Chapter 7: Hypothesis Testing Procedures A

Dr. Daniel B. Rowe
Professor of Computational Statistics
Department of Mathematical and Statistical Sciences
Marquette University

Hypothesis Testing

We make decisions every day in our lives.

Should I believe A or should I believe B (not A)?

Two Competing Hypotheses. A and B.

Null Hypothesis ($\mathbf{H_0}$): No difference, no association, or no effect.

Alternative Hypothesis ($\mathbf{H_1}$): Investigators belief.

The Alternative Hypothesis is always set up to be what you want to build up evidence to prove.

7.1 Introduction to Hypothesis Testing

$$\hat{p} = \frac{x}{n}$$

Example: Friend's Party.

H_0: The party will be boring.

vs.

H_1: The party will be fun.

I wish that every time I had to make a decision, I could calculate a measure and use this measure (test statistic) to decide what to do.

Maybe use $\hat{p}$ the sample proportion of fun parties friend has had?
I might believe the party will be fun if $\hat{p}$ is "large."

7.1 Introduction to Hypothesis Testing

$$\bar{X} = \frac{1}{n}\sum X$$

Example: Men's Weight.

$\leq$

H_0: The mean weight of men is equal to 191 lbs. $\mu = 191$ lbs

vs.

H_1: The mean weight of men is greater than 191 lbs. $\mu > 191$ lbs

I wish that every time I had to make a decision, I could calculate a measure and use this measure (test statistic) to decide what to do.

Maybe use $\bar{X}$ the sample mean weight of men?

I might believe men's mean weight ≥ 191 if $\bar{X}$ is "large."

7.1 Introduction to Hypothesis Testing

Example: H_0: $\mu = 191$ lbs *vs.* H_1: $\mu \geq 191$ lbs

To test the hypothesis, take a sample of n=100 men's weights.

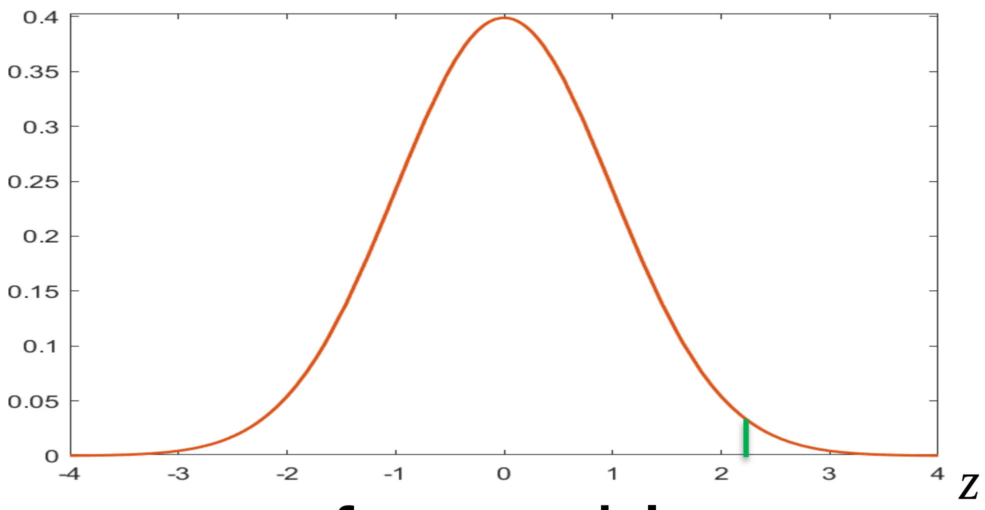
Suppose n=100 , $\bar{X} = 197.1$ lbs, and s=25.6 lbs.

Is 197.1 statistically larger than 191?

In hypothesis testing we assume H_0 is true, then see how likely $\bar{X}$ is.

$$P(\bar{X} > 197.1) = P\left(\frac{\bar{X} - 191}{25.6/\sqrt{100}} > \frac{197.1 - 191}{25.6/\sqrt{100}} \right)$$

$$P(z > 2.38) = 1 - 0.9913 = 0.0087$$ ← very unlikely



Assumed that $\bar{X}$ was normal and used z because n>30. 2.38 from table

7.1 Introduction to Hypothesis Testing

Example: H_0: $\mu = 191$ lbs *vs*. H_1: $\mu \geq 191$ lbs

To test the hypothesis, take a sample of n=100 men's weights.

