Chapter 10: Nonparametric Tests B

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Nonparametric Testing

The hypothesis tests that we learned assume the data (observations) come from a statistical distribution such as the normal distribution or we assume a large sample size. These hypothesis tests are called *parametric tests*. Distributions have parameters such as μ and σ.

However, sometimes our data does not come from the normal distribution or we have a small sample size and we need to resort to alternative distribution free tests called *nonparametric tests*.

7.6 Tests with Matched Samples, Continuous Outcome

RECALL

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

There are three possible pairs.

H_0: μ_d=0 *vs*. H_1: μ_d>0 (prove greater than)

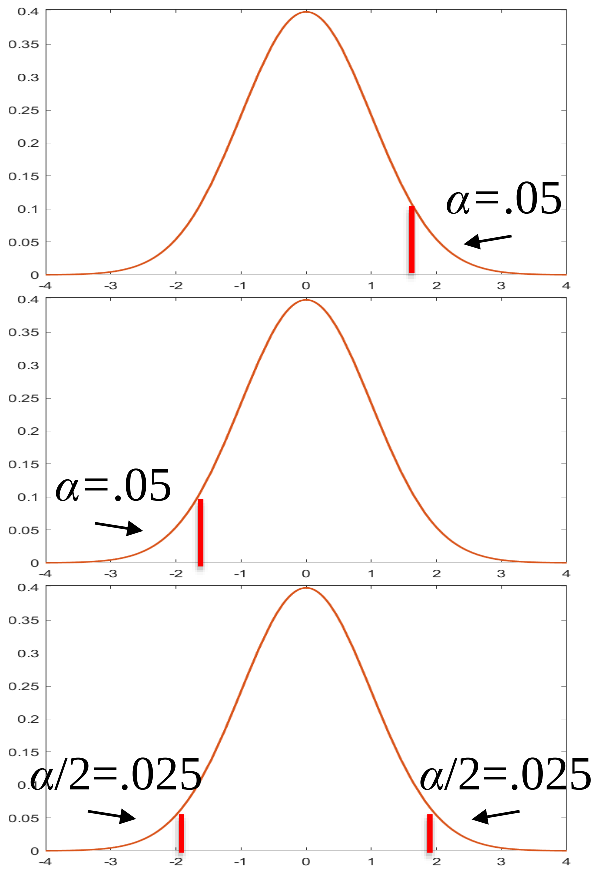
$\leq$ reject for "large" $\bar{X}_d$ or z's

H_0: μ_d=0 *vs*. H_1: μ_d<0 (prove less than)

$\geq$ reject for "small" $\bar{X}_d$ or z's

H_0: μ_d=0 *vs*. H_1: $\mu_d \neq 0$ (prove not equal to)

reject for "large" or "small" $\bar{X}_d$ or z's



10.3 Tests with Matched Samples – Sign Test

We can test if matched samples are likely from the same distribution.
Some interpret this as comparing the medians between two populations.

δ is population version of $d=x_2-x_1$.

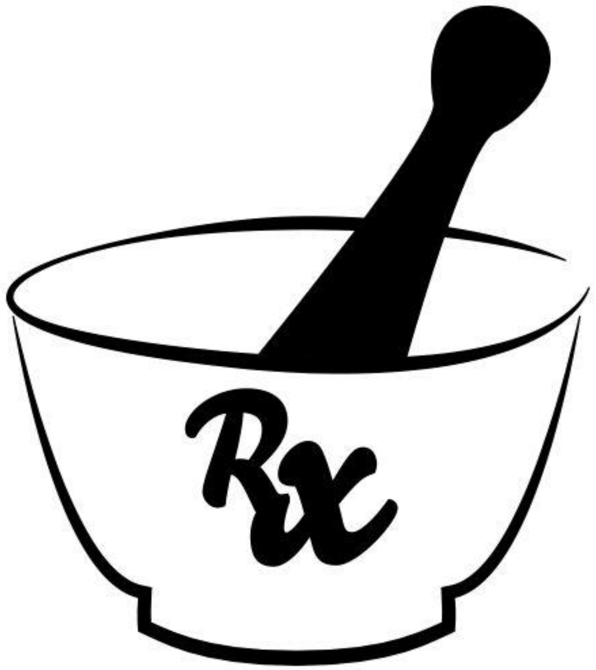
H_0: The median difference is zero (H_0: $\delta=0$)
H_1: The median difference is positive (H_1: $\delta>0$)

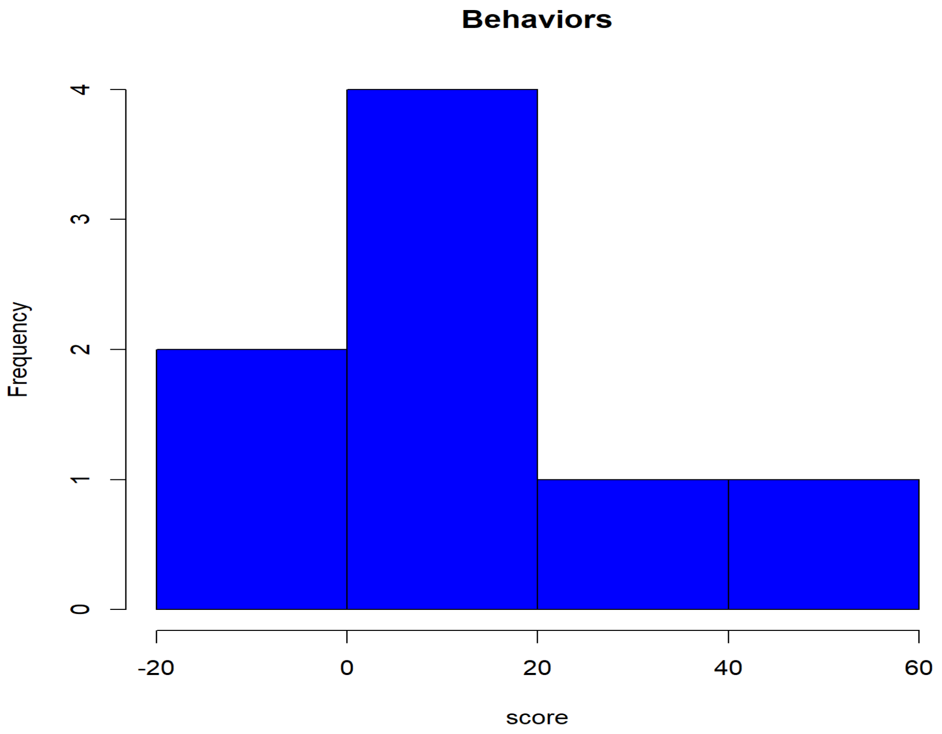
If the median difference of the matched pairs is zero, then half of the time they should be positive and half of the time negative.
If the median difference is greater than zero, then there should be more positives than negatives.
Sign Test.

