

# Chapter 10: Nonparametric Tests II

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# Nonparametric Testing

The hypothesis tests that we learned assume the data (observations) come from a statistical distribution such as the normal distribution or we assume a large sample size. These hypothesis tests are called *parametric tests*. Distributions have parameters such as  $\mu$  and  $\sigma$ .

However, sometimes our data does not come from the normal distribution or we have a small sample size and we need to resort to alternative distribution free tests called *nonparametric tests*.

# 7.6 Tests with Matched Samples, Continuous Outcome

## RECALL

The hypothesis testing process consists of 5 Steps.

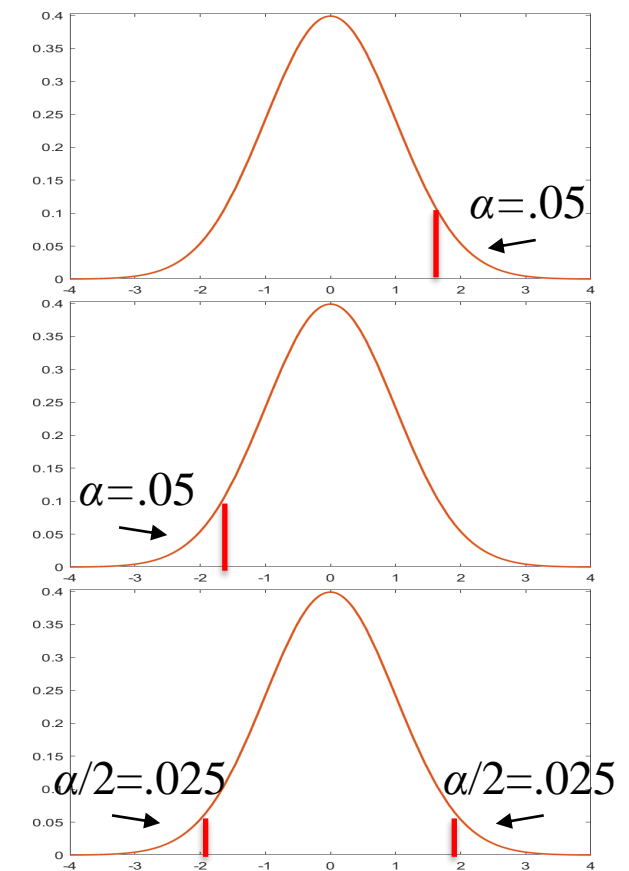
**Step 1:** Set up the hypotheses and determine the level of significance  $\alpha$ .

There are three possible pairs.

$H_0: \mu_d = 0$  vs.  $H_1: \mu_d > 0$  (prove greater than)  
 $\leq$  reject for “large”  $\bar{X}_d$  or  $z$ 's

$H_0: \mu_d = 0$  vs.  $H_1: \mu_d < 0$  (prove less than)  
 $\geq$  reject for “small”  $\bar{X}_d$  or  $z$ 's

$H_0: \mu_d = 0$  vs.  $H_1: \mu_d \neq 0$  (prove not equal to)  
 reject for “large” or “small”  $\bar{X}_d$  or  $z$ 's



## 10.3 Tests with Matched Samples – Sign Test

We can test if matched samples are likely from the same distribution.  
Some interpret this as comparing the medians between two populations.

$H_0$ : The median difference is zero ( $H_0: \delta=0$ )

$\delta$  is population version of  $d=x_2-x_1$ .

$H_1$ : The median difference is positive ( $H_1: \delta>0$ )

If the median difference of the matched pairs is zero, then half of the time they should be positive and half of the time negative.

If the median difference is greater than zero, then there should be more positives than negatives.

