Chapter 10: Nonparametric Tests A

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Nonparametric Testing

The hypothesis tests that we learned assume the data (observations) come from a statistical distribution such as the normal distribution or we assume a large sample size. These hypothesis tests are called *parametric tests*. Distributions have parameters such as μ and σ.

However, sometimes our data does not come from the normal distribution or we have a small sample size and we need to resort to alternative distribution free tests called *nonparametric tests*.

Nonparametric Testing

The cost of fewer assumptions and not assuming a distribution is that nonparametric tests are generally less powerful than parametric tests.

When the alternative hypothesis H_1 is true, they may be less likely to reject H_0. When your data is normal or large, use parametric tests.

There are several hypothesis tests to see if your data comes from a normal distribution. Possible tests are the Kolmogorov-Smirnov test, the Anderson-Darling test, the Shapiro-Wilks test, and the Lilliefors test.

10.1 Introduction to Nonparametric Testing – Sign Test

If I knew the median MD_0 of a population (distribution) that observations or data was sampled from, then half of the time the observations are above the median and half the time below.

This means that the probability an observation is above the median MD_0 is $p=1/2$ and the probability below the median MD_0 is $p=1/2$.

So I should be able to count x, how many above MD_0 and see how likely I am to get that number or more from a binomial distribution.
$P(X \geq x)$ from Binomial with n and $p=1/2$.

7.1 Introduction to Hypothesis Testing

$$t = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} \qquad df=n\text{-}1$$

The hypothesis testing process consists of 5 Steps.

RECALL

Step 1: Set up the hypotheses and determine the level of significance.

There are three possible pairs.

H_0: $\mu = \mu_0$ *vs*. H_1: $\mu > \mu_0$ (prove greater than, **upper tailed test**)

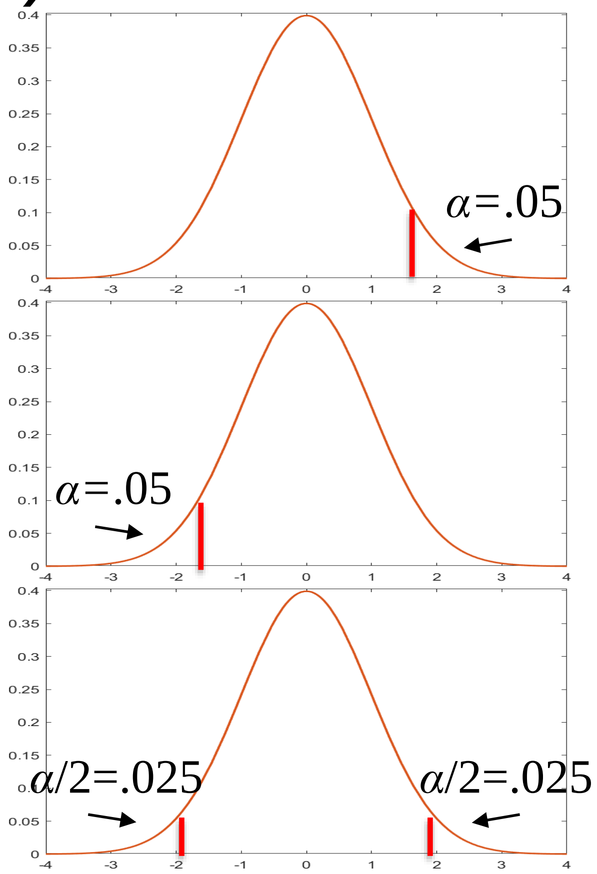
$\leq$ reject for “large" $\bar{X}$ or z’s

H_0: $\mu = \mu_0$ *vs*. H_1: $\mu < \mu_0$ (prove less than, **lower tailed test**)

$\geq$ reject for “small” $\bar{X}$ or z’s

H_0: $\mu = \mu_0$ *vs*. H_1: $\mu \neq \mu_0$ (prove not equal to, **two tailed test**)

reject for “large” or “small” $\bar{X}$ or z’s



10.1 Introduction to Nonparametric Testing – Sign Test

Sign Test for median (MD), nonparametric version of t-test.

Step 1: Set up the hypotheses and determine the level of significance.

There are three possible pairs.

H_0: $MD=MD_0$ *vs*. H_1: $MD>MD_0$ (prove greater than, **upper tailed test**)

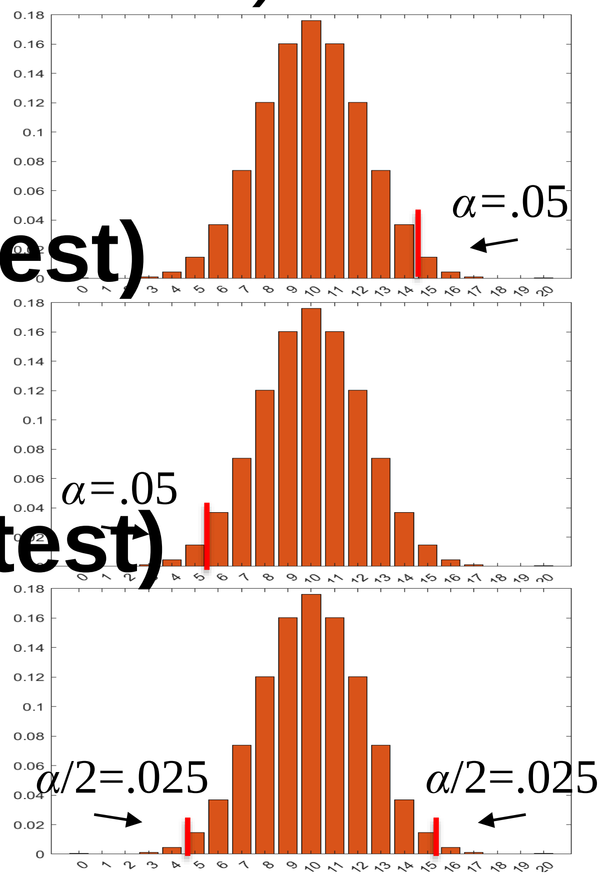
$\leq$ reject for "large" MD's

H_0: $MD=MD_0$ *vs*. H_1: $MD<MD_0$ (prove less than, **lower tailed test**)

$\geq$ reject for "small" MD's

H_0: $MD=MD_0$ *vs*. H_1: $MD\neq MD_0$ (prove not equal to, **two tailed test**)

reject for "large" or "small" MD's



10.1 Introduction to Nonparametric Testing – Sign Test

We compare each value with the conjectured median MD_0.