10.3 Tests with Matched Samples – Sign Test

Example: Effectiveness of new drug designed to reduce repetitive behavior. A total of n=8 children in the study. Observed for 3 hours before and after treatment. Test at the α=0.05 level whether the median difference in repetitive behaviors is positive.

b	a	d
85	75	10
70	50	20
40	50	-10
65	40	25
80	20	60
75	65	10
55	40	15
20	25	-5





10.3 Tests with Matched Samples – Sign Test

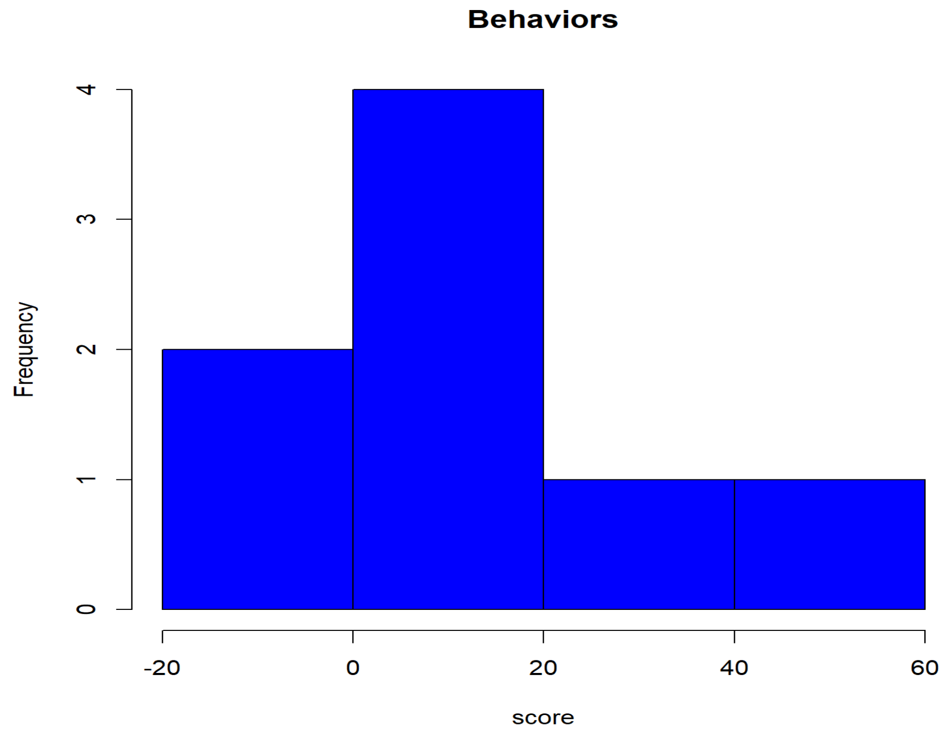
The first thing you should do is to test if the data is consistent with a normal distribution. Test at the α=0.05 level.

b	a	d
85	75	10
70	50	20
40	50	-10
65	40	25
80	20	60
75	65	10
55	40	15
20	25	-5

```
install.packages('nortest')
library(nortest)

before <- c(85,70,40,65,80,75,55,20)
after    <- c(75,50,50,40,20,65,40,25)
difference <- before - after
hist(difference,col='blue',main='Behaviors',xlab='score'))
ad.test(dfference)
# if p-value < 0.05 data not normal

Anderson-Darling normality test
data:  difference
A = 0.39956, p-value = 0.2744
```



Because p-value > 0.05, normal.

But let's use nonparametric.

10.3 Tests with Matched Samples – Sign Test

Sign Test for median difference (δ), nonparametric version of t-test.

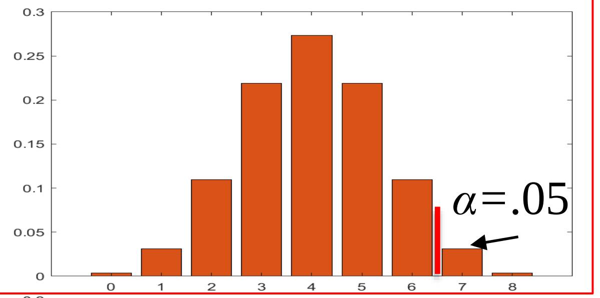
Step 1: Set up the hypotheses and determine the level of significance.

There are three possible pairs.

For our example.

