Chapter 6: Confidence Interval Estimates

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6.1 Introduction to Estimation

A **Point Estimate** for a population parameter is a single-valued estimate of that parameter. i.e. X for μ or s^2 for σ^2 .

A **Confidence Interval** (CI) estimate is a range of values for a population parameter with a confidence attached (i.e., 95%).

A CI starts with the point estimate and builds in what is called the **Margin of Error**. The margin of error incorporates probabilities.

$$\overline{X} \pm that depends on a probability$$

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6.1 Introduction to Estimation

We will have confidence intervals for several scenarios.

Number of Samples	Outcome Variable	Parameter to be Estimated
One sample	Continuous	Mean: µ
Two independent samples	Continuous	Difference in means: $\mu_1 - \mu_2$
Two dependent, matched samples	Continuous	Mean difference : $\mu_d = \mu_1 - \mu_2$
One sample	Dichotomous	Proportion (e.g., prevalence, cu incidence): <i>p</i>
Two independent samples	Dichotomous	Difference or ratio of proportion attributable risk, relative risk, or

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Imulative

is (e.g., $: p_1 - p_2,$ dds ratio) p_1 / p_2



6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is "large" then X has a normal distribution. We can make probability statements. First, we assume σ is known.



Want to find a and b.



6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is "large" then X has a normal distribution. We convert from X to z and find our values for area (probability).



σ known P(c < z < d) = 0.95P(a < X < b) = 0.95

We get that c=-1.96 and d=+1.96.

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Z.



 σ known

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 $\overline{X} - 1.95 \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.95 \frac{\sigma}{\sqrt{n}}$

6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if *n* is "large" then \overline{X} has a normal distribution. Performing a little algebra.



from table

c = -1.96 d = 1.96





Ζ.



6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if *n* is "large" then *X* has a normal distribution.

Therefore a 95% CI for μ is $\overline{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$.

More generally, a $100(1-\alpha)\%$ CI for μ is:

 σ known

$$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$P(c < z < d) = 0.95$$



from table where z_{α} is the *z* value that has area of $\alpha/2$ larger. May see $z(\alpha/2)$.









6.2 Confidence Intervals for One Sample, **Continuous Outcome**

However, we never truly know what σ is, so we estimate it by s. But when we do this our distribution changes from normal-z to Student-t.



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6.2 Confidence Intervals for One Sample, Continuous Outcome

The Student-*t* distribution is like a superset of the normal distribution.



Confidence Level	80%	90%	95%	98%	99 %
Two-Sided Test α	.20	.10	.05	.02	.01
One-Sided Test a	.10	.05	.025	.01	.005
df					
1	3.078	6.314	12.71	31.82	63.66
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
7 30	1.310	1.697	2.042	2.457	2.750
$$ ∞	1.282	1.645	1.960	2.326	2.576

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6.2 Confidence Intervals for One Sample, Continuous Outcome

df = n-1

Example: Find the value of $t_{0.025,10}$.

The (critical) value of *t* that has an area of 0.025 larger than it when we have 10 degrees of freedom is 2.228.

This is the value we use for a 95% CI when α =0.05 and *n*=11.

Book says $n \ge 30$ use bottom *z* value.

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95%	98%	99 %
.05	.02	.01
.025	.01	.005
12.71	31.82	63.66
4.303	6.965	9.925
3.182	4.541	5.841
2.776	3.747	4.604
2.571	3.365	4.032
2.447	3.143	3.707
2.365	2.998	3.499
2.306	2.896	3.355
2.262	2.821	3.250
2.228	2.764	3.169
2.201	2.718	3.106
2.179	2.681	3.055
2.160	2.650	3.012
2.145	2.624	2.977
2.131	2.602	2.947
2.120	2.583	2.921
2.110	2.567	2.898
2.101	2.552	2.878
2.093	2.539	2.861
2.086	2.528	2.845
2.080	2.518	2.831
2.074	2.508	2.819
2.069	2.500	2.807
2.064	2.492	2.797
2.060	2.485	2.787
2.056	2.479	2.779
2.052	2.473	2.771
2.048	2.467	2.763
2.045	2.462	2.756
2.042	2.457	2.750
1.960	2.326	2.576