If a data value is larger than the hypothesized median, replace with a $+$.

If a data value is smaller than the hypothesized median, replace with a $-$.

If the data value is equal to the hypothesized median, replace with a 0.

The test statistic is x the number of $+$'s.

Reject H_0 based on binomial probabilities, $p=1/2$.

10.1 Introduction to Nonparametric Testing – Sign Test

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.
The test statistic is a single (decision) number.

$$x = (\text{the number of observations} > MD_0)$$

Use the test statistic x that depends on data and null hypothesis with a critical x_a value from a binomial distribution with probability of success p=1/2.

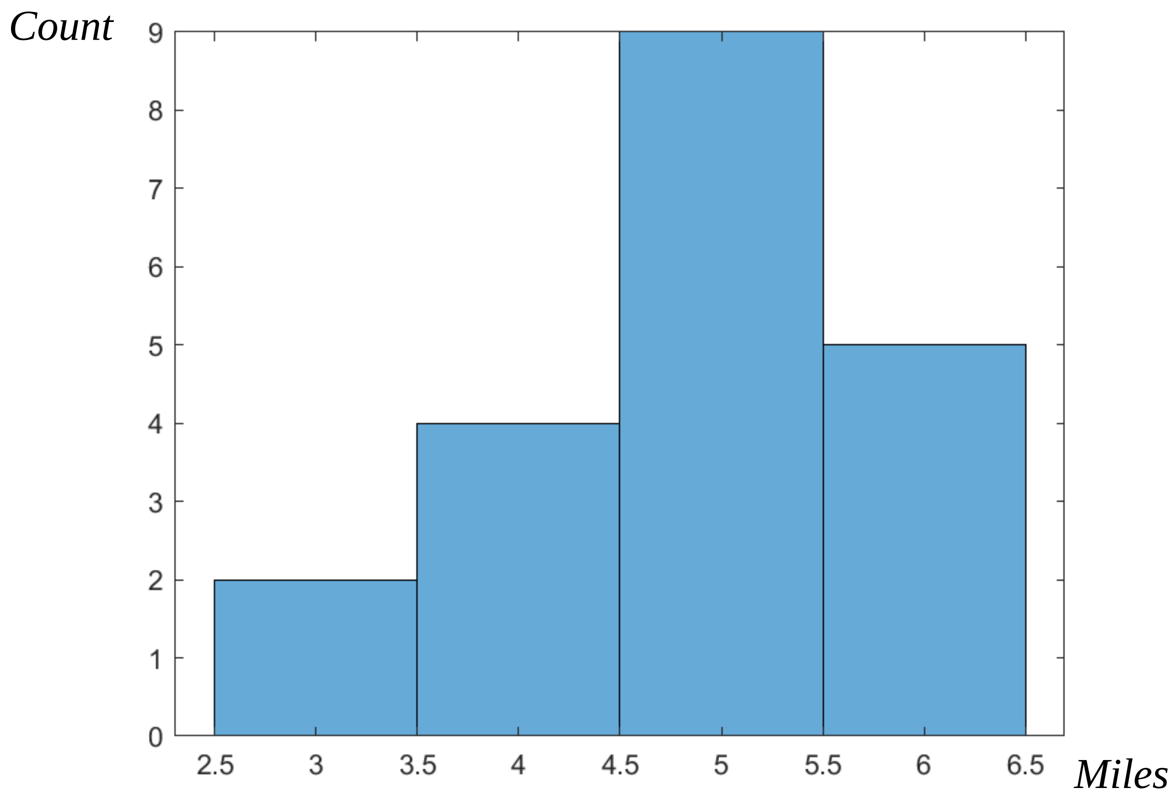
x_a = x value with area a larger than it from binomial, n, p=1/2.

$$a = \alpha \text{ or } \alpha/2$$

10.1 Introduction to Nonparametric Testing – Sign Test

Example: Mark is training for a 10K run. The last 20 days he recorded the number of miles he ran each day. A friend of his advised him that he should be averaging more than 4 miles per day. Test at the α=0.05 level whether the median number of daily miles is greater than 4.