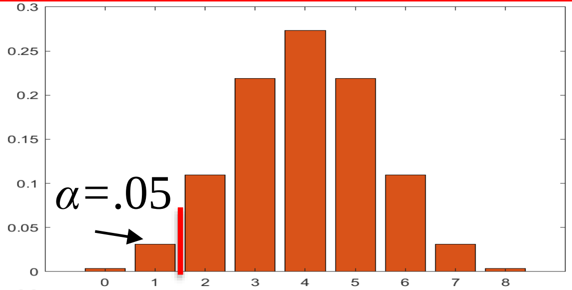
H_0: δ=0 *vs*. H_1: $\delta > 0$ (prove greater than, **upper tailed test**)

$\leq$ reject for "large" median differences



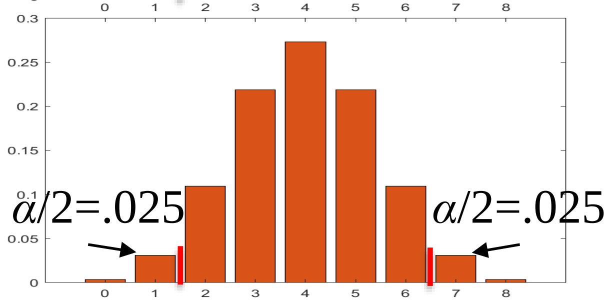
H_0: δ= 0 *vs*. H_1: $\delta < 0$ (prove less than, **lower tailed test)**

$\geq$ reject for "small" median differences



H_0: $\delta = 0$ *vs*. H_1: $\delta \neq 0$ (prove not equal to, **two tailed test)**

reject for "large" or "small" median differences



10.3 Tests with Matched Samples – Sign Test

H_0: $\delta \leq 0$ *vs*. H_1: $\delta > 0$

We compare each difference with the conjectured median difference 0.

If a difference is larger than the hypothesized difference 0, replace with a +.

If a difference is smaller than the hypothesized difference 0, replace with a –.

If the difference is equal to the hypothesized difference 0, replace with a 0.

The test statistic is x the number of +'s.

Reject H_0 based on binomial probabilities, $p=1/2$.

10.3 Tests with Matched Samples – Sign Test

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.
The test statistic is a single (decision) number.

H_0: $\delta \leq 0$ *vs*. H_1: $\delta > 0$

$$x = (\text{the number of differences} > 0)$$

When using Table 6, "the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller**." Page 240 of Sullivan.

So it would be # d's <0 using Table 6.

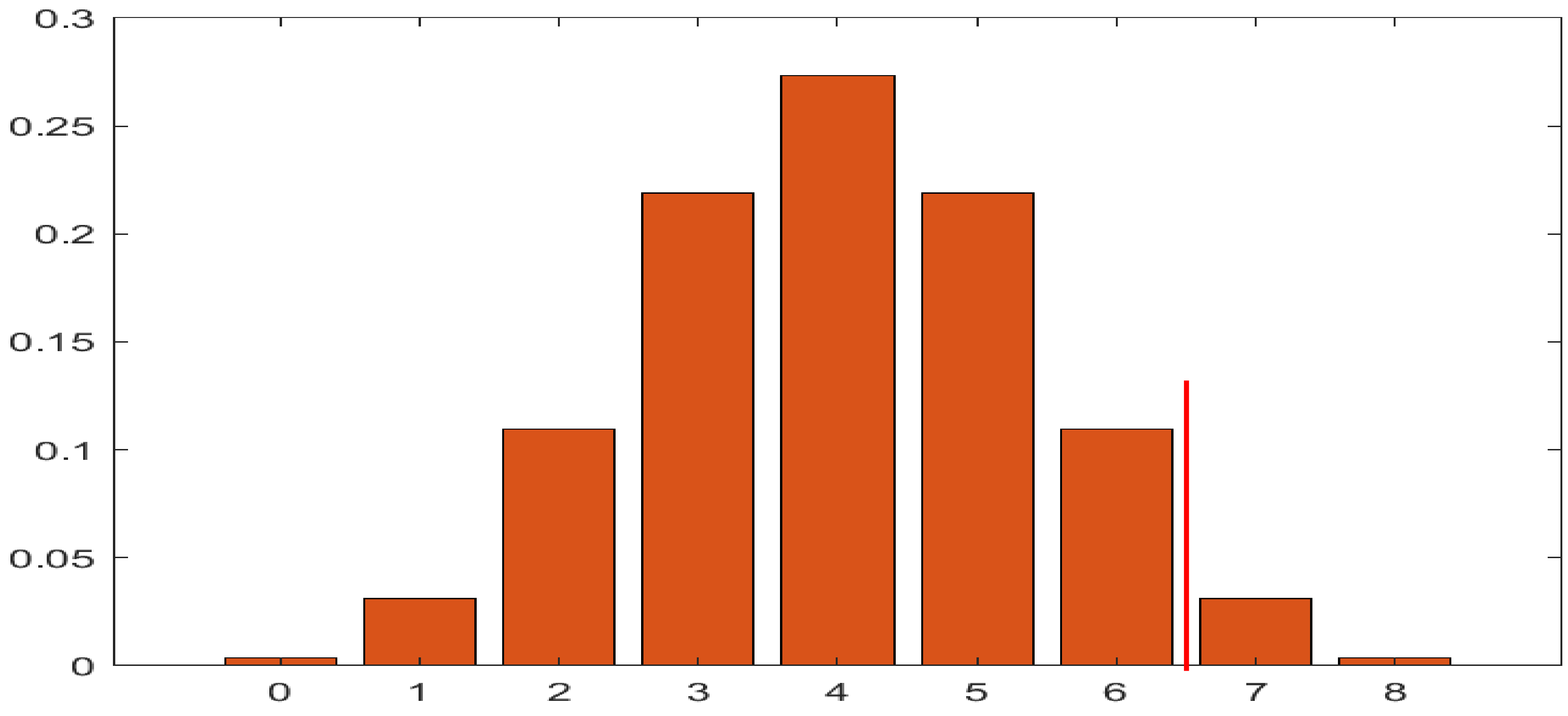
Use the test statistic x that depends on data and null hypothesis with a critical x_a value from a cumulative binomial distribution p=1/2.

