Bivariate Probability Density Functions

Dr. Daniel B. Rowe
Professor of Computational Statistics
Department of Mathematical and Statistical Sciences
Marquette University

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Motivation

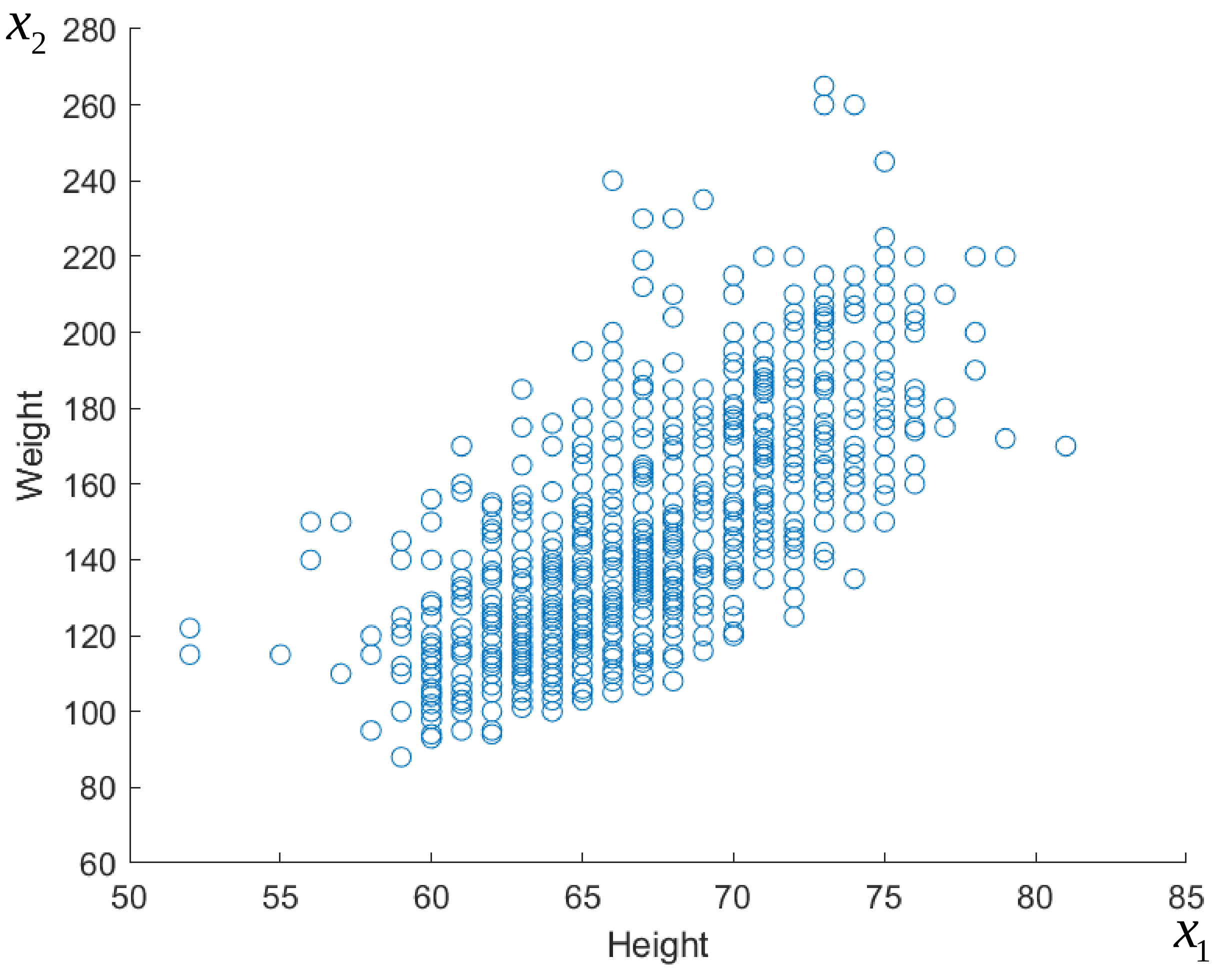
Dr. Rowe kindly asked his previous introductory statistics classes to voluntarily provide their