6.2 Confidence Intervals for One Sample, **Continuous Outcome**

 $100(1-\alpha)\%$

Example: Suppose we wish to compute a 95% CI for true systolic BP. A random sample of n=10 is take with sample mean $\overline{X}=121.2$ mm Hg and sample standard deviation s=11.1 mm Hg.



n=10, df=9 and $\alpha=0.05$



6.2 Confidence Intervals for One Sample, **Continuous Outcome**

Example: Suppose we wish to compute a 95% CI for true systolic BP. A random sample of n=10 is take with sample mean $\overline{X}=121.2$ mm Hg and sample standard deviation s=11.1 mm Hg.

The equation (when σ unknown) is $\overline{X} \pm t_{\frac{\alpha}{2}, df} \frac{s}{\sqrt{n}}$, df=n-1. We find the critical t value in the table We find the critical *t* value in the table.

Down to row *df*=9 and over to column CI=95% (or two side-test α =0.05, or one side-test α =0.025).

$$121.2 \pm (2.262) \frac{11.1}{\sqrt{10}} \rightarrow 113.3 \text{ to } 129.1$$
 mm Hg

			¥		
Confidence Level	80%	90%	95%	98%	99 %
Two-Sided Test α	.20	.10	.05	.02	.01
One-Sided Test a	.10	.05	.025	.01	.005
df					
1	3.078	6.314	12.71	31.82	63.66
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→ 9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169



 $100(1-\alpha)\%$

n=10, df=9 and $\alpha=0.05$



6.3 Confidence Intervals for One Sample, **Dichotomous Outcome**

Dichotomous when we have Yes/No or Heads/Tails or 1/0 outcomes.

Think of data	Yes	Νο	Yes	Yes	Νο	Yes
	Heads	Tails	Heads	Heads	Tails	Heads
	1	0	1	1	0	1

If we estimate the probability p of Yes or Heads or 1 as $\hat{p} = \frac{x}{-}$ where x is the number of Yes or Heads or 1. n

We think of this as an average $\hat{p} = \frac{1+0+1+1+0+1+0}{7}$

If *n* is "large," then the central limit theorem applies.







The Central Limit Theorem (CLT) says, that if *n* is large, i.e. n>30, then the average has an approximately normal distribution with mean μ and standard deviation σ/\sqrt{n} no matter what original distribution the data x_1, \ldots, x_n came from.



6.3 Confidence Intervals for One Sample, Dichotomous Outcome

Going through the same process as for μ we will find that:

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \text{or} \quad \hat{p} - z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

This CI only applies when *n* is large and $\min[n\hat{p}, n(1-\hat{p})] \ge 5$.

We use the *z* table as we did for CI for μ when *n* large.

No small *n* CI here. Do not use Student-*t*.







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6.3 Confidence Intervals for One Sample, **Dichotomous Outcome**

Example: A sample of *n*=3532 hypertensive patients were examined and it was found that x=1219 were treated. Generate a 95% CI for p.



 0.345 ± 0.016

0.320 < *p* < 0.361

A 95% CI for the probability a randomly selected person is treated.





6.4 Confidence Intervals for Two Independent Samples, **Continuous Outcome**

We often have two independent samples,

 x_1, \ldots, x_{n_1} from population 1 with mean μ_1 and standard deviation σ_1 and x_1, \ldots, x_{n_2} from population 2 with mean μ_2 and standard deviation σ_2 .

Because of the central limit theorem, if n_1 and n_2 are "large," then \overline{X}_1 and \overline{X}_2 have normal distributions with means $\mu_{\overline{X}_1} = \mu_1$ and $\mu_{\overline{X}_2} = \mu_2$ and standard deviations $\sigma_{\bar{X}_1} = \sigma_1 / \sqrt{n_1}$ and $\sigma_{\bar{X}_2} = \sigma_2 / \sqrt{n_2}$.