data
5
3
5
3
4
4
6
6
6
4
6
5
5
5
4
5
5
5
5
6



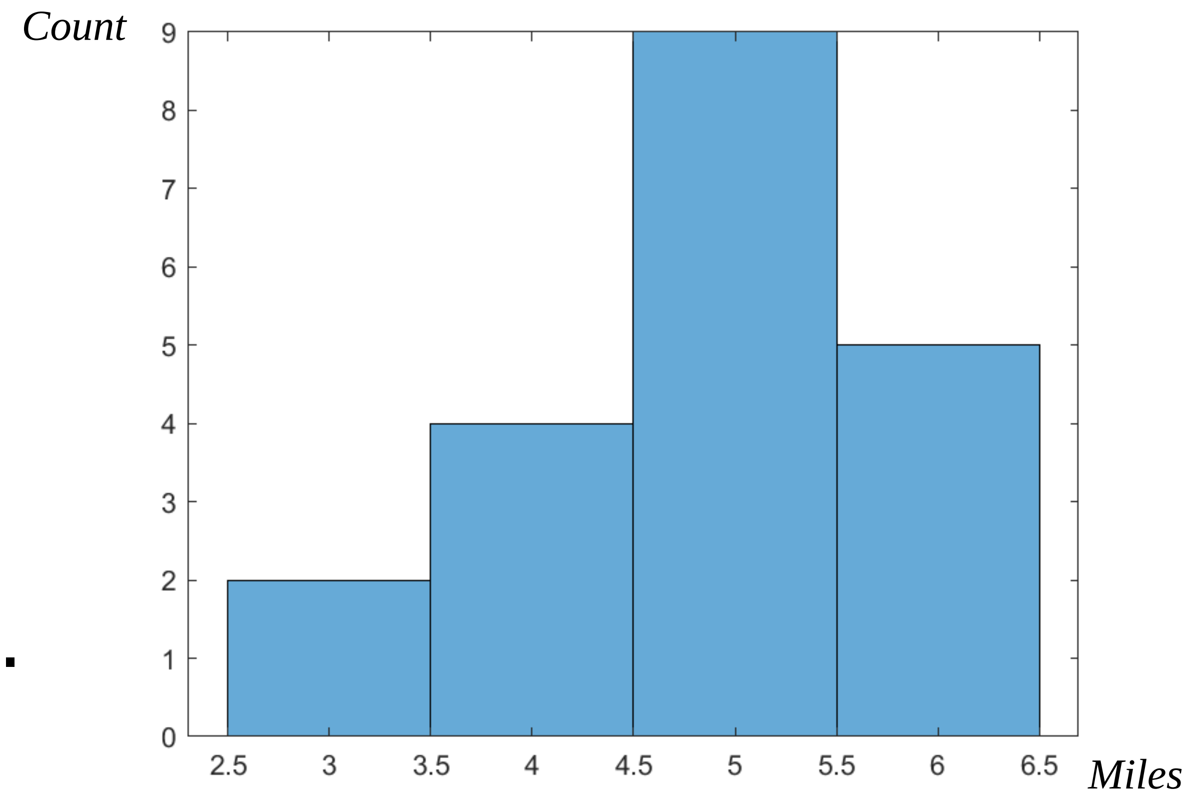
10.1 Introduction to Nonparametric Testing – Sign Test

The first thing you should do is to test if the data is consistent with a normal distribution. Test at the α=0.05 level.

```
install.packages('nortest')
library(nortest)

data <- c(5,3,5,3,4,4,6,6,6,4,6,5,5,5,4,5,5,5,5,6)
ad.test(data)
# if p-value < 0.05 data not normal

Anderson-Darling normality test
data:  data
A = 1.1415, p-value = 0.00418
```

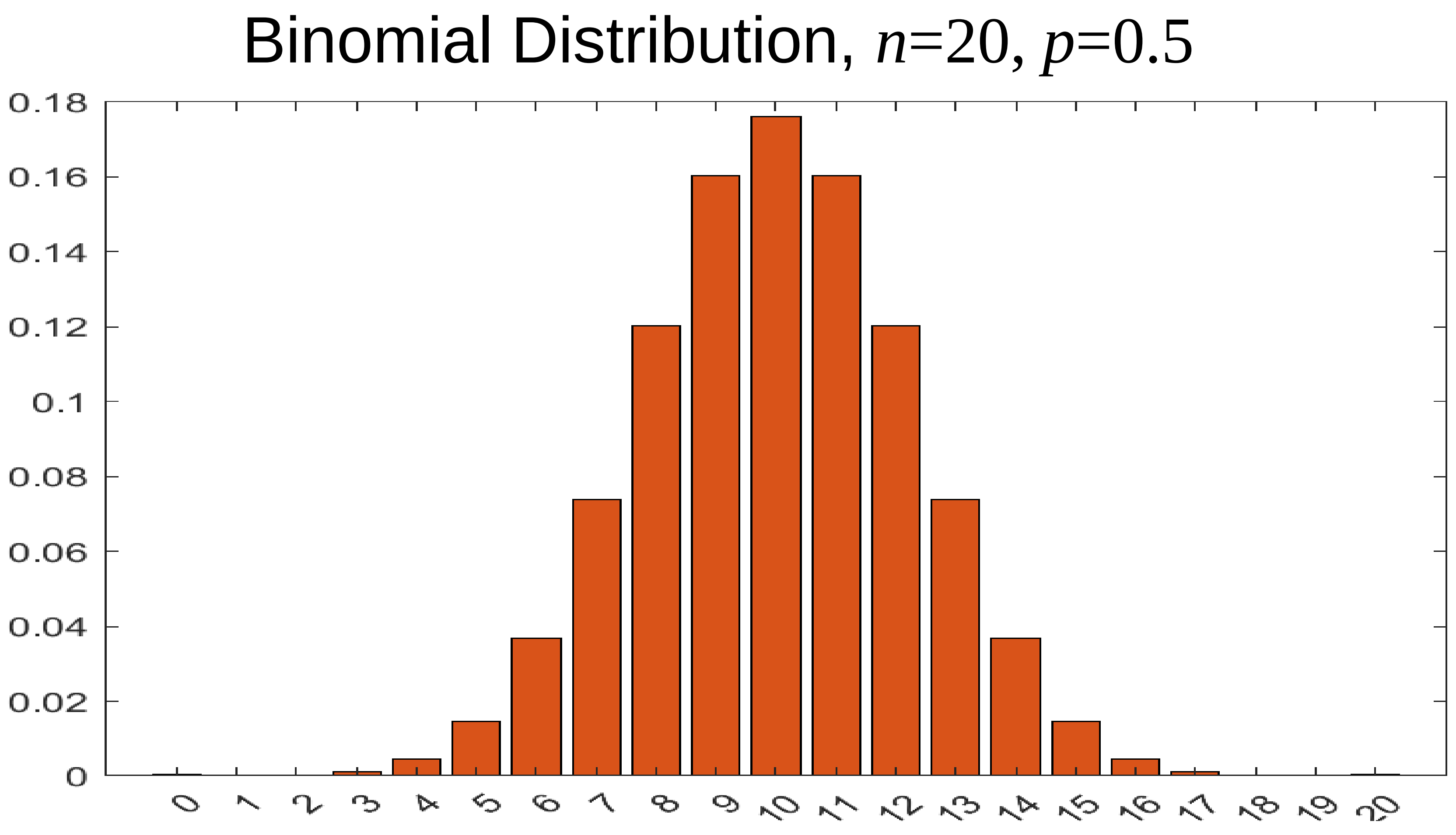


data
5
3
5
3
4
4
6
6
6
4
6
5
5
5
4
5
5
5
5
6

Because p-value < 0.05, not normal.