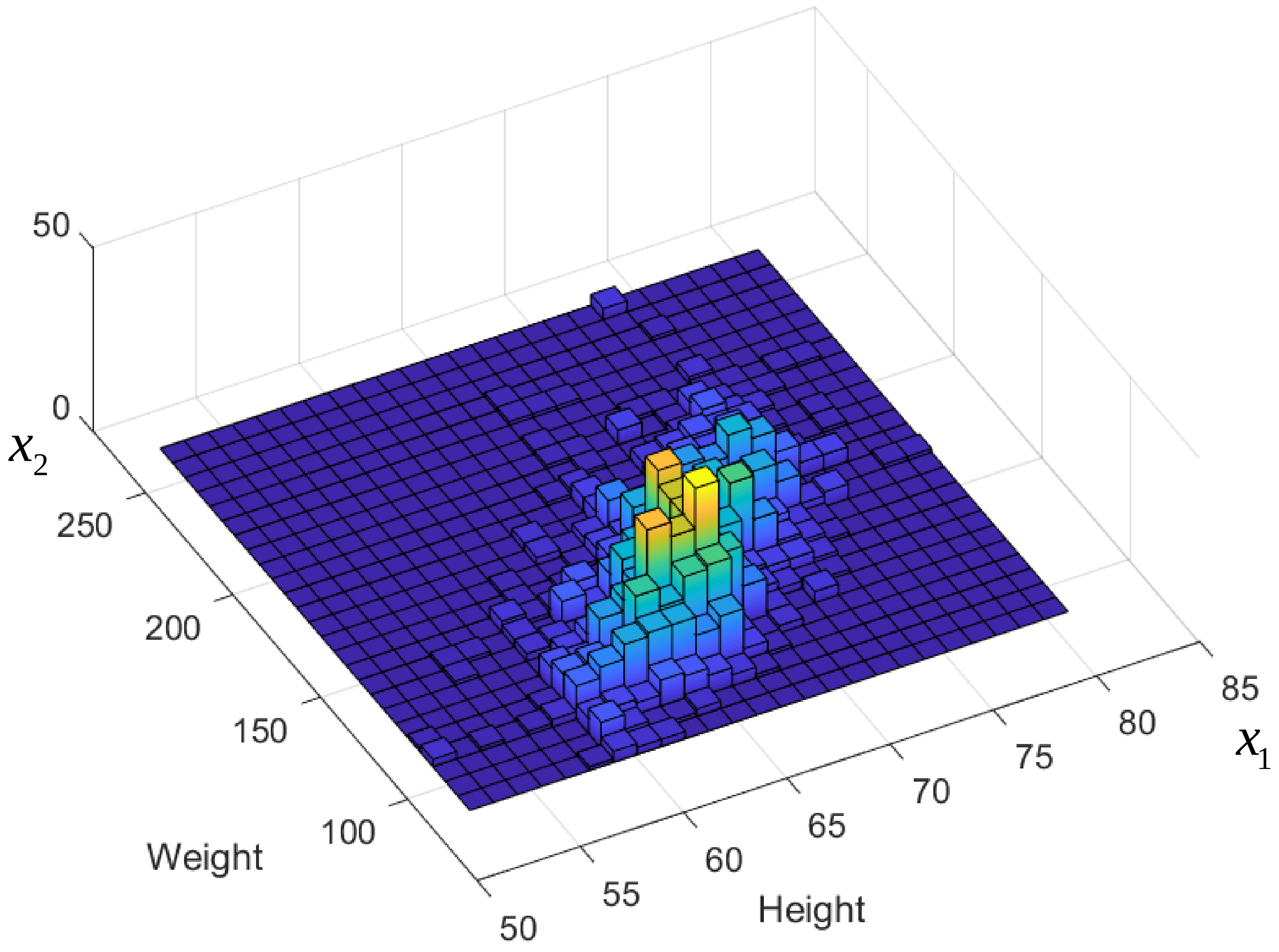
x_1=height
x_2=weight
x_3=birth gender.

This was done for 16 classes.

There were n=1366 usable data points.