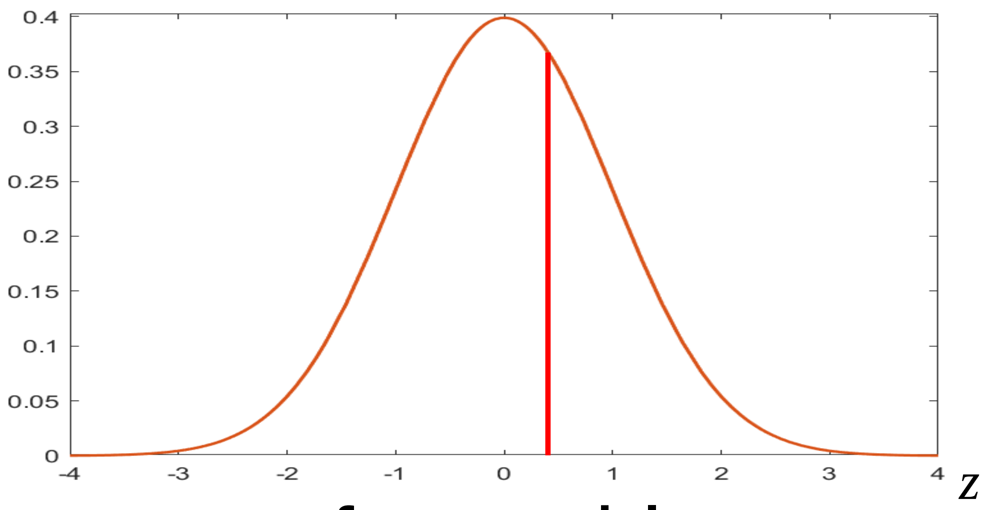
Suppose n=100 , $\bar{X} = 192.1$ lbs, and s=25.6 lbs.

Is 192.1 statistically larger than 191?

In hypothesis testing we assume H_0 is true, then see how likely $\bar{X}$ is.

$$P(\bar{X} > 192.1) = P\left(\frac{\bar{X} - 191}{25.6/\sqrt{100}} > \frac{192.1 - 191}{25.6/\sqrt{100}} \right)$$

$$P(z > 0.43) = 1 - 0.6664 = 0.3336$$ ← somewhat unlikely



Assumed that $\bar{X}$ was normal and used z because n>30. 0.43 from table

7.1 Introduction to Hypothesis Testing

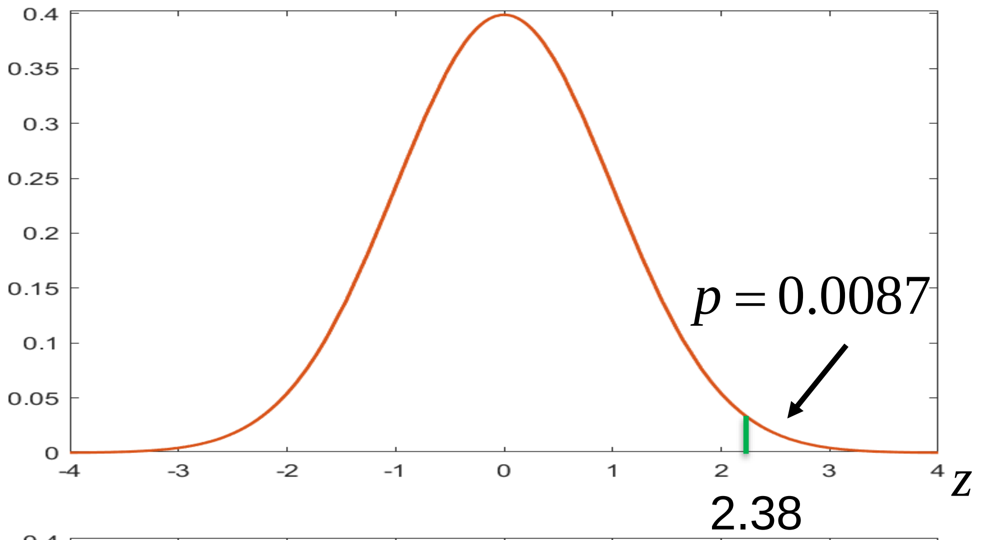
Where do we draw the line?