Can't use parametric t-test.

10.1 Introduction to Nonparametric Testing – Sign Test



$$P(X=x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

Value with 0.05 or 0.025 less than.

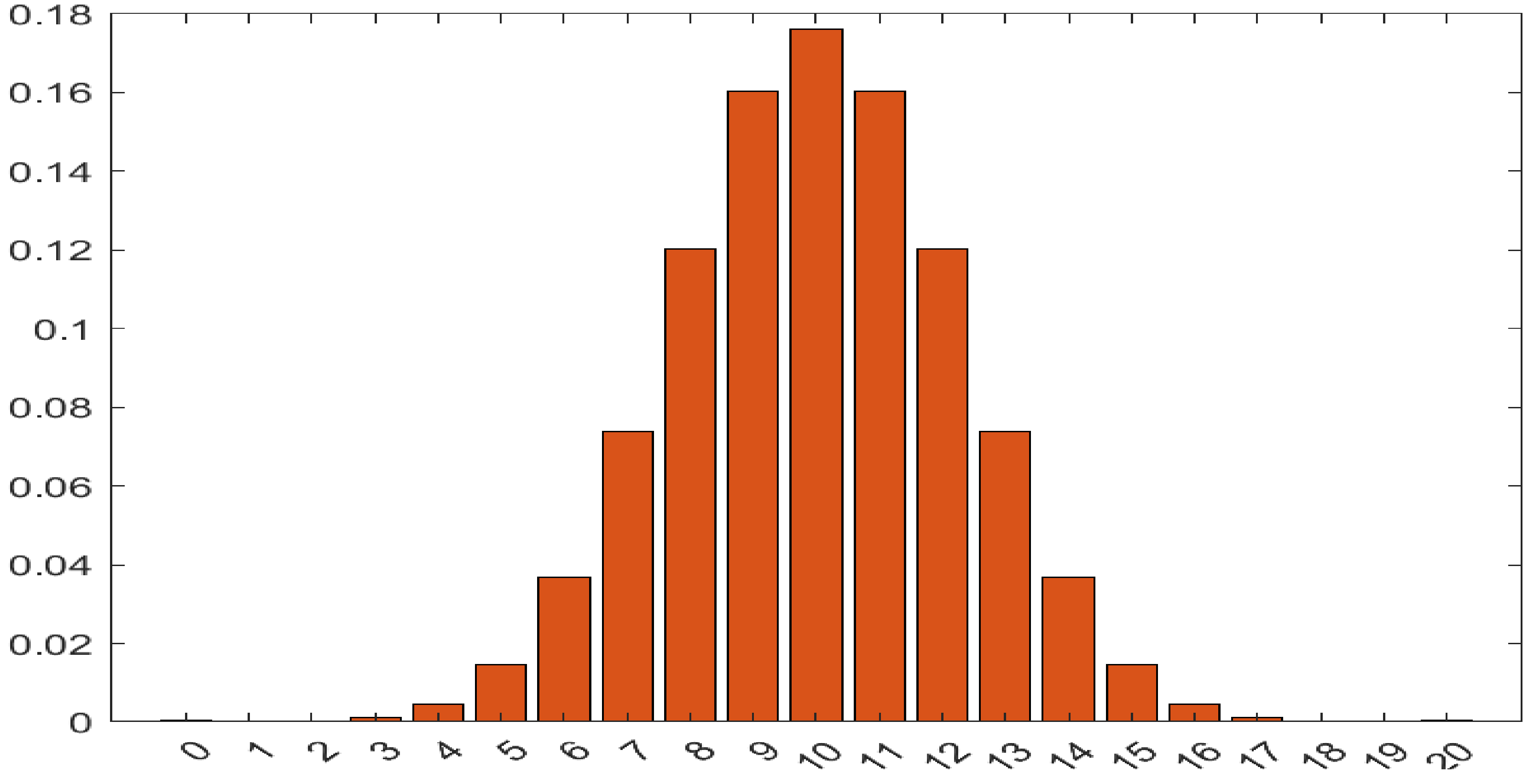
Value with 0.05 or 0.025 greater than.

x	P(X=x)	CumSum	CumSumR
0	0.000	0.000	1.000
1	0.000	0.000	1.000
2	0.000	0.000	1.000
3	0.001	0.001	1.000
4	0.005	0.006	0.999
5	0.015	**0.021**	0.994
6	0.037	0.058	0.979
7	0.074	0.132	0.942
8	0.120	0.252	0.868
9	0.160	0.412	0.748
10	0.176	0.588	0.588
11	0.160	0.748	0.412
12	0.120	0.868	0.252
13	0.074	0.942	0.132
14	0.037	0.979	0.058
15	0.015	0.994	**0.021**
16	0.005	0.999	0.006
17	0.001	1.000	0.001
18	0.000	1.000	0.000
19	0.000	1.000	0.000
20	0.000	1.000	0.000

See also Table 6

10.1 Introduction to Nonparametric Testing – Sign Test

Binomial Distribution, n=20, p=0.5



$$P(X=x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

Value with 0.05 or 0.025 greater than

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	.05	.025	.01	.005
n				
1				
2				
3				
4				
5	0			
6	0	0		
7	0	0	0	
8	1	0	0	0
9	1	1	0	0
10	1	1	0	0
11	2	1	1	0
12	2	2	1	1
13	3	2	1	1
14	3	2	2	1
15	3	3	2	2
16	4	3	2	2
17	4	4	3	2
18	5	4	3	3
19	5	4	4	3
20	5	5	4	3
21	6	5	4	4
22	6	5	5	4
23	7	6	5	4
24	7	6	5	5
25	7	7	6	5

