2 \mid \theta)} = \frac{\beta^{\alpha}}{2\Gamma(\alpha)} x_1^{-\alpha-2} e^{-\frac{|x_2-\mu|+\beta}{x_1}}$$

toggle forward

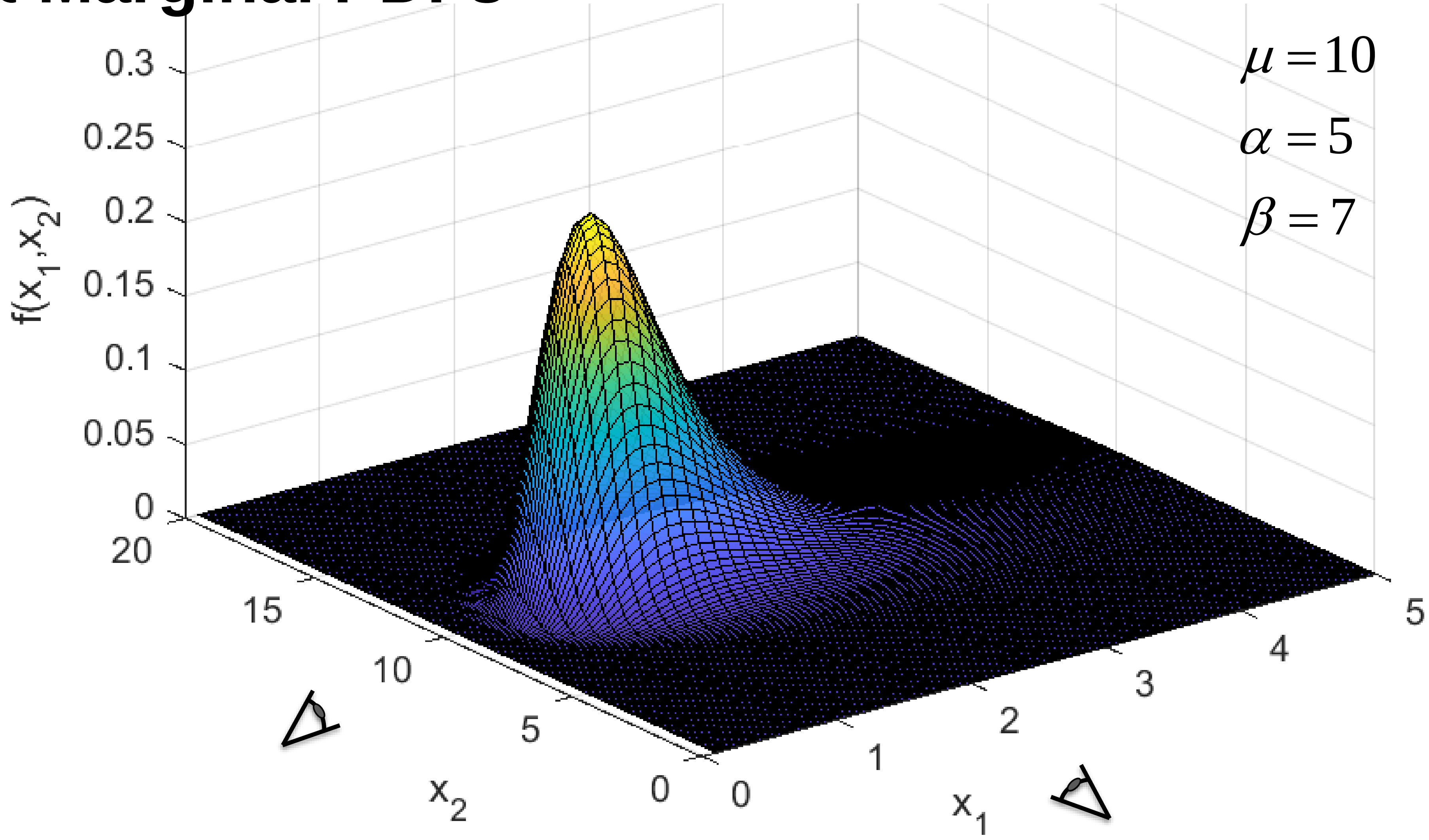
Bivariate N-IG Conditional & Marginal PDFs