x_a = x value with cumulative probability a larger than it from binomial, n, p=1/2. $a = \alpha$ or $\alpha/2$

10.3 Tests with Matched Samples – Sign Test

$H_0: \delta \le 0$ *vs.* $H_1: \delta > 0$

Binomial Distribution, n=8, p=0.5



Value with 0.05 less than. Compliment.

p-value

Value with 0.05 larger than.

x	P(X=x)	CumSum	CumSumR
0	0.004	0.004	1.000
1	0.031	0.035	0.996
2	0.109	0.145	0.965
3	0.219	0.363	0.856
4	0.273	0.637	0.637
5	0.219	0.856	0.363
6	0.109	0.965	0.145
7	0.031	0.996	0.035
8	0.004	1.000	0.004

See also Table 6

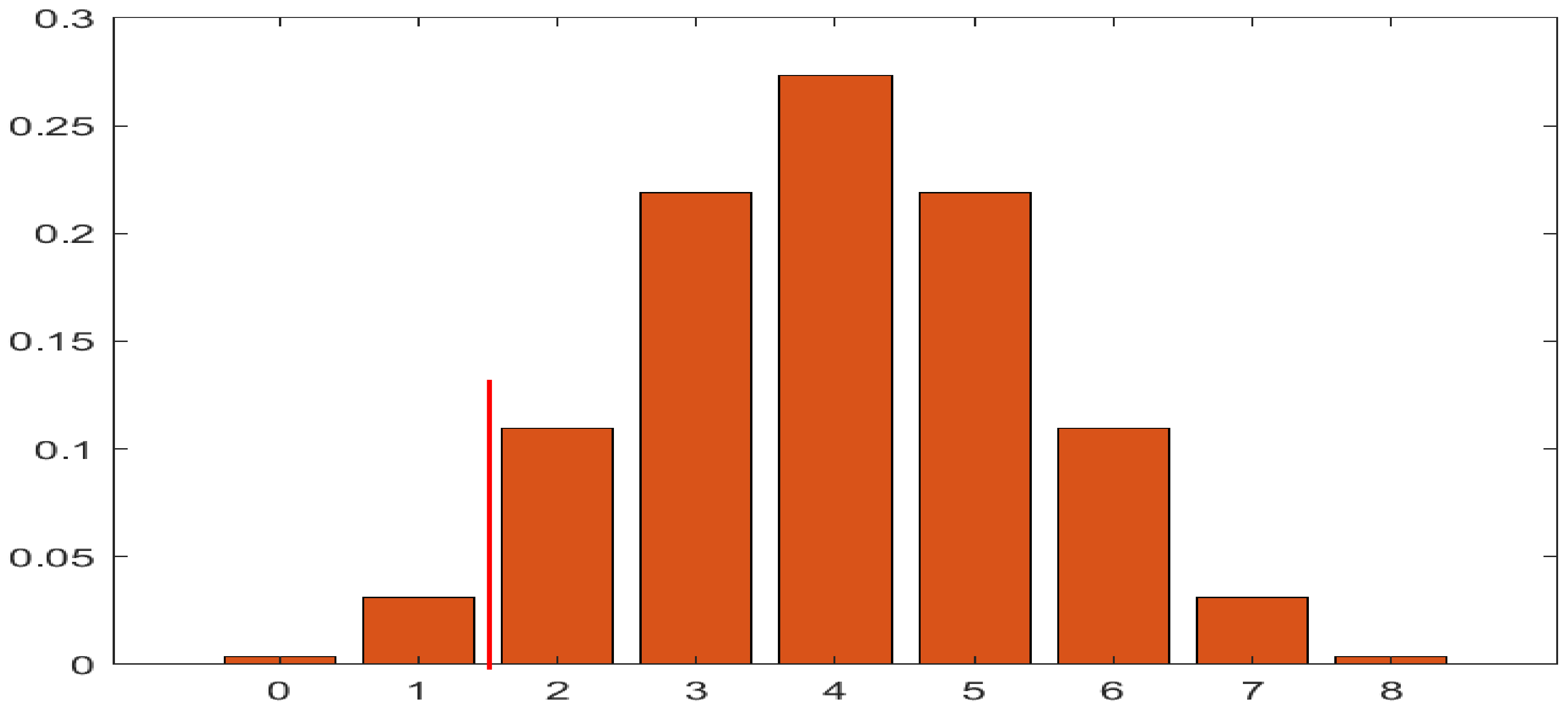
$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

When using Table 6, the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller**. Page 240 of Sullivan.

Here $H_1: \delta > 0$ is the problem, reject for large # d's.

10.3 Tests with Matched Samples – Sign Test

Binomial Distribution, n=8, p=0.5



$$P(X=x)=\frac{n!}{x!(n-x)!}p^{x}(1-p)^{n-x}$$

Value with 0.05 less than. Compliment.

Table 6

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	**.05**	**.025**	**.01**	**.005**
n				
1				
2				
3				
4				
5	0			
6	0	0		
7	0	0	0	
8	1	0	0	0
9	1	1	0	0
10	1	1	0	0

x	P(X=x)	CumSum
0	0.004	0.004
1	0.031	0.035

Symmetry

Book turn it into $H_1: \delta < 0$ problem, reject for small # d's.