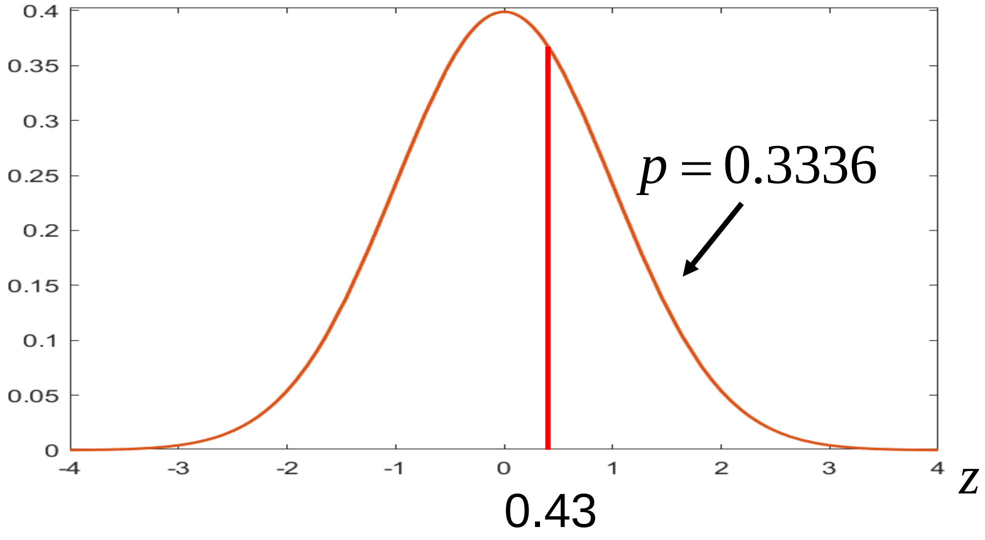
Suppose n=100 , $\bar{X}$ =197.1 lbs, and s=25.6 lbs.

$P(z > 2.38) = 1 - 0.9913 = 0.0087$ ← very unlikely

Suppose n=100 , $\bar{X}$ =192.1 lbs, and s=25.6 lbs.

$P(z > 0.43) = 1 - 0.6664 = 0.3336$ ← somewhat unlikely





We need a scientific way to select a cut-off α (probability) or z-*value* (critical value).

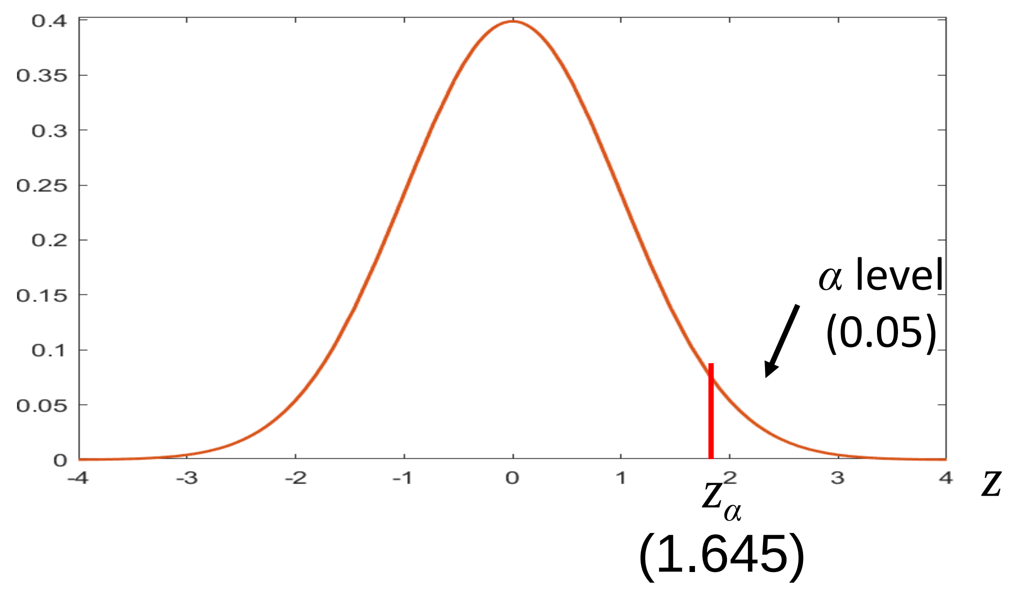
cut-off(s) called **critical value(s)** and depend on significance level α