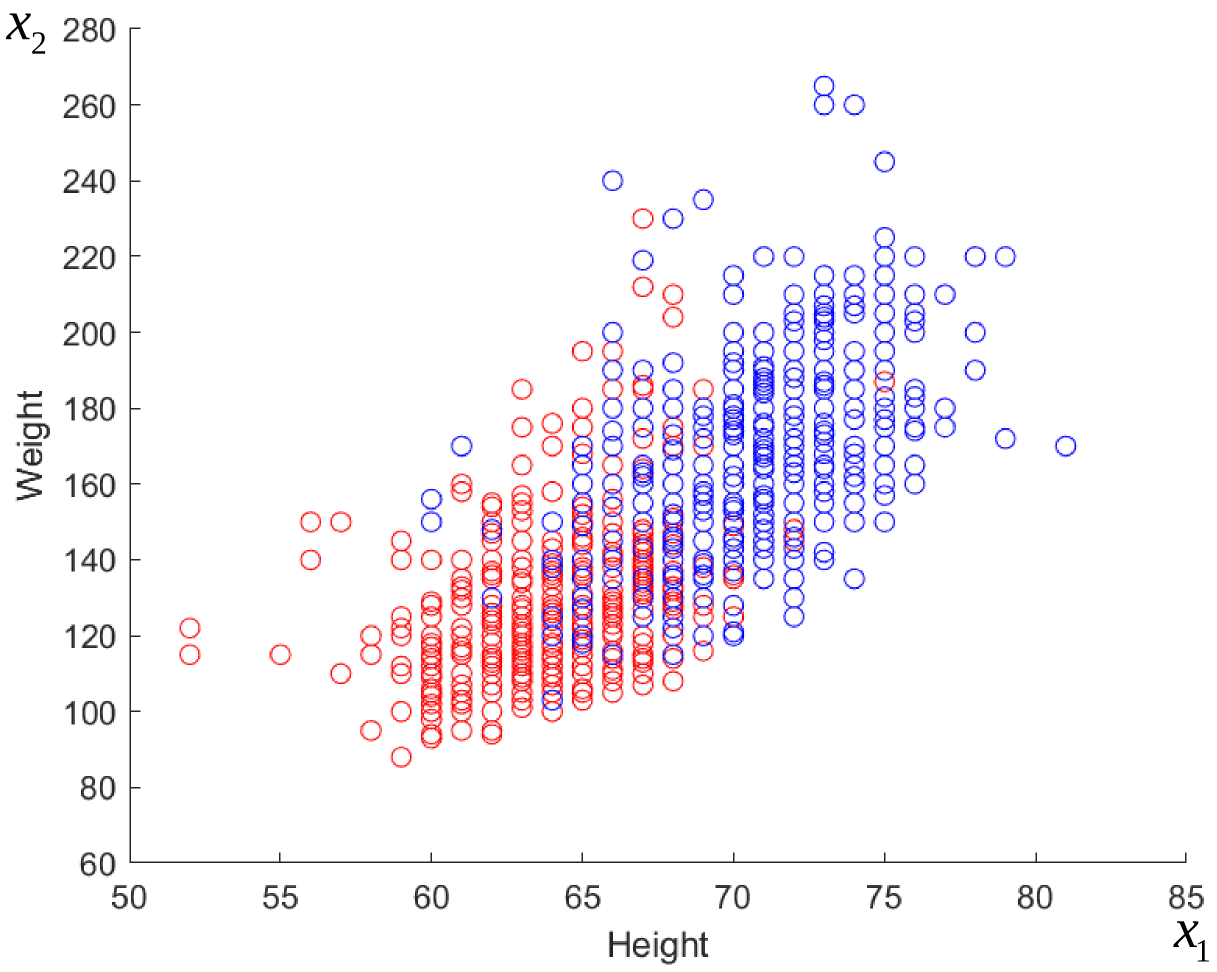
Motivation

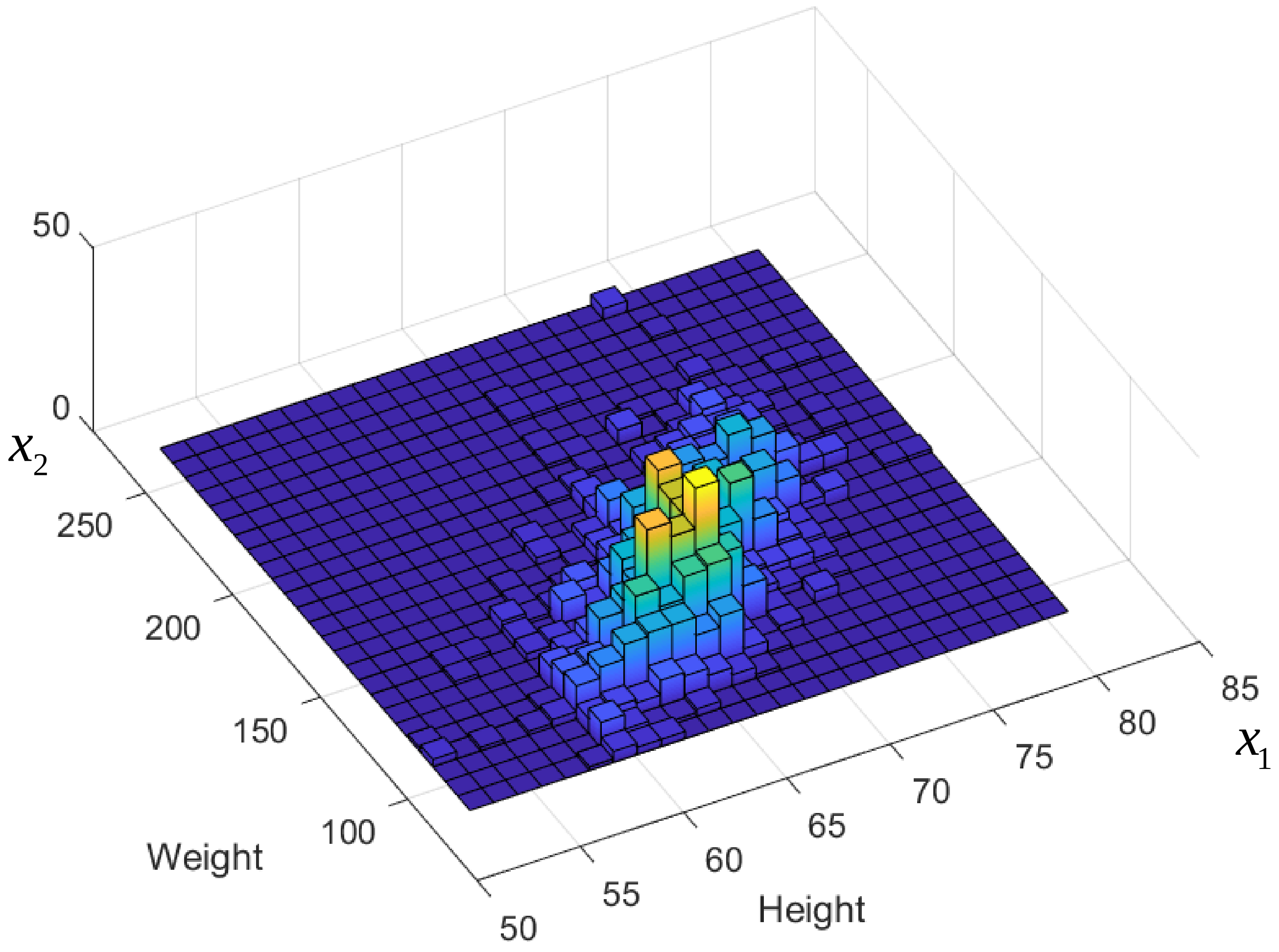




$$\underset{2\times 1}{x} = \begin{pmatrix} height \\ weight \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Motivation





○ male

○ female

$$\underset{2\times 1}{x} = \begin{pmatrix} height \\ weight \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Properties and Summaries

Assume that the 2D continuous random variable (RV) $x=(x_1,x_2)'$ can take on values

$x_1 \in (a_1,b_1) \quad x_2 \in (a_2,b_2)$

then, the probability density function (PDF) is given by

$f(x|\theta)$ defined for $x_1 \in (a_1,b_1) \quad x_2 \in (a_2,b_2)$

where x can be defined within a 2D infinite interval

and θ are any parameters that the PDF depends on.