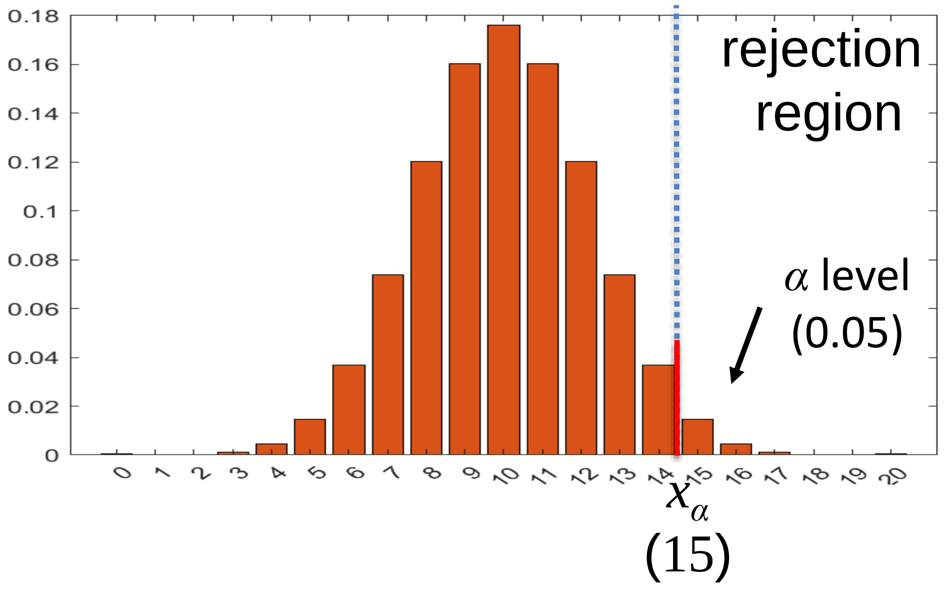
Table 6

10.1 Introduction to Nonparametric Testing – Sign Test

The hypothesis testing process consists of 5 Steps.

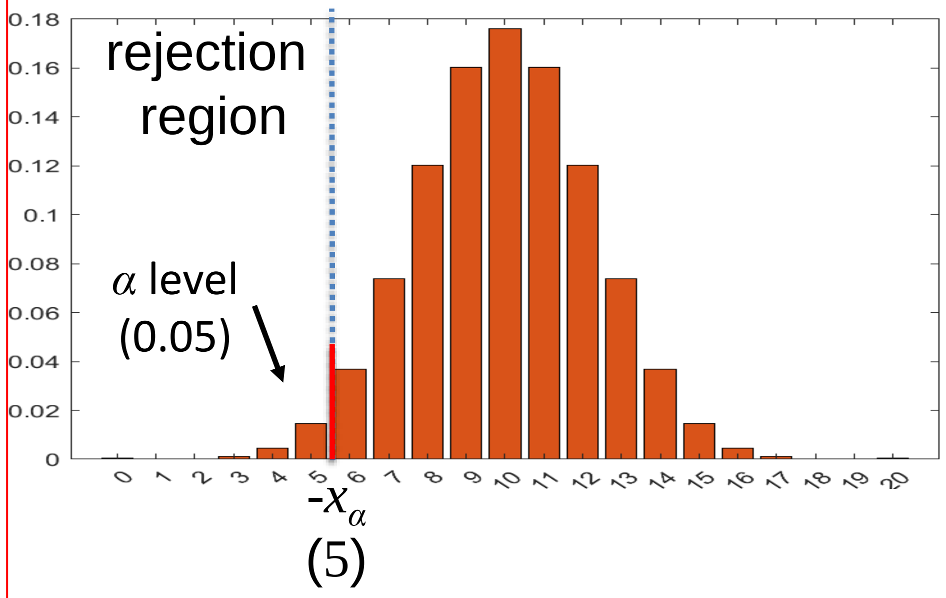
Step 3: Set-up the decision rule.

$\mathrm{H}_0{:}\,MD{=}4$ *vs*. $\mathrm{H}_1{:}\,MD{>}4$



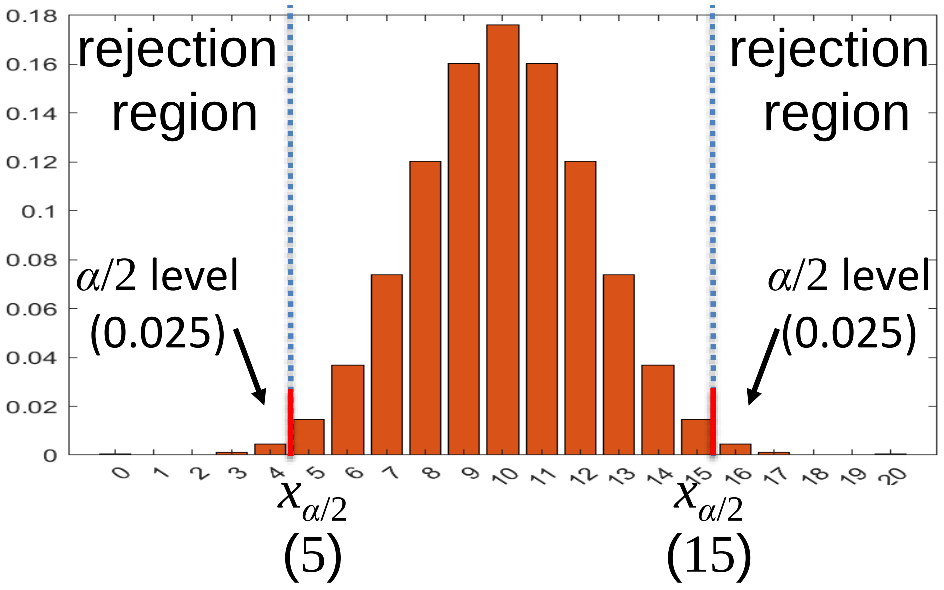
Reject H_0 if $P(X{\geq}x_\alpha) \leq \alpha$