7.1 Introduction to Hypothesis Testing

We will select a **Level of Significance** α.

$$\alpha = P(Reject\ \mathrm{H}_0 \,|\, \mathrm{H}_0\ is\ true)$$

Find z-$value$, z_α that corresponds to this α level.



Reject the Null Hypothesis H_0 in favor of the Alternative Hypothesis H_1

When z-$value > z_\alpha$ or p-$value < \alpha$.

This will be our scientific way to determine whether to believe the null hypothesis H_0 or alternative hypothesis H_1.

7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.

Step 2: Select the appropriate test statistic.

Step 3: Set-up the decision rule.

Step 4: Compute the test statistic.

Step 5: Conclusion.

7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.
State the null and the alternative hypotheses.
H_0: Null Hypothesis (no change, no difference)
vs.
H_1: Research Hypothesis (investigators belief, what we want to prove)

Select a level of significance α. $\alpha=0.05$

7.1 Introduction to Hypothesis Testing

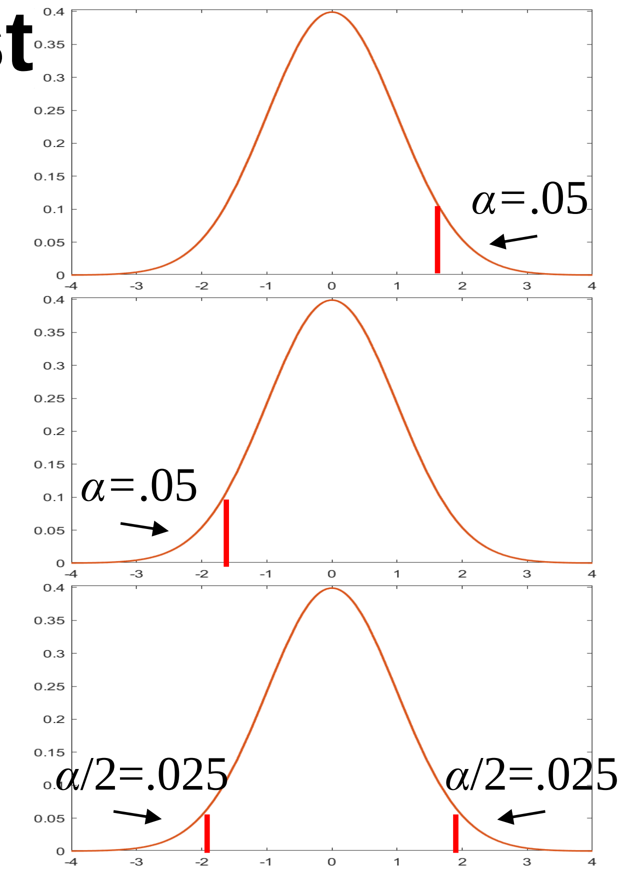
The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.
There are three possible pairs.

$\mathrm{H}_0: \mu = \mu_0$ *vs.* $\mathrm{H}_1: \mu > \mu_0$ (prove greater than, **upper tailed test**)
$\quad\quad \leq$ reject for “large" $\bar{X}$ or z’s

$\mathrm{H}_0: \mu = \mu_0$ *vs.* $\mathrm{H}_1: \mu < \mu_0$ (prove less than, **lower tailed test**)
$\quad\quad \geq$ reject for “small” $\bar{X}$ or z’s

$\mathrm{H}_0: \mu = \mu_0$ *vs.* $\mathrm{H}_1: \mu \neq \mu_0$ (prove not equal to, **two-tailed test**)
reject for “large” or “small” $\bar{X}$ or z’s



7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number.

n large

$$z = \frac{\bar{X} - \mu_0}{s / \sqrt{n}}$$

n small

$$t = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} \qquad df = n-1$$

Use the test statistic that depends on data and null hypothesis with a critical value z_a (or $t_{a,df}$) that depends on significance level α to make decision.