Properties and Summaries

A bivariate (2D) PDF

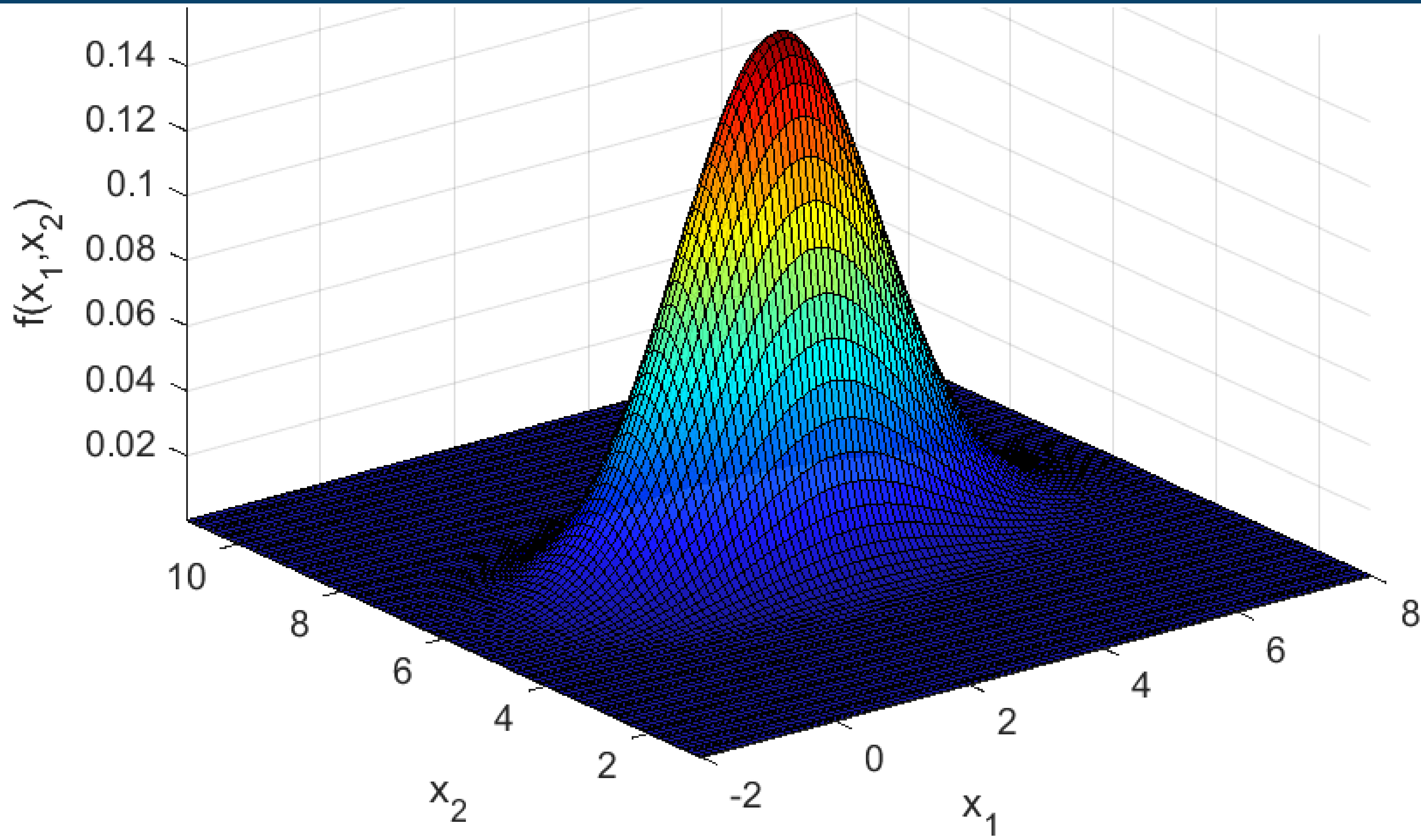
of two continuous random

variables (x_1,x_2) depending

upon parameters θ satisfies

1) $0 \leq f(x|\theta), \forall(x_1,x_2)$

2) $\iint_{x_1 x_2} f(x_1,x_2|\theta)dx_1 dx_2 = 1$.



Given θ, we completely know $f(x_1,x_2|\theta)$.

Properties and Summaries

Let $\underset{2\times 1}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a 2-dimensional (or p-dimensional)

random variable with PDF of $\underset{2\times 1}{x}$ being $f(x|\theta)$, then

$$E(x|\theta) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underset{2\times 1}{\mu} \qquad \operatorname{cov}(x|\theta) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \underset{2\times 2}{\Sigma}$$

which come from marginal PDFs.

Properties and Summaries

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The marginal PDFs of x_1 and x_2 are

$$E(x|\theta) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$f(x_1|\theta) = \int_{x_2} f(x_1, x_2|\theta) dx_2$$

$$f(x_2|\theta) = \int_{x_1} f(x_1, x_2|\theta) dx_1$$

With marginal means

$$\mu_1 = \int_{x_1} x_1 f(x_1|\theta) dx_1$$

$$\mu_2 = \int_{x_2} x_2 f(x_2|\theta) dx_2$$

Properties and Summaries

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

And with marginal variances and covariance

$$\mathrm{cov}(x|\theta) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$\sigma_1^2 = \int_{x_1} (x_1 - \mu_1)^2 \int_{x_2} f(x_1, x_2 | \theta) dx_2 dx_1 = \int_{x_1} (x_1 - \mu_1)^2 f(x_1 | \theta) dx_1$$

$$\sigma_2^2 = \int_{x_2} (x_2 - \mu_2)^2 \int_{x_1} f(x_1, x_2 | \theta) dx_1 dx_2 = \int_{x_2} (x_2 - \mu_2)^2 f(x_2 | \theta) dx_2$$

$$\sigma_{12} = \int_{x_1} \int_{x_2} (x_1 - \mu_1)(x_2 - \mu_2) f(x_1, x_2 | \theta) dx_1 dx_2$$

Properties and Summaries

Conditional PDFs can also be found

$$f(x_1 \mid x_2, \theta) = \frac{f(x_1, x_2 \mid \theta)}{f(x_2 \mid \theta)}$$

conditional PDFs

marginal PDF

joint PDFs

$$f(x_2 \mid x_1, \theta) = \frac{f(x_1, x_2 \mid \theta)}{f(x_1 \mid \theta)}$$

marginal PDF

And since they are PDFs, conditional summary measures can also be found.