6.4 Confidence Intervals for Two Independent Samples, **Continuous Outcome**

We can go through the same process as for one sample. Therefore a 95% CI for μ_1 - μ_2 is

$$(\bar{X}_{1} - \bar{X}_{2}) \pm t_{\alpha} S_{P} \sqrt{\frac{1}{n_{1}} + \frac{1}{n_{2}}} \text{ for } n_{1}, n_{2} \leq 30 \qquad df = n_{1} + n_{2} - 2$$

from t table

$$(\bar{X}_1 - \bar{X}_2) \pm z_{\alpha} S_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$
 for $n_1, n_2 \ge 30$.

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The Central Limit Theorem (CLT) says, that if *n* is large, i.e. n>30, then the average has an approximately normal distribution with mean μ and standard deviation σ/\sqrt{n} no matter what original distribution the data x_1, \ldots, x_n came from.

6.4 Confidence Intervals for Two Independent Samples, **Continuous Outcome**

Example: A sample of *n*=10 males and females had systolic blood pressure measured. The data are: males: $n_1=6$, $\overline{X}_1=117.5$ mm Hg $s_1=9.7$ mm Hg and females $n_2=4, X_2=126.8 \text{ mm Hg}, s_2=12 \text{ mm Hg}.$ Generate a 95% CI for μ_1 - μ_2 .

$$\overline{X}_{1} - \overline{X}_{2} \pm t_{\frac{\alpha}{2}, df} S_{P} \sqrt{\frac{1}{n_{1}} + \frac{1}{n_{2}}}, \quad df = n_{1} + n_{2} - 2, \quad S_{P} = \sqrt{\frac{(n_{1} - 1)(s_{1})^{2} + (n_{2} - 1)(s_{2} - 1)(s$$

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 $S_P^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$



Confidence Level	80%	90%	95%
Two-Sided Test α	.20	.10	.05
One-Sided Test a	.10	.05	.025
df			
1	3.078	6.314	12.71
2	1.886	2.920	4.303
3	1.638	2.353	3.182
4	1.533	2.132	2.776
5	1.476	2.015	2.571
6	1.440	1.943	2.447
7	1.415	1.895	2.365
8	1.397	1.860	2.306

5.08 to 6.48 mm Hg

6.4 Confidence Intervals for Two Dependent Samples, **Continuous Outcome**

We often encounter two samples where there are matched pairs. This is often the case for before vs. after, twins, couples, etc. We subtract x_1 from sample 1 and x_2 from sample 2 for each pair.

The differences are labeled generically $d=x_1-x_2$ and so the sample of differences is d_1, \ldots, d_n . Because of the central limit theorem, \overline{X}_A has a mean of μ_d and standard deviation of σ_d/\sqrt{n} .

Once we have these differences we treat them exactly the same as we did in Section 6.2 CIs for One Sample, Continuous Outcome.



 $\overline{X}_d \pm t_{\underline{\alpha}, df}$ -



6.4 Confidence Intervals for Two Independent Samples, Continuous Outcome

Example: Difference in Systolic blood pressure between two visits.

 $\overline{X}_{d} \pm t_{\frac{\alpha}{2}, df} \frac{s_{d}}{\sqrt{n}}$ Compute a 95% Cl. $\overline{X}_{d} = \frac{-79.0}{15} = -5.3 \text{ mm Hg}$ $s_{d} = \sqrt{\frac{2296.95}{15-1}} = \sqrt{164.07} = 12.8 \text{ mm Hg}$

 $-5.3 \pm (2.145) \frac{12.8}{\sqrt{15}} \longrightarrow -5.3 \pm 7.1 \longrightarrow -12.4 \ to \ 1.8 \ mm \ Hg$

Subject ntification Number	Examination 6	Examination 7	Difference	Difference – X _d	(Difference – \overline{X}_d)²
1	168	141	-27	-21.7	470.89
2	111	119	8	13.3	176.89
3	139	122	-17	-11.7	136.89
4	127	127	0	5.3	28.09
5	155	125	-30	-24.7	610.09
6	115	123	8	13.3	176.89
7	125	113	-12	-6.7	44.89
8	123	106	-17	-11.7	136.89
9	130	131	1	6.3	39.69
10	137	142	5	10.3	106.09
11	130	131	1	6.3	39.69
12	129	135	6	11.3	127.69
13	112	119	7	12.3	151.29
14	141	130	-11	-5.7	32.49
15	122	121	-1	4.3	18.49
			-79.0	0.5	2296.95

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Confidence Level	80%	90%	95%
Two-Sided Test α	.20	.10	.05
One-Sided Test α	.10	.05	.025
df			
1	3.078	6.314	12.71
2	1.886	2.920	4.303
13	1.350	1.771	2.160
14	1.345	1.761	2.145
15	1.341	1.753	2.131

6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

The CI for a difference in proportions, risk difference is:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

We go through the same process.