Normal view



Inverse gamma view



$$\mu = 10$$

$$\alpha = 5$$

$$\beta = 7$$

$$\underset{\text{joint}}{f(x_1, x_2 \,|\, \theta)} = \frac{\beta^{\alpha}}{\sqrt{2\pi}\Gamma(\alpha)} x_1^{-\alpha-3/2} e^{-\frac{(x_2-\mu)^2+2\beta}{2x_1}}$$

toggle backward

Bivariate N-IG Conditional & Marginal PDFs

```
mu=10; alpha=5; beta=7;
figure;
x=(0:.1:5);, y=(0:.1:20);
lenx=length(x); leny=length(y);
k=beta^alpha/(2*gamma(alpha));
for i=1:lenx
    for j=1:leny
        X(j,i)=x(i);
        Y(j,i)=y(j);
        Z(j,i)=x(i)^(-alpha-2)*exp(-(abs(y(j)-mu)+beta)/x(i));
    end
end
Z=k*Z;
surf(X,Y,Z,'FaceAlpha',0.9,'FaceColor','interp')
xlabel('x_1'),ylabel('x_2'),zlabel('f(x_1,x_2)')
xlim([0,5]),ylim([0,20]), zlim([0,.35])
az=90; el=0; view(az,el) %Laplace view
az=0; el=0; view(az,el)  %inverse gamma view
```

Discussion

Questions?

Homework 5

1. Assume Marquette Undergrads heights have

μ_h=67 in, μ_w=150 lbs, and T=[4, 6;6,16] and Σ=$\nu T/(\nu$-2) with ν=7.

a) Generate 10^4 h-w 2×1 observations from the bivariate Student-t PDF.

b) Calculate the sample means, variances, covariance, and correlation.

c) Make an (x_1,x_2) bivariate histogram and marginal histograms for x_1 and x_2. (For marginal just ignore other.)

d*) For an interval of heights $69\ in \pm 0.5\ in$ make a histogram. Compare to Homework 4 #1e*

* For students in 5790.

Homework 5

2*. Derive with pencil and paper the marginal PDF $f(s_1|\nu,\delta,T)$ of s_1 from the bivariate Student-t PDF $f(s_1,s_2|\nu,\delta,T)$.
Compare to simulated observation based marginal in #1.

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

3**. Present work to derive with pencil and paper the conditional PDF of s_2 given s_1,ν,δ,T from the bivariate Student-t PDF $f(s_1,s_2|\nu,\delta,T)$. Compare to simulated observation based conditional in 1d*.

$$\mathrm{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}$$

$$f(s_1,s_2|\nu,\delta,\mathrm{T}) = \frac{\Gamma(\frac{\nu+p}{2})|\mathrm{T}|^{-1/2}}{(\nu\pi)^{p/2}\Gamma(\frac{\nu}{2})} \frac{1}{[1+\frac{1}{\nu}(s-\delta)'\mathrm{T}^{-1}(s-\delta)]^{(\nu+p)/2}}$$

* For students in MSSC 5790.

** For students that have had MSSC 6010 and 6020.

Homework 5

$$\mu = 10$$
$$\alpha = 5$$
$$\beta = 7$$

4**. Assume μ=10, α=5, β=7.

$$f(x_2 \mid x_1, \mu) = \frac{1}{\sqrt{2\pi x_1}} e^{-\frac{(x_2-\mu)^2}{2x_1}} \qquad f(x_1 \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x_1^{-\alpha-1} \mathrm{e}^{-\beta/x_1}$$

a) Generate 10^4 x_1 observations from the inverse gamma PDF.

b) Given each x_1 in a), generate an x_2 observation from the normal pdf.

c) Make an (x_1,x_2) bivariate histogram and marginal histograms for x_1 and x_2. (For marginal just ignore other.)

d) Compare to normal-inverse gamma surface plot.

** For students that have had MSSC 6010 and 6020.

Homework 5

5*. Derive with pencil and paper the marginal PDF $f(x_2|\mu,\alpha,\beta)$ of x_2 from the bivariate normal-inverse gamma PDF $f(x_1,x_2|\mu,\alpha,\beta)$.
Compare to simulated observation based marginal in #4.

6**. Present work to derive with pencil and paper the conditional PDF of x_2 given x_1,μ,α,β from the bivariate normal-inverse gamma PDF $f(x_1,x_2|\mu,\alpha,\beta)$. Does this agree with your simulation? Justify.

$$f(x_1,x_2\,|\,\theta)=\frac{\beta^{\alpha}}{\sqrt{2\pi}\Gamma(\alpha)}x_1^{-\alpha-3/2}e^{-\frac{(x_2-\mu)^2+2\beta}{2x_1}}$$

$\mu=10$

$\alpha=5$

$\beta=7$

* For students in MSSC 5790.

** For students that have had MSSC 6010 and 6020.

Homework 5

7***. Repeat # 4., 5., and 6. for Laplace-inverse gamma pdfs.

*** For students that want to show off.

$$\mu = 10$$
$$\alpha = 5$$
$$\beta = 7$$

$$f(x_2 \mid x_1, \mu) = \frac{1}{2x_1} e^{-\frac{|x_2 - \mu|}{x_1}}$$

$$f(x_2 \mid \mu, \alpha, \beta) = \frac{1}{2} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \frac{1}{[\beta + | x_2 - \mu |]^{\alpha+1}}$$

$$f(x_1 \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x_1^{-\alpha-1} e^{-\beta / x_1}$$