Bivariate Normal PDFs

The bivariate normal PDF can be written as

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}Q\right]$$

$$Q = \frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$\sigma_1, \sigma_2 > 0, \ -1 < \rho < 1 \qquad -\infty < x_1, x_2, \mu_1, \mu_2 < \infty$$

$$\rho = \sigma_{12} / (\sigma_1\sigma_2) \qquad \sigma_{12} = \mathrm{cov}(x_1, x_2)$$

Bivariate Normal PDFs

$$\sigma_1 > 0,\ \ \sigma_2 > 0$$

If there is no correlation between x_1 and x_2, ρ=0 then

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right] \right\}$$

$$Q = \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]$$

$$-\infty < x_1, x_2, \mu_1, \mu_2 < \infty$$

$$\sigma_1, \sigma_2 > 0$$

Bivariate Normal PDFs

$$\sigma_1 > 0,\ \ \sigma_2 > 0$$

If there is no correlation between x_1 and x_2, ρ=0 then

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right\}$$

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} \right] \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[-\frac{(x_2-\mu_2)^2}{2\sigma_2^2} \right]$$

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = f_{X_1}(x_1 \mid \mu_1, \sigma_1^2)\, f_{X_2}(x_2 \mid \mu_2, \sigma_2^2)$$

Bivariate Normal PDFs

It can be shown that

$$Q = \frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$Q = \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$Q = (x-\mu)'\Sigma^{-1}(x-\mu)$$

$$\rho = \sigma_{12} / (\sigma_1\sigma_2)$$

Bivariate Normal PDFs

It can be shown that

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_1^2 \end{vmatrix}$$

$$= \sigma_1^2 \sigma_2^2 - \sigma_{12}\sigma_{12}$$

$$= \sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2 \qquad \rho = \sigma_{12} / (\sigma_1 \sigma_2)$$

$$= \sigma_1^2 \sigma_2^2 (1 - \rho^2)$$

Bivariate Normal PDFs

$$|\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$Q = (x-\mu)'\Sigma^{-1}(x-\mu)$$

This means that

$$f_X(x_1, x_2 \mid \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}Q\right]$$

$$Q = \frac{1}{(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

becomes

$$f(x \mid \mu, \Sigma) = (2\pi)^{-2/2} |\Sigma|^{-1/2} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

$$x, \mu \in \mathbb{R}^2$$

$$\Sigma > 0$$

$\uparrow$ set of pos def matrices

$$-1 < \rho < 1$$

which is the parameterization that Statisticians use.

Bivariate Normal PDFs

If a random variable $\underset{2\times 1}{x}$ has a normal distribution with

mean vector $\underset{2\times 1}{\mu}$ and variance-covariance matrix $\underset{2\times 2}{\Sigma}$, then

mean vector mean vector

$$f(x\mid\mu,\Sigma)=(2\pi)^{-p/2}\,|\Sigma|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

covariance matrix covariance matrix

$x,\mu\in\mathbb{R}^2$

$p=2$

$\Sigma>0$

set of pos def matrices

mean vector covariance matrix

and we write $\underset{2\times 1}{x}\sim N(\underset{2\times 1}{\mu},\underset{2\times 2}{\Sigma})$. The covariance matrix $\underset{2\times 2}{\Sigma}$, has to

be of full rank (there is an inverse in PDF).

make sure you know what this means

Bivariate Normal PDFs

$$|\Sigma| = \sigma_1^2\sigma_1^2(1-\rho^2)$$

From the bivariate normal PDF,

$$Q = (x-\mu)'\Sigma^{-1}(x-\mu)$$

$$f(\underset{2\times1}{x}|\underset{2\times1}{\mu},\underset{2\times2}{\Sigma}) = (2\pi)^{-2/2}|\Sigma|^{-1/2}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

The marginal normal PDFs of x_1 and x_2 can be shown to be

$$f(x_i|\mu_i,\sigma_i^2) = \frac{e^{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}}}{\sqrt{2\pi\sigma_i^2}}$$

$$-\infty < x_i < \infty$$
$$-\infty < \mu_i < \infty$$
$$\sigma_i > 0$$

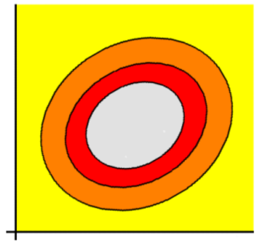
$$i = 1,2$$

with marginal mean μ_i and variance σ_i.