$a = \alpha$ or $\alpha/2$

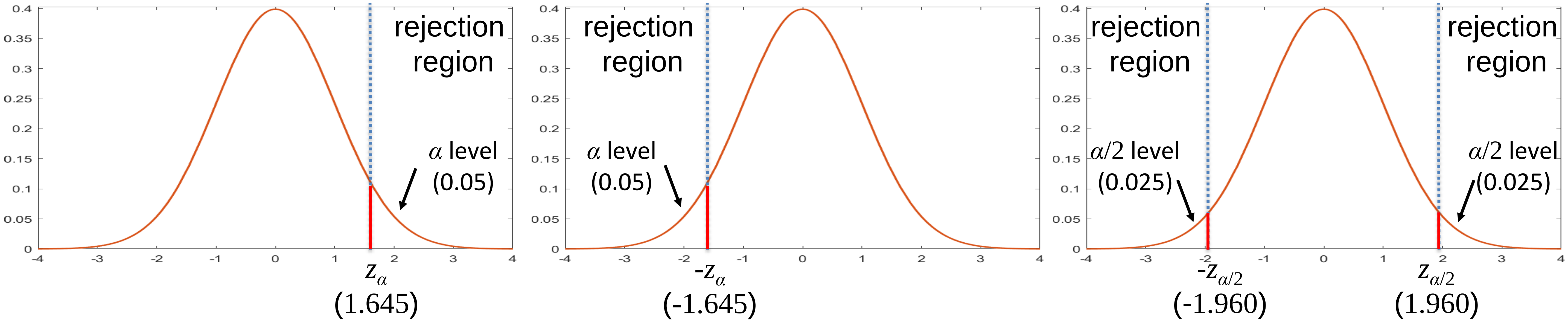
We will test hypotheses on various parameters with various test statistics.

7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

$H_0: \mu = \mu_0$ vs. $H_1: \mu > \mu_0$ $H_0: \mu = \mu_0$ vs. $H_1: \mu < \mu_0$ $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$



Reject H_0 if $z \geq z_\alpha$ Reject H_0 if $z \leq -z_\alpha$ Reject H_0 $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$

7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.

Use sample data $x_1,\ldots,x_n$ and hypothesized value μ_0 to compute z (or t).

Compare test statistic z (or t) to critical value(s) $z_{\alpha/2}$ (or $t_{\alpha/2,df}$) with rule.

Step 5: Conclusion.

Make a decision, reject H_0 or not to reject H_0.

Interpret the results.

7.1 Introduction to Hypothesis Testing

There are two types of error we can make.

Type I error rate.

$$\alpha = P(\text{Type I Error}) = P(\text{Reject } \mathrm{H}_0 | \mathrm{H}_0 \text{ is true})$$

Sometimes called the false positive rate.

Type II error rate.

$$\beta = P(\text{Type II Error}) = P(\text{Do Not Reject } \mathrm{H}_0 | \mathrm{H}_0 \text{ is false})$$

Sometimes called the false negative rate.

	H_0 True	H_0 False
Fail to Reject H_0	Correct Decision $(1\text{-}\alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1\text{-}\beta)$

7.2 tests with One Sample, Continuous Outcome

Covers same material as Section 7.1 but additional small sample test with t statistic.

TABLE 7-4	Test Statistic for Testing $H_0: \mu = \mu_0$	
$n \geq 30$	$z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$	(Find critical value in Table 1C)
$n < 30$	$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$	(Find critical value in Table 2, $df = n - 1$)

7.3 Tests with One Sample, Dichotomous Outcome

To test hypothesis on a proportion, we follow the same 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.

$H_0: p = p_0$ vs. $H_1: p > p_0$, $H_0: p = p_0$ vs. $H_1: p < p_0$, $H_0: p = p_0$ vs. $H_1: p \neq p_0$

Step 2: Select the appropriate test statistic.