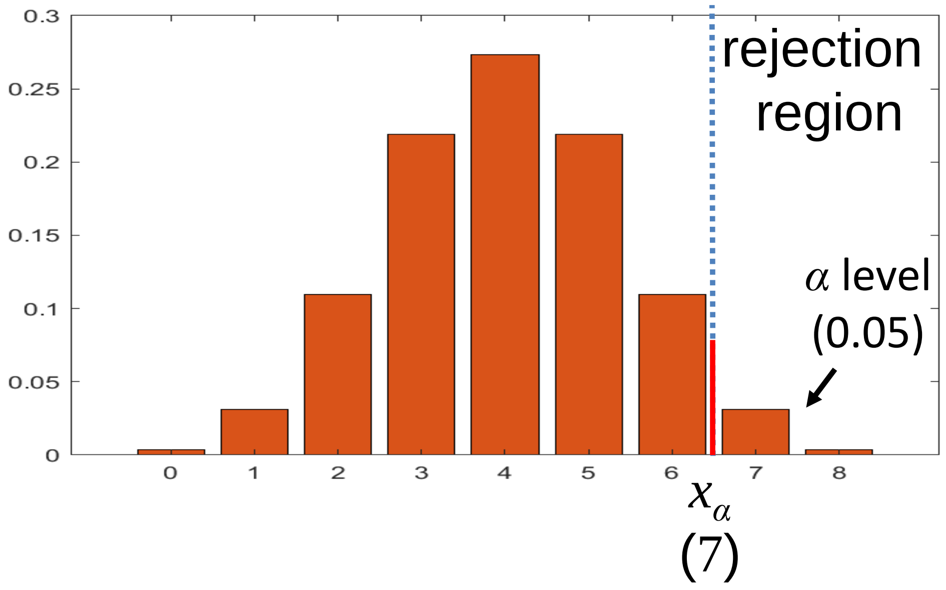
When using Table 6, the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller**. Page 240 of Sullivan.

10.3 Tests with Matched Samples – Sign Test

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

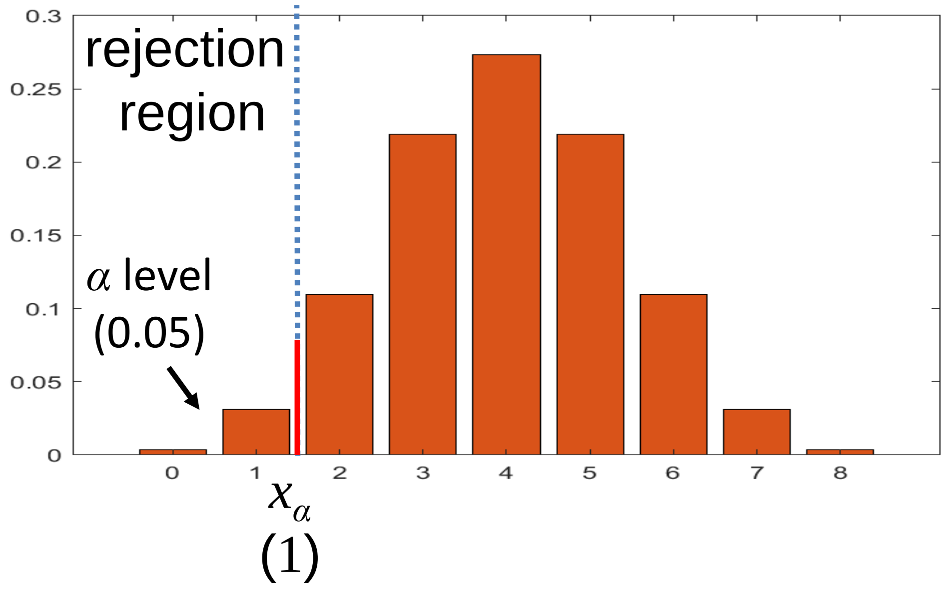
$H_0: \delta = 0$ *vs.* $H_1: \delta > 0$



Reject H_0 if $P(X \geq x_\alpha) \leq \alpha$

Reject for large x,

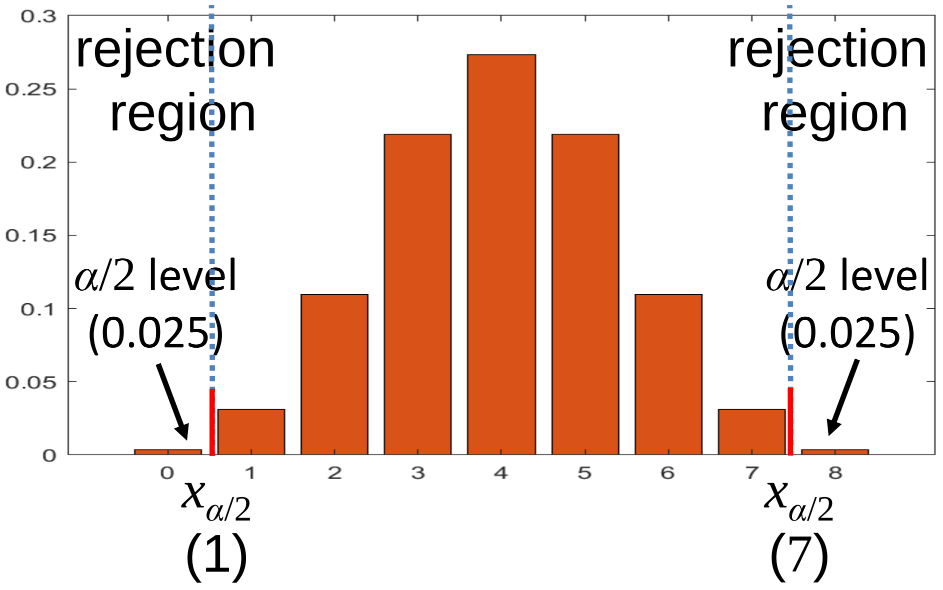
$H_0: \delta = 0$ *vs.* $H_1: \delta < 0$



Reject H_0 if $P(X \leq x_\alpha) \leq \alpha$

Reject for small x.