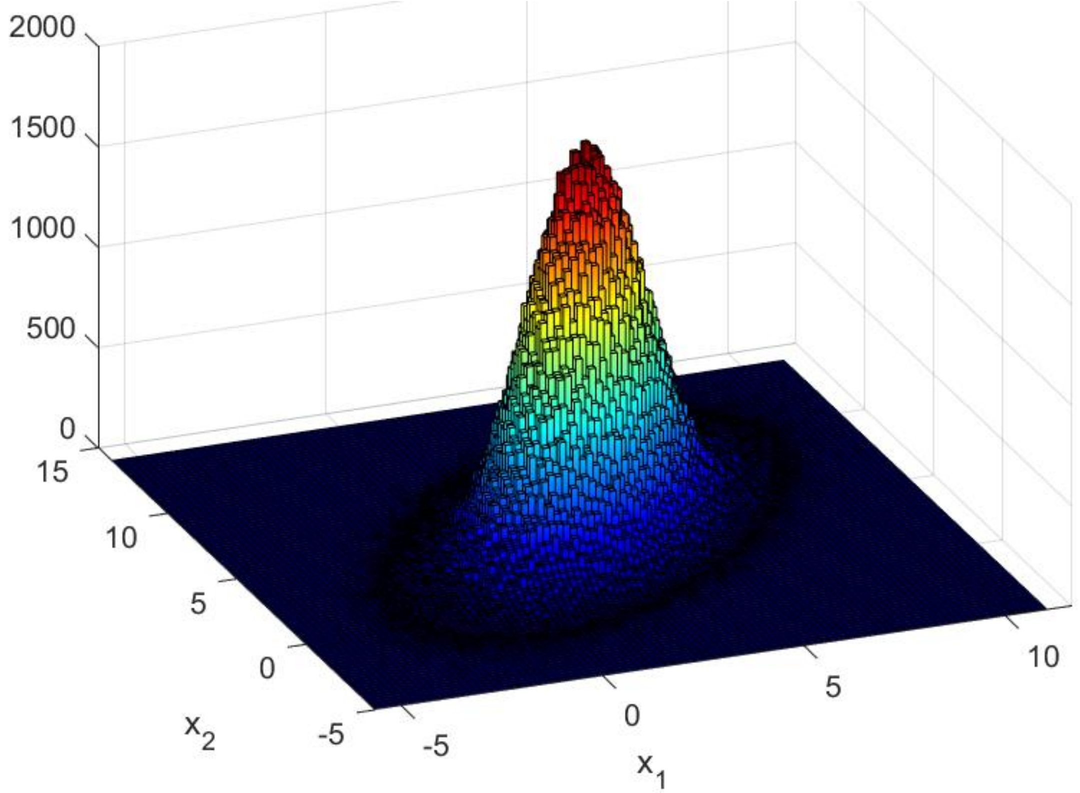
Bivariate Normal PDFs

$$f(x\,|\,\mu,\Sigma)=(2\pi)^{-2/2}\,|\,\Sigma\,|^{-1/2}\,e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

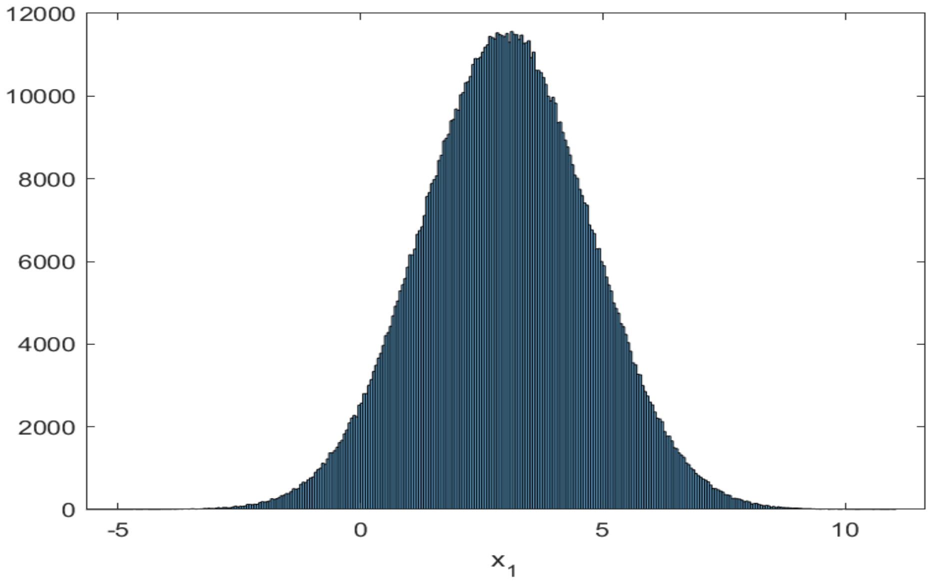
$$\underset{2\times 1}{\mu}=\begin{pmatrix}3\\5\end{pmatrix}\qquad \underset{2\times 2}{\Sigma}=\begin{pmatrix}3 & 2\\2 & 4\end{pmatrix}$$

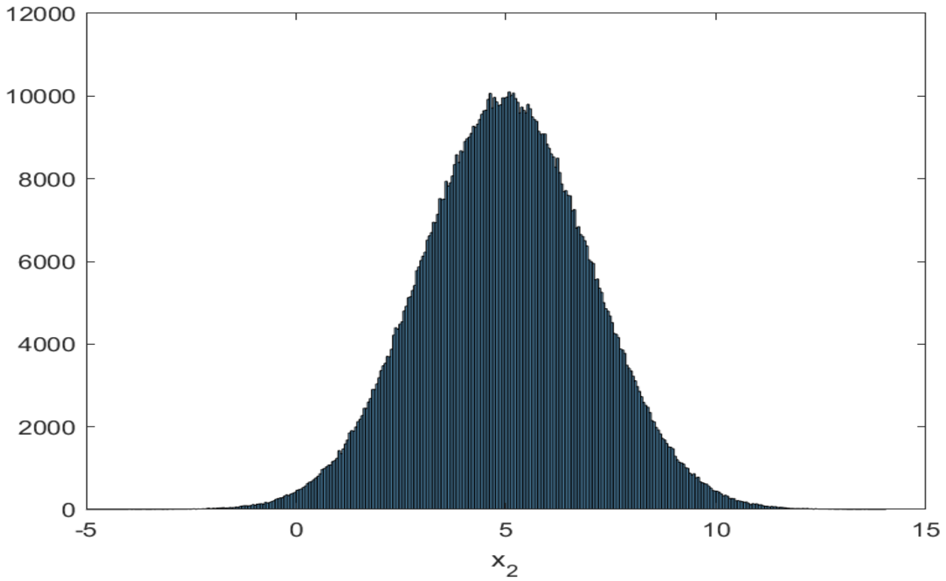


cross section



marginals are normal





Bivariate Normal PDFs

```
rng('default')
n=10^6;
mu=[3,5];,Sigma=[3,2;2,4];
X = mvnrnd(mu,Sigma,n);

figure;
hist3(X,[100,100],'CDataMode','auto','FaceColor','interp','EdgeColor',[0,0,0])
xlim([mu(1,1)-5*sqrt(3) mu(1,1)+5*sqrt(3)]),ylim([mu(1,2)-5*sqrt(4) mu(1,2)+5*sqrt(4)])
xlabel('x_1'), ylabel('x_2')
colormap(jet)

figure;
histogram(X(:,1))
xlim([mu(1,1)-5*sqrt(3) mu(1,1)+5*sqrt(3)])
xlabel('x_1')

figure;
histogram(X(:,2))
xlim([mu(1,2)-5*sqrt(4) mu(1,2)+5*sqrt(4)])
xlabel('x_2')
```