Assume n is large. $z = (\hat{p} - p_0) \Big/ \sqrt{p_0(1 - p_0)/n}$ $\hat{p} = \dfrac{x}{n}$

Step 3: Set-up the decision rule.

Reject H_0 if $z \geq z_\alpha$, Reject H_0 if $z \leq z_\alpha$, Reject H_0 $z \geq z_{\alpha/2}$ or $z \leq z_{\alpha/2}$

Step 4: Compute the test statistic.

$$z = a\ number$$

Step 5: Conclusion.

Compare test statistic to critical value(s). Make a decision.

7.3 Tests with One Sample, Dichotomous Outcome

Example: Is proportion of children using dental service different from 0.86?

Step 1: Null and Alternative Hypotheses.

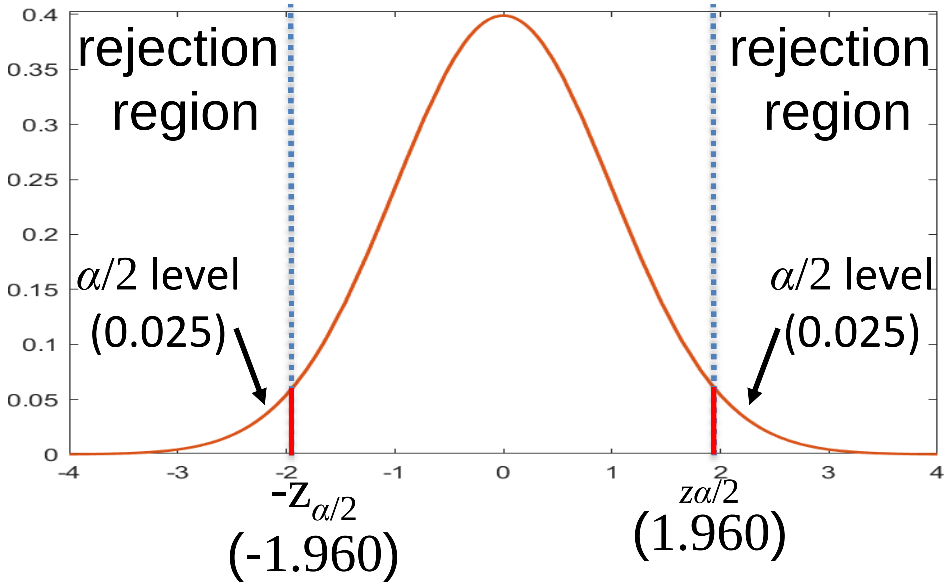
H_0: $p = 0.86$ *vs*. H_1: $p \neq 0.86$

Step 2: Test Statistic.

$$z = (\hat{p} - p_0) \Big/ \sqrt{p_0(1 - p_0)/n}$$

Step 3: Decision Rule. α=0.05

Reject H_0 if $z \leq$ -1.960 or $z \geq 1.960$.



Step 4: Compute test statistic. n=125, x=64, $\hat{p} = x/n = 0.512$.

$$z = (0.512 - 0.86) \Big/ \sqrt{0.86(1-0.86)/125} = -11.21$$

Step 5: Conclusion

Because $z \leq$ -1.96, reject and conclude proportion different from 0.86.

7.4 Tests with One Sample, Categorical and Ordinal Outcomes

There are cases with more than two Yes/No categories. Binomial

Assume that we have n items classified into one of k categories.

We have a hypothesis about the true proportions for each category.

We want to test to see if our hypothesis is correct, or something different.

We can do this with a scientific statistical hypothesis test.

7.4 Tests with One Sample, Categorical and Ordinal Outcomes

To test hypothesis on a proportion, we follow the same 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.

$H_0: p_1 = p_{01}, \ldots, p_k = p_{0k}$ vs. H_1: H_0 false (only one pair)

Step 2: Select the appropriate test statistic.

$$\chi^2 = \sum (O - E)^2 / E \qquad df = k\text{-}1 \qquad E_i = np_{0i}$$

Step 3: Set-up the decision rule.

Reject H_0 if $\chi^2 \geq \chi^2_{\alpha,df}$.

Step 4: Compute the test statistic.