$H_0: \delta = 0$ *vs.* $H_1: \delta \neq 0$



Reject H_0 $P(x_\alpha \leq X) \leq \dfrac{\alpha}{2}$

Reject for large or small.

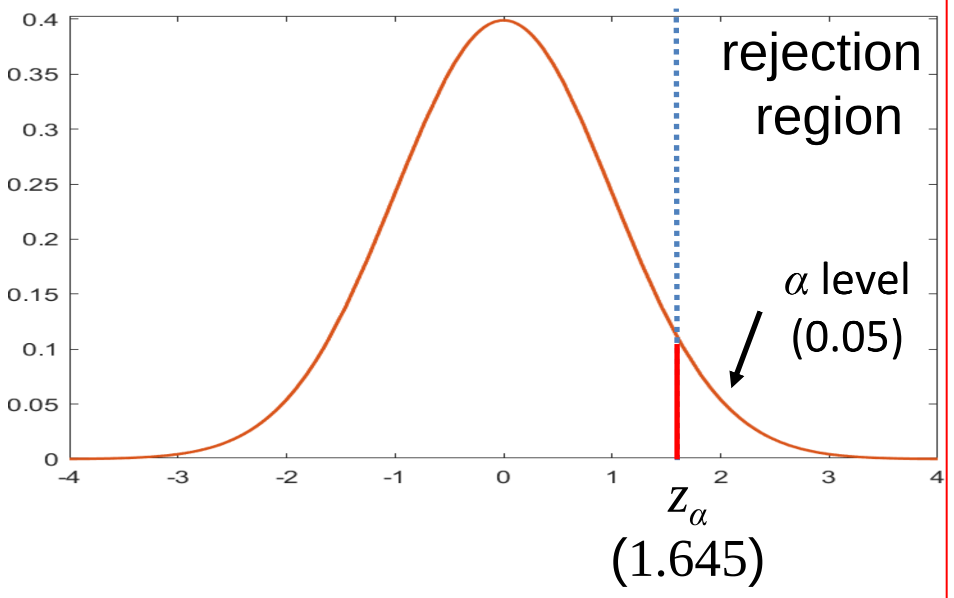
7.1 Introduction to Hypothesis Testing

RECALL

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

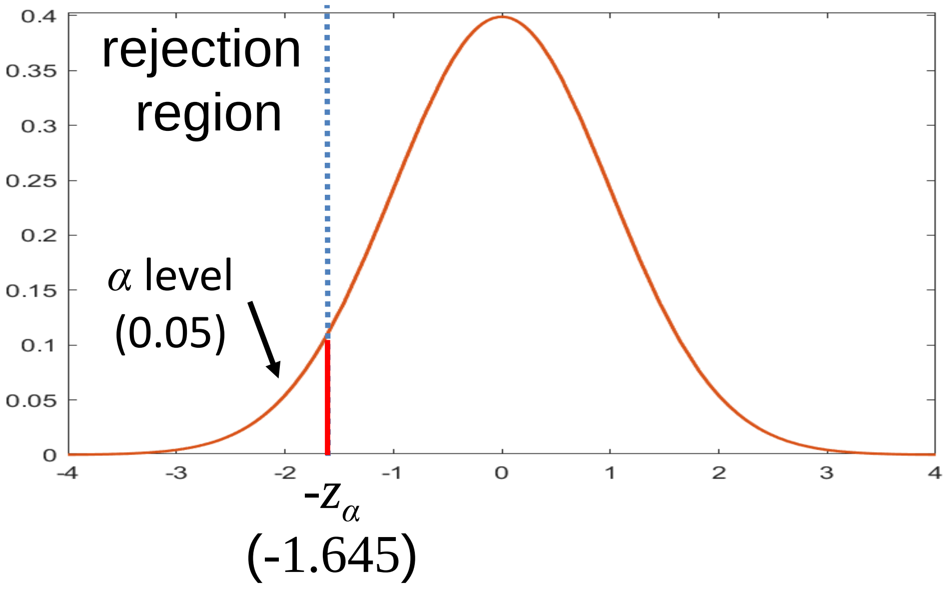
$\mathrm{H_0}: \mu_d = 0$ *vs.* $\mathrm{H_1}: \mu_d > 0$



Reject $\mathrm{H_0}$ if $z \geq z_\alpha$

Reject for large $\bar{X}_d$.

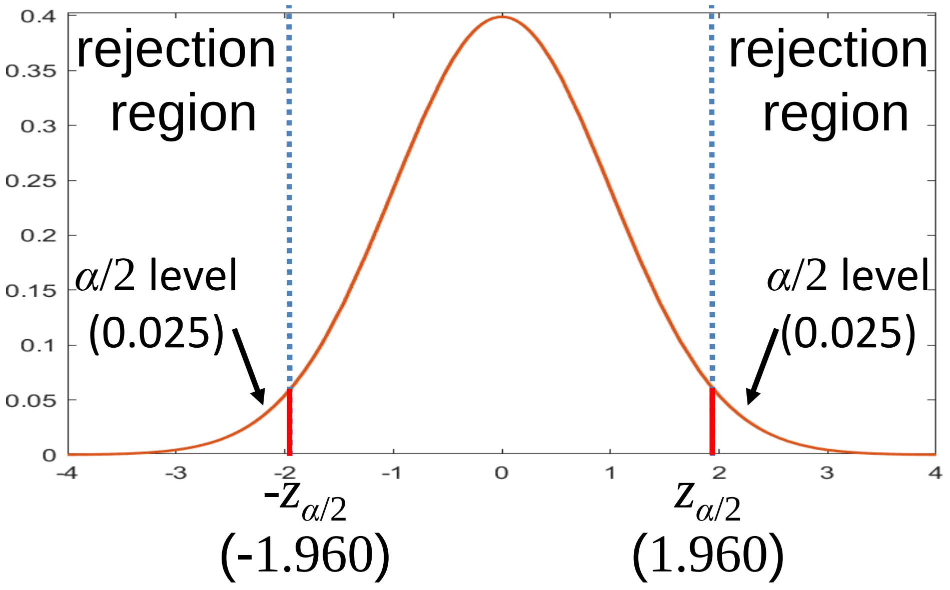
$\mathrm{H_0}: \mu_d = 0$ *vs.* $\mathrm{H_1}: \mu_d < 0$