Bivariate Normal PDFs

$$|\Sigma| = \sigma_1^2\sigma_1^2(1-\rho^2)$$

And from the bivariate normal PDF,

$$Q = (x-\mu)'\Sigma^{-1}(x-\mu)$$

$$f(\underset{2\times 1}{x}|\underset{2\times 1}{\mu},\underset{2\times 2}{\Sigma}) = (2\pi)^{-2/2}|\Sigma|^{-1/2}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

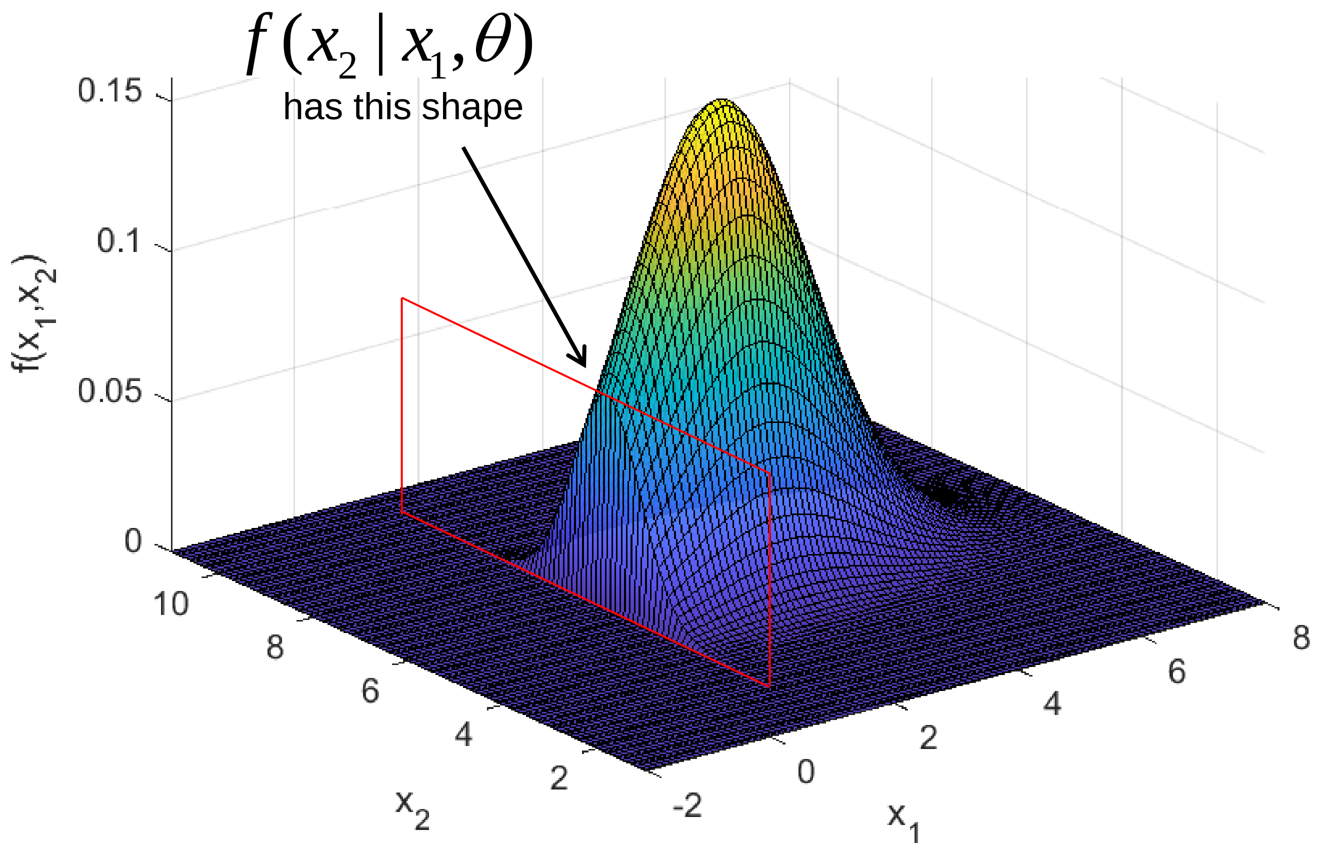
The conditional PDFs of $x_2|x_1$ can be shown to be

$$f(x_2|x_1,\mu_1,\sigma_1^2,\mu_2,\sigma_2^2,\rho) = \frac{\exp\left\{-\frac{\left[x_2-\mu_2-\frac{\rho\sigma_2}{\sigma_1}(x_1-\mu_1)\right]^2}{2\sigma_2^2(1-\rho^2)}\right\}}{(1-\rho^2)^{1/2}\sqrt{2\pi\sigma_2^2}}\,, \qquad \begin{array}{c} -\infty < x_i < \infty \\ -\infty < \mu_i < \infty \\ \sigma_i > 0 \end{array}\,.$$

Note that becomes marginal when $\rho=0$.
Can also compute conditional PDF summaries.