$\mathrm{H}_0{:}\,MD{=}4$ *vs*. $\mathrm{H}_1{:}\,MD{<}4$



Reject H_0 if $P(X{\leq}x_\alpha) \leq \alpha$

$\mathrm{H}_0{:}\,MD{=}4$ *vs*. $\mathrm{H}_1{:}\,MD{\neq}4$



Reject H_0 $P(x_\alpha{\leq}X) \leq \dfrac{\alpha}{2}$

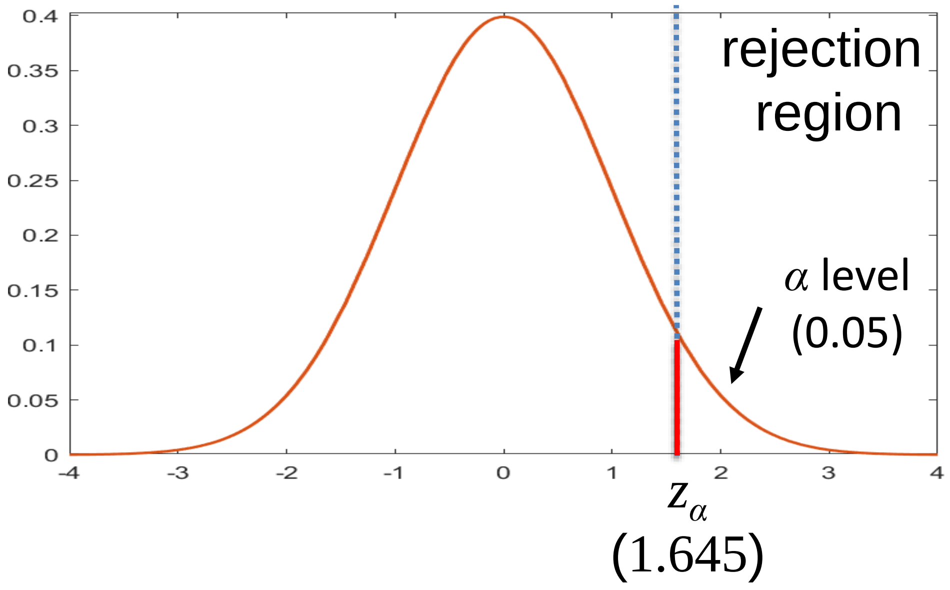
7.1 Introduction to Hypothesis Testing

RECALL

The hypothesis testing process consists of 5 Steps.

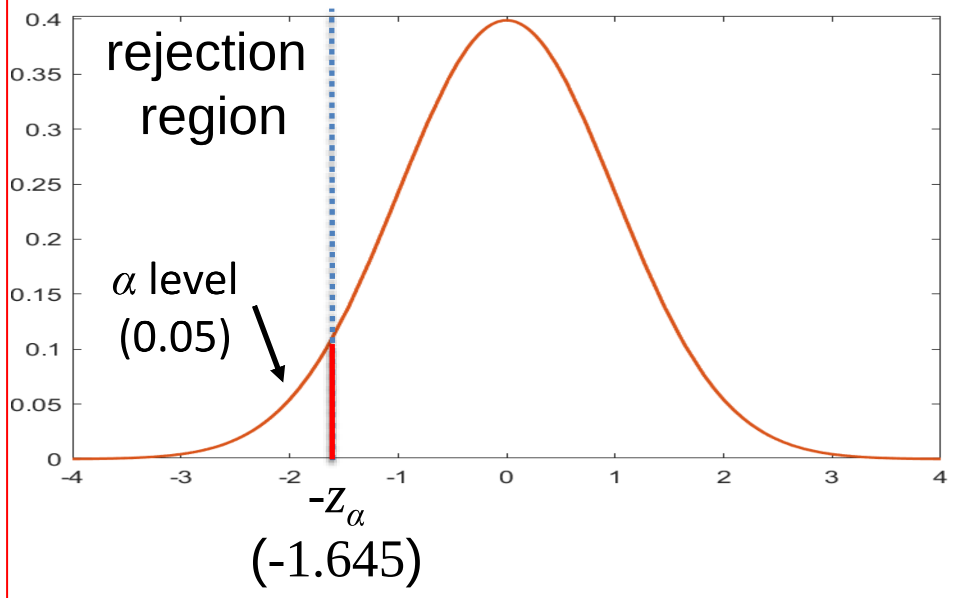
Step 3: Set-up the decision rule.