Reject $\mathrm{H_0}$ if $z \leq z_\alpha$

Reject for small $\bar{X}_d$.

$\mathrm{H_0}: \mu_d = 0$ *vs.* $\mathrm{H_1}: \mu_d \neq 0$



Reject $\mathrm{H_0}$ $z \leq z_{\alpha/2}$ or $z \geq z_{\alpha/2}$

Reject for large or small.

10.3 Tests with Matched Samples – Sign Test

If diff < MD_0, $-$.
If diff = MD_0, 0.
If diff > MD_0, +.

b	a	d	sorted	sign
85	75	10	-10	-1
70	50	20	-5	-1
40	50	-10	10	+1
65	40	25	10	+1
80	20	60	15	+1
75	65	10	20	+1
55	40	15	25	+1
20	25	-5	60	+1

Step 4: Compute the test statistic.

x = (the number of differences > 0)

x=6 (8-6=2 if we're going to use the table.)

Step 5: Because $x=6 \leq x_\alpha=7$ or $p\text{-}value=0.145 > \alpha=0.05$.
($x=2 > x_\alpha=1$ when using Table 6) do not reject H_0

x	P(X=x)	CumSum	CumSumR
6	0.109	0.965	0.145
7	0.031	0.996	0.035
8	0.004	1.000	0.004

See also Table 6 →

Table 6

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	.05	.025	.01	.005
7	0	0	0	
8	1	0	0	0
9	1	1	0	0
10	1	1	0	0

x	P(X=x)	CumSum
0	0.004	0.004
1	0.031	0.035

x=min(+'s,-'s)

Note:

If we used normal, we would not reject H_0 , $t=1.533<t_{0.05,7}=1.895$.

$$t=\frac{\bar{X}-\mu_0}{s/\sqrt{n}} \quad df=n\text{-}1 \quad \bar{X}=15.6250 \quad s=21.4539$$

10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

An alternative for the test for population median difference is the Wilcoxon Signed Rank test.

H_0: $\delta \leq 0$ *vs*. H_1: $\delta > 0$
δ is population version of d.

Step 1:

H_0: The median difference is zero (H_0: $\delta=0$)

H_1: The median difference is positive (H_1: $\delta>0$)

We will calculate a test statistic W the smaller of $W+$ and W_- .

$W+$ = sum of positive ranks

W_- = sum of negative ranks ⟶ $W=\min(W+, W_-)$

If the median difference of the matched pairs is zero, then the sum of the positive ranks should be the same as the sum of the negative ranks.

10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

$H_0: \delta \leq 0$ vs. $H_1: \delta > 0$

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number summarizing information.

$$W_+ = \text{sum of positive ranks} = 32$$

$$W_- = \text{sum of negative ranks} = 4$$

$$W = \min(W_1, W_2) = \min(4, 32) = 4$$

b	a	d	sorted	sign	rank	SgnRnk
85	75	10	-10	-1	3	-3
70	50	20	-5	-1	1	-1
40	50	-10	10	+1	3	3
65	40	25	10	+1	3	3
80	20	60	15	+1	5	5
75	65	10	20	+1	6	6
55	40	15	25	+1	7	7
20	25	-5	60	+1	8	8

n=8

Reject H_0 for small W.

IF Signed Ranks

SgnRnk	SgnRnk	SgnRnk	SgnRnk
1	-4	-7	-8
2	-3	-5	-7
3	-2	-3	-6
4	-1	-1	-5
5	5	2	2
6	6	4	4
7	7	6	6
8	8	8	8
$W=0$	$W=10$	$W=16$	$W=26$

10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

Step 3: Set-up the decision rule.

n=8, α=0.05

If we did Two Sided Test

Reject H_0 if $W \le W_{\alpha,n}$.

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	.05	.025	.01	.005
n				
5	1			
6	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3

Table 7

Step 4: Compute test statistic.

Already done, W=4.

Step 5: Conclusion.

Reject H_0 because

$W=4 \le W_{0.05,8}=4$. Interpret.

10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

Step 3: Set-up the decision rule.

n=8, α=0.05

If we did One Sided Test

Reject H_0 if $W \leq W_{\alpha,n}$.

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	.05	.025	.01	.005
n				
5	1			
6	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3

Table 7

Step 4: Compute test statistic.

Already done, W=4.

Step 5: Conclusion.

Reject H_0 because

$W=4 \leq W_{0.05,8}=6$. Interpret.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

Step 1:

We can test if three or more population medians are different.

H_0: The k population medians are equal

H_1: The k population medians are not all equal

We will go through the same 5 hypothesis steps

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

RECALL

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

H_0: $\mu_1 = \mu_2 \ldots = \mu_k$ *vs*. H_1: at least two μ's different

reject for "large" disparities or $F = MSB/MSE$.

We will assume the means are equal and calculate two different variances.