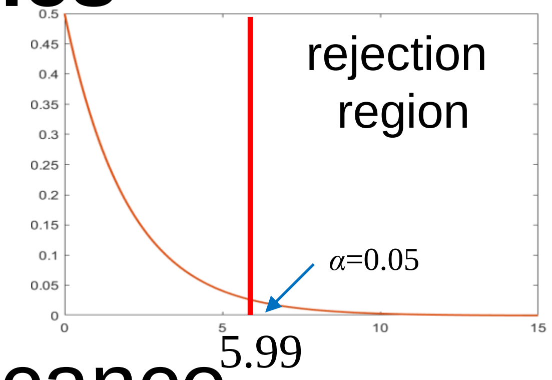
$$\chi^2 = a\ number$$

Step 5: Conclusion.

Compare test statistic to critical value. Make a decision.

7.4 Tests with One Sample, Categorical and Ordinal Outcomes

Book has df=1 figure.



Example: Health Survey. $n = 470$

Step 1: Set up the hypotheses and determine the level of significance.

H_0: $p_1 = 0.60$, $p_2 = 0.25$, $p_3 = 0.15$ *vs.* H_1: H_0: false (only one pair) $\alpha=0.05$

Step 2: Select the appropriate test statistic.

$$\chi^2 = \sum (O-E)^2 / E \qquad df=k\text{-}1 \qquad E_i = np_{0i}$$

Step 3: Set-up the decision rule.

Reject H_0 if $\chi^2 \geq \chi^2_{0.05,2}$=5.99. ← Table 3 next slide

	No Regular Exercise	Sporadic Exercise	Regular Exercise	Total
(O)	255	125	90	470
(E)	470(0.60) = 282	470(0.25) = 117.5	470(0.15) = 70.5	470

Step 4: Compute the test statistic.

$$\chi^2 = \frac{(255-282)^2}{282} + \frac{(125-177.5)^2}{177.5} + \frac{(90-70.5)^2}{70.5} = 8.46$$

Step 5: Conclusion.

Since χ^2=8.46 $\geq \chi^2_{0.05,2}$ =5.99, reject H_0 conclude p's not what we hypothesize.

7.4 Tests with One Sample, Categorical and Ordinal Outcomes

TABLE 3. Critical Values of the χ2 Distribution

Table entries represent values from χ^2 distribution with upper tail area equal to α.
$P(\chi^2_{df} > \chi^2) = \alpha$, e.g., $P(\chi^2_3 > 7.81) = 0.05$

	α				
df	.10	.05	.025	.01	.005
1	2.71	3.84	5.02	6.63	7.88
2	4.61	5.99	7.38	9.21	10.60
3	6.25	7.81	9.35	11.34	12.84
4	7.78	9.49	11.14	13.28	14.86
5	9.24	11.07	12.83	15.09	16.75
6	10.64	12.59	14.45	16.81	18.55
7	12.02	14.07	16.01	18.48	20.28
8	13.36	15.51	17.53	20.09	21.95
9	14.68	16.92	19.02	21.67	23.59
10	15.99	18.31	20.48	23.21	25.19

df	.10	.05	.025	.01	.005
11	17.28	19.68	21.92	24.72	26.76
12	18.55	21.03	23.34	26.22	28.30
13	19.81	22.36	24.74	27.69	29.82
14	21.06	23.68	26.12	29.14	31.32
15	22.31	25.00	27.49	30.58	32.80
16	23.54	26.30	28.85	32.00	34.27
17	24.77	27.59	30.19	33.41	35.72
18	25.99	28.87	31.53	34.81	37.16
19	27.20	30.14	32.85	36.19	38.58
20	28.41	31.41	34.17	37.57	40.00
21	29.62	32.67	35.48	38.93	41.40
22	30.81	33.92	36.78	40.29	42.80
23	32.01	35.17	38.08	41.64	44.18
24	33.20	36.42	39.36	42.98	45.56

Questions?

Homework 7

Read Chapter 7.

Problems # 4, *, 9

* A doctor believes that less than 20% of patients have a certain disease.
In a random sample of $n=100$ patients, $x=17$ had the disease.
Test the hypotheses $H_0: p \geq 0.20$ *vs*. $H_1: p < 0.20$ at $\alpha=0.025$.