Bivariate Normal PDFs

The conditional PDF of $x_2|x_1$ has the general slice shape but normalized to integrate to 1.



$$f(x_2 \mid x_1, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho) = \frac{\exp\left\{ -\frac{\left[x_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(x_1 - \mu_1) \right]^2}{2\sigma_2^2(1-\rho^2)} \right\}}{(1-\rho^2)^{1/2}\sqrt{2\pi\sigma_2^2}}$$

$$-\infty < x_i < \infty$$
$$-\infty < \mu_i < \infty$$
$$\sigma_i > 0$$

Bivariate Normal PDFs

Theorem:

If x is a 2-D (or p-D) random variable from $f(x|\mu,\Sigma)$, with

$E(x|\mu,\Sigma) = \underset{p\times 1}{\mu}$ and $cov(x|\mu,\Sigma) = \underset{p\times p}{\Sigma}$ think of p=2

then we form $\underset{r\times 1}{y} = \underset{r\times p}{A}\ \underset{p\times 1}{x} + \underset{r\times 1}{\delta}$ where dimensions match

and $\underset{r\times p}{A}$ full column rank (A: $r\times p$, $r\leq p$), then

$E(y|\mu,\Sigma,\delta,A) = \underset{r\times p}{A}\,\underset{p\times 1}{\mu} + \underset{r\times 1}{\delta}$ and $cov(y|\mu,\Sigma,\delta,A) = \underset{r\times p}{A}\ \underset{p\times p}{\Sigma}\ \underset{p\times r}{A'}$.

Bivariate Normal PDFs

Theorem: Marginal

If x is a 2-D (or p-D) random variable from $f(x|\mu,\Sigma)$, with

$E(x|\mu,\Sigma) = \underset{p\times 1}{\mu}$ and $cov(x|\mu,\Sigma) = \underset{p\times p}{\Sigma}$ think of p=2

then we form $\underset{r\times 1}{y} = \underset{r\times p}{A}\ \underset{p\times 1}{x} + \underset{r\times 1}{\delta}$ where dimensions match

and partition $\underset{p\times 1}{x} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$, $\underset{p\times 1}{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$, $\underset{p\times p}{\Sigma} = \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \underbrace{\Sigma_{BA}}_{p_A\times 1} & \underbrace{\Sigma_{BB}}_{p_B\times 1} \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$

where $p_A+p_B=p$, then the marginal PDFs of $\underset{p_A\times 1}{x_A}$ and $\underset{p_B\times 1}{x_B}$ are

$x_A \sim N(\mu_A,\Sigma_{AA})$ and $x_B \sim N(\mu_B,\Sigma_{BB})$!

Bivariate Normal PDFs

Theorem: Conditional

If x is a 2-D (or p-D) random variable from $f(x|\mu,\Sigma)$, with

$$E(x|\mu,\Sigma) = \underset{p\times 1}{\mu} \quad \text{and} \quad cov(x|\mu,\Sigma) = \underset{p\times p}{\Sigma}$$

think of p=2

then we form $\underset{r\times 1}{y} = \underset{r\times p}{A}\ \underset{p\times 1}{x} + \underset{r\times 1}{\delta}$ where dimensions match

and partition $\underset{p\times 1}{x} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$, $\underset{p\times 1}{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$, $\underset{p\times p}{\Sigma} = \begin{pmatrix} \underbrace{\begin{matrix}\Sigma_{AA} \\ \Sigma_{BA}\end{matrix}}_{p_A\times 1} & \underbrace{\begin{matrix}\Sigma_{AB} \\ \Sigma_{BB}\end{matrix}}_{p_B\times 1} \end{pmatrix} \begin{matrix} \}\ p_A\times 1 \\ \}\ p_B\times 1 \end{matrix}$

where $p_A+p_B=p$, then the conditional PDFs of $\underset{p_A\times 1}{x_A}$ and $\underset{p_B\times 1}{x_B}$ are

$x_A|x_B \sim N(\mu_A+\Sigma_{AB}\Sigma_{BB}^{-1}(x_B-\mu_B), \Lambda_{AA})$ and $x_B|x_A \sim N(\mu_B+\Sigma_{BA}\Sigma_{AA}^{-1}(x_A-\mu_A), \Lambda_{BB})$

where $\Lambda_{BB}= \Sigma_{BB}-\Sigma_{BA}\Sigma_{AA}^{-1}\Sigma_{AB}$ and $\Lambda_{AA}= \Sigma_{AA}-\Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{BA}$.

Discussion

Questions?

Homework 4

1. Assume Marquette Undergrads heights have true μ_h=67 in and σ_h=2 in while their true weights have μ_w=150 lbs and σ_w=4 lbs with ρ=.75.

a) Generate 10^4 h/w 2×1 vectors from a bivariate normal PDF.

b) Calculate sample means, variances, and covariance. Compare to truth.

c) Make an (x_1, x_2) bivariate histogram and marginal histograms for x_1 and x_2. (For marginal just ignore other.)

d) Plot the theoretical conditional distribution of weights x_2 given a height of x_1=69 in.

e*) For an interval of heights $69\ in \pm 0.5\ in$ make a histogram. Compare.

* For students in 5790.

Homework 4

2*. Derive by pencil and paper the theoretical marginal PDF

$f(x_1|\mu_1,\sigma_1,\mu_2,\sigma_2,\rho)$ of heights x_1 given all the parameters.

Start with equation for bivariate.

3*. Derive by pencil and paper the theoretical conditional PDF

$f(x_2|x_{1,}\mu_1,\sigma_1,\mu_2,\sigma_2,\rho)$ of weights x_2 given a height of x_1 and all the parameters.

Start with equation for bivariate.

4* Superimpose plots of theoretical PDFs on histograms in #1.

* For students in 5790.