If the means are truly equal, the two different variances will be the same.

If the means are noy equal, the two different variances will be different.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

A clinical study is designed to assess differences in albumin levels.

5% Protein	10% Protein	15% Protein
3.1	3.8	4.0
2.6	4.1	5.5
2.9	2.9	5.0
	3.4	4.8
	4.2	

The question of interest is whether there is a difference in albumin levels among the three different diets.

The data is not normally distributed with same variance for ANOVA.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

H_0: $MD_1=MD_2$ … $=MD_k$ *vs*. H_1: at least two MD's different
reject for "large" disparities H.

We will assume the medians are equal and see how different from equal.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number.

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

R_j is sum of ranks for sample j.

Use the test statistic that depends on data and null hypothesis with a critical value H_{α,n_1,n_2,n_3}) that depends on significance level α to make decision.

Table 8 in book

We will test a single hypotheses on medians with the test statistic.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

H_0: $MD_1 = MD_2 \ldots = MD_k$ vs. H_1: at least two different

Select a level of significance α.

Table 8 in book

Reject H_0 if $H \geq H_{\alpha, n_1, n_2, n_3}$.

Sample size order doesn't matter.

Table 8

Three groups				
n_1	n_2	n_3	$\alpha = .05$	$\alpha = .01$
2	2	2		
3	2	1		
3	2	2	4.714	
3	3	1	5.143	
3	3	2	5.361	
3	3	3	5.600	7.200
4	2	1		
4	2	2	5.333	
4	3	1	5.208	
4	3	2	5.444	6.444
4	3	3	5.791	6.745
4	4	1	4.967	6.667
4	4	2	5.455	7.036
4	4	3	5.598	7.144
4	4	4	5.692	7.654
5	2	1	5.000	
5	2	2	5.160	6.533
5	3	1	4.960	
5	3	2	5.251	6.909
5	3	3	5.648	7.079
5	4	1	4.985	6.955
5	4	2	5.273	7.205
5	4	3	5.656	7.445

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.

Use sample data n_1 from population 1 and n_2 from population 2 and n_3 from population 3 to compute test statistic H.

Compare test statistic H to critical value(s) H_{α,n_1,n_2,n_3} with rule.

Table 8 in book

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

Step 5: Conclusion.

Make a decision, reject H_0 or not to reject H_0.

Interpret the results.

$N = n_1 + n_2 + n_3$.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

Example: Statistical difference in albumin for 3 diets?

Step 1: Null and Alternative Hypotheses.

H_0: $MD_1=MD_2=MD_3$ *vs*. H_1: at least two different

Step 2: Test Statistic. $N=n_1+n_2+n_3$.

$$H=\left(\frac{12}{N(N+1)}\sum_{j=1}^{k}\frac{R_j^2}{n_j}\right)-3(N+1)$$

Step 3: Decision Rule. $\alpha=0.05$, $n_1=3$, $n_2=5$, $n_3=4$

Reject H_0 if $H \geq H_{\alpha,n_1,n_2,n_3}=5.656$.

Table 8

Three groups				
n_1	n_2	n_3	$\alpha = .05$	$\alpha = .01$
5	4	3	5.656	7.445

Sample size order doesn’t matter.

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

Example: Statistical difference in albumin for 3 diets?

Step 4: Compute test statistic.

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

			Total Sample (Ordered Smallest to Largest)			Ranks		
5% Protein	10% Protein	15% Protein	5% Protein	10% Protein	15% Protein	5% Protein	10% Protein	15% Protein
3.1	3.8	4.0	2.6			1		
2.6	4.1	5.5	2.9	2.9		2.5	2.5	
2.9	2.9	5.0	3.1			4		
	3.4	4.8		3.4			5	
	4.2			3.8			6	
					4.0			7
				4.1			8	
				4.2			9	
					4.8			10
					5.0			11
					5.5			12

$R_1 = 7.5$ $\quad R_2 = 30.5$ $\quad R_3 = 40$

$$H = \frac{12}{12(12+1)} \left(\frac{7.5^2}{3} + \frac{30.5^2}{5} + \frac{40^2}{4} \right) - 3(12+1) = 7.52$$

10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

Example: Statistical difference in albumin for 3 diets?

Step 1: Null and Alternative Hypotheses.

H_0: $MD_1 = MD_2 = MD_3$ *vs*. H_1: at least two different

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

Step 2: Test Statistic.

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

Step 3: Decision Rule. $\alpha=0.05$, $n_1=3$, $n_2=5$, $n_3=4$

Reject H_0 if $H \geq 5.656$.

Table 8

Three groups				
n_1	n_2	n_3	$\alpha = .05$	$\alpha = .01$
5	4	3	5.656	7.445

Sample size order doesn't matter.

Step 4: Compute test statistic.

$$H = 7.52$$

Step 5: Conclusion

Reject H_0 because $7.52 > 5.656$, and conclude difference in median albumin.

10.5 Summary

Sign Test (one sample)

x = number of observations $\begin{matrix} > \\ < \end{matrix}$ MD_0

Mann-Whitney U Test

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

$$U = \min(U_1, U_2)$$

Sign Test (two sample)

x = number of differences $\begin{matrix} > \\ < \end{matrix}$ 0

Wilcoxon Signed Rank Test

(two sample)

$W = \min(W+, W-)$

$W+$ = sum of positive ranks

$W-$ = sum of negative ranks

Kruskal-Wallis Test

(three or more samples)

$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

R_j = sum of ranks for sample j.

Questions?

Homework 10

Read Chapter 10.

Problems # 6 (Sign Test), 7 (Wilcoxon Signed Rank Test),
8 (Kruskal-Wallis Test) the $n_1=n_2=n_3=n_4=5$ critical value is 7.377.