$H_0: \mu = \mu_0$ vs. $H_1: \mu > \mu_0$



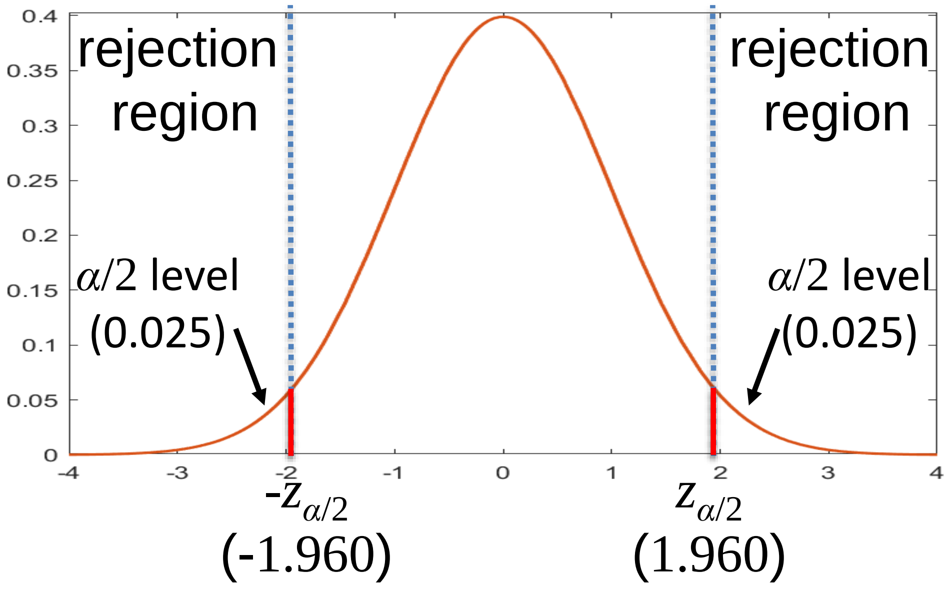
Reject H_0 if $z \geq z_\alpha$

$H_0: \mu = \mu_0$ vs. $H_1: \mu < \mu_0$



Reject H_0 if $z \leq z_\alpha$

$H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$



Reject H_0 $z \leq z_{\alpha/2}$ or $z \geq z_{\alpha/2}$

10.1 Introduction to Nonparametric Testing – Sign Test

Step 4: Compute the test statistic.

If value < MD_0, –.
If value = MD_0, 0.
If value > MD_0, +.

$$x = (\text{the number of observations} > MD_0{=}4)$$
$$x = 14$$

data	sorted	sign
5	3	-1
3	3	-1
5	4	0
3	4	0
4	4	0
4	4	0
6	5	+1
6	5	+1
6	5	+1
4	5	+ 1
6	5	+1
5	5	+1
5	5	+1
5	5	+1
4	5	+1
5	6	+1
5	6	+1
5	6	+1
5	6	+1
6	6	+1

Step 5: Because x=14 < x_α=15, do not reject H_0.

x	P(X=x)	CumSum	CumSumR
5	0.015	0.021	0.994
6	0.037	0.058	0.979
14	0.037	0.979	0.058
15	0.015	0.994	0.021

See also Table 6

Two-Sided Test α	.10	.05	.02	.01
One-Sided Test α	.05	.025	.01	.005
19	5	4	4	3
20	5	5	4	3
21	6	5	4	4

Table 6

Note:

If we used normal, we would reject H_0 , t=4.07>$t_{0.05,19}$=2.093.

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \quad df=n\text{-}1 \quad \bar{X} = 4.8500 \quad s = 0.9333$$

10.1 Introduction to Nonparametric Testing

In nonparametric hypothesis testing, we often focus on the order of data and use ranks, not the actual data values.

Data: 0 2 3 5 7 9 | Mean: $\bar{X} = 3.5$
Ranks: 1 2 3 4 5 6 | Sum: sum=21

Data: 0 2 3 7 7 9 | $\frac{4+5}{2}=4.5$ | Mean: $\bar{X} = 4.33$
Ranks: 1 2 3 4.5 4.5 6 | Sum: sum=21

Data: 0 2 3 7 7 7 | $\frac{4+5+6}{3}=5$ | Mean: $\bar{X} = 3.5$
Ranks: 1 2 3 5 5 5 | Sum: sum=21

In all cases the means are different but the sum of the ranks is $n(n+1)/2$.

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

We can test if two samples are likely from the same distribution.
Some interpret this as comparing he medians between two populations.

H_0: The two populations are equal
H_1: The two populations are not equal

We will go through the same 5 hypothesis steps

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

Example: Phase II clinical trial, n=10 children. Difference in episodes?

Step 1: Set up the hypotheses and determine α.

Group 1	Group 2
Placebo	NewDrug
7	3
5	6
6	4
4	2
12	10

$n_1 = 5 \quad n_2 = 5$

H_0: The two populations are equal

vs.

H_1: The two populations are not equal

α=0.05

Some interpret this as comparing the medians between two populations.

This test can be two sided or one sided.

i.e. H_0:MD_1=MD_2 vs. H_1:$MD_1 \neq MD_2$ or H_0:MD_1=MD_2 vs. H_1:$MD_1 > MD_2$.

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

Step 2: Select the appropriate test statistic.

Pool data and assign ranks. Test statistic based on ranks.

		Total Sample (Ordered Smallest to Largest)		Ranks	
Placebo	New Drug	Placebo	New Drug	Placebo	New Drug
7	3		1		1
5	6		2		2
6	4		3		3
4	2	4	4	4.5	4.5
12	1	5		6	
		6	6	7.5	7.5
		7		9	
		12		10	
				R_1=37	R_2=18

Ranks	
Placebo	New Drug
	1
	2
	3
4.5	4.5
6	
7.5	7.5
9	
10	

$$R=\frac{n(n+1)}{2}=55$$

If H_0 is true, we expect R_1=R_2.

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number summarizing information.

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1 = (5)(5) + \frac{5(5+1)}{2} - 37 = 3$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2 = (5)(5) + \frac{5(5+1)}{2} - 18 = 22$$

$$U = \min(U_1, U_2) = \min(3, 22) = 3$$

Reject H_0 for small U.

Rankings

Group 1	Group 2
	1
	2
	3
	4
	5
6	
7	
8	
9	
10	

$U = 0$

Group 1	Group 2
	1
2	
	3
4	
	5
6	
	7
8	
	9
10	

$U = 25$

complete separation

complete alternating

$$0 \le U \le n_1 n_2$$

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

Step 3: Set-up the decision rule.

$n_1=5$, $n_2=5$

If we did Two Sided Test

Reject H_0 if $U \leq U_{0.05,n_1,n_2}$.

Step 4: Compute test statistic.

Already done, $U=3$.

Step 5: Conclusion.

Do not reject H_0 because

$U=3>U_{0.05,5,5}=2$. Interpret.