6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

The CI for the natural log of relative risk, ln(RR) is:

$$ln(RR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{(n_1 - X_1) / X_1}{n_1} + \frac{(n_2 - X_2) / X_2}{n_2}}$$

CI for relative risk (RR) is:

exp(Lower Limit), exp(Upper Limit)

We go through the same process.





$RR = \frac{\hat{p}_1}{\hat{p}_2}$

6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

CI for the natural log of odds ratio, ln(OR) is:

$$ln(OR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{X_1} + \frac{1}{n_1 - X_1} + \frac{1}{X_2} + \frac{1}{n_2 - X_2}}$$

CI for odds ratio, OR is:

exp(Lower Limit), exp(Upper Limit)

We go through the same process.



$OR = \frac{\hat{p}_1 / (1 - \hat{p}_1)}{\hat{p}_2 / (1 - \hat{p}_2)}$



6.7 Summary

Number of Groups, Outcome: Parameter	Confidence Interval, n<30	Confidence Interval, n≥30
One sample, continuous: CI for μ	$\overline{X} \pm t_{\frac{\alpha}{2}, df} \frac{s}{\sqrt{n}}$	$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$
One sample, dichotomous: CI for <i>p</i>	(Not taught in this class.)	$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Two independent samples, continuous: CI for μ_1 - μ_2	$(\overline{X}_{1} - \overline{X}_{2}) \pm t_{\frac{\alpha}{2}, df} S_{P} \sqrt{\frac{1}{n_{1}} + \frac{1}{n_{2}}}$ $S_{P} = \sqrt{\frac{(n_{1} - 1)s_{1}^{2} + (n_{2} - 1)s_{2}^{2}}{n_{1} + n_{2} - 2}}$ $df = n_{1} + n_{2} - 2$	$(\overline{X}_{1} - \overline{X}_{2}) \pm z_{\frac{\alpha}{2}} S_{P} \sqrt{\frac{1}{n_{1}} + \frac{1}{n_{2}}}$ $S_{P} = \sqrt{\frac{(n_{1} - 1)s_{1}^{2} + (n_{2} - 1)s_{2}^{2}}{n_{1} + n_{2} - 2}}$
Two matched samples, continuous: CI for $\mu_d = \mu_1 - \mu_2$	$\overline{X}_d \pm t_{\frac{\alpha}{2}, df} \frac{s_d}{\sqrt{n}}$	$\overline{X}_d \pm z_{\frac{\alpha}{2}} \frac{s_d}{\sqrt{n}}$
Two independent samples, dichotomous: CI for $RD=(p_1-p_2)$	(Not taught in this class.)	$(\hat{p}_1 - \hat{p}_2) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$
CI for $ln(RR)=ln(p_1/p_2)$	(Not taught in this class.)	$\ln(RR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{(n_1 - X_1) / X_1}{n_1} + \frac{(n_2 - X_2) / X_2}{n_2}}$
CI for $RR=p_1/p_2$	(Not taught in this class.)	<pre>exp(Lower Limit), exp(Upper Limit)</pre>
CI for $ln(OR) =$ $ln([p_1/(1-p_1)]/[p_2/(1-p_2)])$	(Not taught in this class.)	$ln(OR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{X_1} + \frac{1}{n_1 - X_1} + \frac{1}{X_2} + \frac{1}{n_2 - X_2}}$
CI for $OR = [p_1/(1-p_1)]/[p_2/(1-p_2)]$	(Not taught in this class.)	exp(Lower Limit), exp(Upper Limit)

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Questions?







Homework 6

Read Chapter 6.

Problems # 1, 3, 5, 7