Two-Sided Test $\alpha = 0.05$ $n_1 \leq n_2$

n_2 \ n_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2								0	0	0	0	1	1	1	1	1	2	2	2	2
3					0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
4				0	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	13
5			0	1	2	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20
6			1	2	3	5	6	8	10	11	13	14	16	17	19	21	22	24	25	27
7			1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
8		0	2	4	6	8	10	13	15	17	19	22	24	26	29	31	34	36	38	41
9		0	2	4	7	10	12	15	17	20	23	26	28	31	34	37	39	42	45	48
10		0	3	5	8	11	14	17	20	23	26	29	33	36	39	42	45	48	52	55
11		0	3	6	9	13	16	19	23	26	30	33	37	40	44	47	51	55	58	62
12		1	4	7	11	14	18	22	26	29	33	37	41	45	49	53	57	61	65	69
13		1	4	8	12	16	20	24	28	33	37	41	45	50	54	59	63	67	72	76
14		1	5	9	13	17	22	26	31	36	40	45	50	55	59	64	67	74	78	83
15		1	5	10	14	19	24	29	34	39	44	49	54	59	64	70	75	80	85	90
16		1	6	11	15	21	26	31	37	42	47	53	59	64	70	75	81	86	92	98
17		2	6	11	17	22	28	34	39	45	51	57	63	67	75	81	87	93	99	105
18		2	7	12	18	24	30	36	42	48	55	61	67	74	80	86	93	99	106	112
19		2	7	13	19	25	32	38	45	52	58	65	72	78	85	92	99	106	113	119
20		2	8	13	20	27	34	41	48	55	62	69	76	83	90	98	105	112	119	127

10.2 Tests with Two Independent Samples – Mann-Whitney U Test

Step 3: Set-up the decision rule.

n_1=5, n_2=5

If we did One Sided Test

Reject H_0 if $U \le U_{0.05,n_1,n_2}$.

Step 4: Compute test statistic.

Already done, U=3.

Step 5: Conclusion.

Reject H_0 because

U=3<$U_{0.05,5,5}$=4. Interpret.

One-Sided Test α = 0.05 $n_1 \le n_2$

	n_1																			
n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2					0	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4
3			0	0	1	2	2	3	3	4	5	5	6	7	7	8	9	9	10	11
4			0	1	2	3	4	5	6	7	8	9	10	11	12	14	15	16	17	18
5		0	1	2	4	5	6	8	9	11	12	13	15	16	18	19	20	22	23	25
6		0	2	3	5	7	8	10	12	14	16	17	19	21	23	25	26	28	30	32
7		0	2	4	6	8	11	13	15	17	19	21	24	26	28	30	33	35	37	39
8		1	3	5	8	10	13	15	18	20	23	26	28	31	33	36	39	41	44	47
9		1	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54
10		1	4	7	11	14	17	20	24	27	31	34	37	41	44	48	51	55	58	62
11		1	5	8	12	16	19	23	27	31	34	38	42	46	50	54	57	61	65	69
12		2	5	9	13	17	21	26	30	34	38	42	47	51	55	60	64	68	72	77
13		2	6	10	15	19	24	28	33	37	42	47	51	56	61	65	70	75	80	84
14		2	7	11	16	21	26	31	36	41	46	51	56	61	66	71	77	82	87	92
15		3	7	12	18	23	28	33	39	44	50	55	61	66	72	77	83	88	94	100
16		3	8	14	19	25	30	36	42	48	54	60	65	71	77	83	89	95	101	107
17		3	9	15	20	26	33	39	45	51	57	64	70	77	83	89	96	102	109	115
18		4	9	16	22	28	35	41	48	55	61	68	75	82	88	95	102	109	116	123
19	0	4	10	17	23	30	37	44	51	58	65	72	80	87	94	101	109	116	123	130
20	0	4	11	18	25	32	39	47	54	62	69	77	84	92	100	107	115	123	130	138

10.5 Summary

Sign Test (one sample)

x = number of observations > MD_0

Mann-Whitney U Test

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

$$U = \min(U_1, U_2)$$

Sign Test (two sample)

x = number of observations > 0

Wilcoxon Signed Rank Test

$W = \min(W+, W-)$

$W+$ = sum of positive ranks

$W-$ = sum of negative ranks

Kruskal-Wallis Test

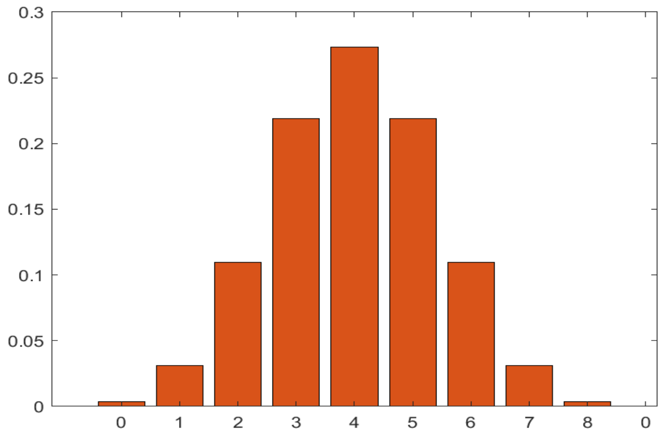
$$H = \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

Questions?

Homework 10

Read Chapter 10.

Problems * (below), 3 (Mann-Whitney U Test)



Binomial(n=8, p=1/2)

x	P(X=x)	P(X<=x)	P(X>=x)
0	0.0039	0.0039	1
1	0.0313	0.0352	0.9961
2	0.1094	0.1445	0.9648
3	0.2188	0.3633	0.8555
4	0.2734	0.6367	0.6367
5	0.2188	0.8555	0.3633
6	0.1094	0.9648	0.1445
7	0.0313	0.9961	0.0352
8	0.0039	1	0.0039

* A basketball team has played 8 games with scores
82, 78, 89, 84, 74, 91, 80, 77
Test H_0: MD=85 *vs*. H_1: MD<85 at the α=0.10 level.
One sample Sign Test. Go through the 5 steps.