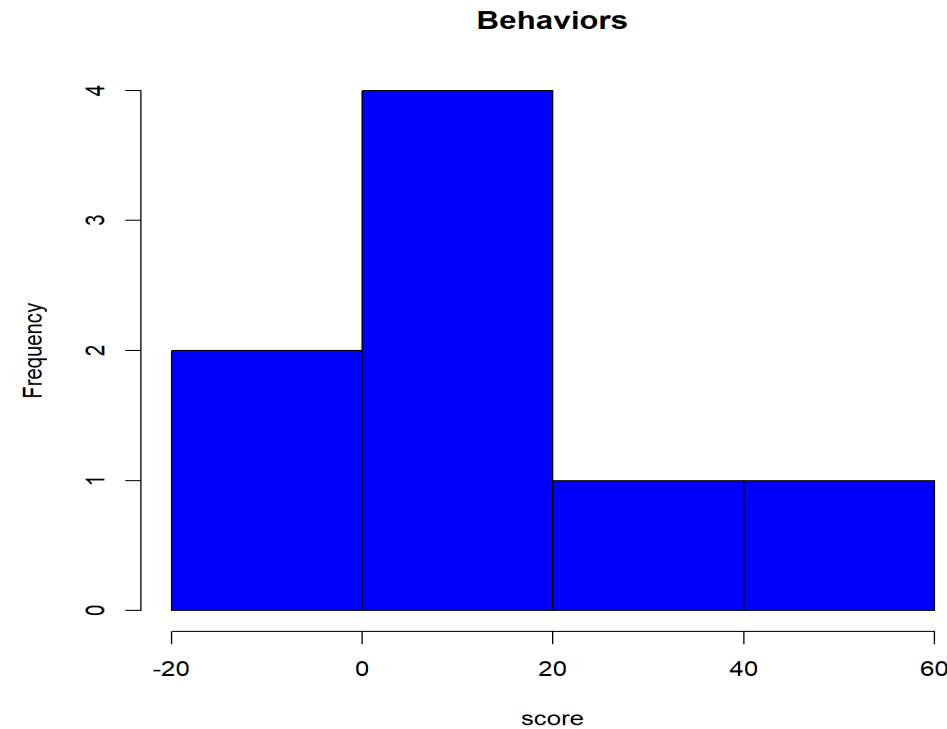
Sign Test.

# 10.3 Tests with Matched Samples – Sign Test

**Example:** Effectiveness of new drug designed to reduce repetitive behavior.

A total of  $n=8$  children in the study. Observed for 3 hours before and after treatment. Test at the  $\alpha=0.05$  level whether the median difference in repetitive behaviors is positive.

b	a	d
85	75	10
70	50	20
40	50	-10
65	40	25
80	20	60
75	65	10
55	40	15
20	25	-5



## 10.3 Tests with Matched Samples – Sign Test

The first thing you should do is to test if the data is consistent with a normal distribution. Test at the  $\alpha=0.05$  level.

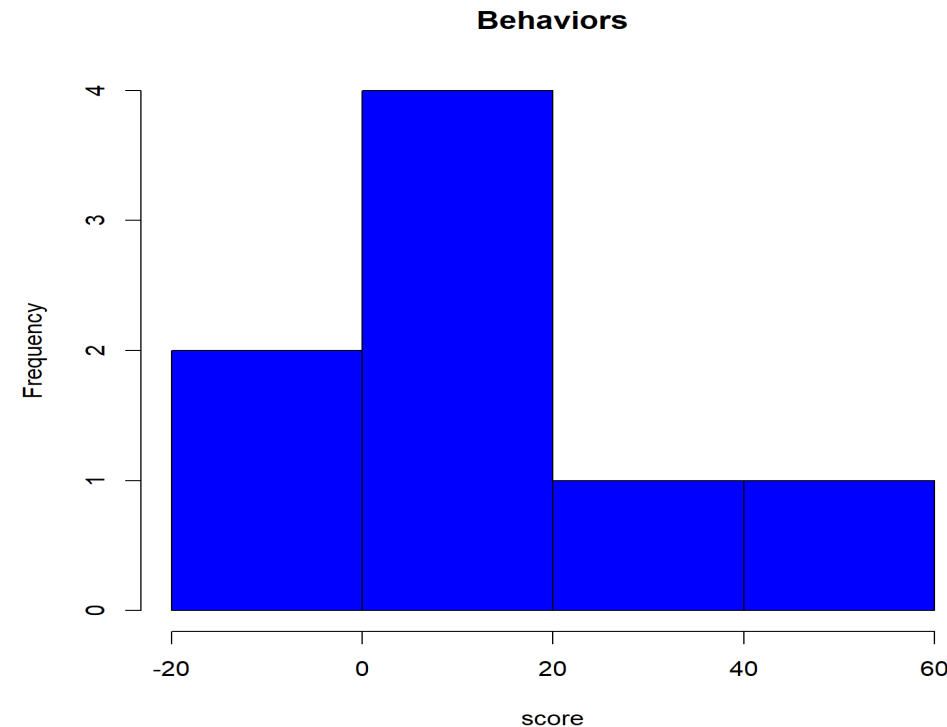
```
install.packages('nortest')
library(nortest)

before <- c(85,70,40,65,80,75,55,20)
after  <- c(75,50,50,40,20,65,40,25)
difference <- before - after
hist(difference,col='blue',main='Behaviors',xlab='score'))
ad.test(difference)
# if p-value < 0.05 data not normal
```

Anderson-Darling normality test  
 data: difference  
 A = 0.39956, p-value = 0.2744

Because  $p\text{-value} > 0.05$ , normal.  
 But let's use nonparametric.

b	a	d
85	75	10
70	50	20
40	50	-10
65	40	25
80	20	60
75	65	10
55	40	15
20	25	-5



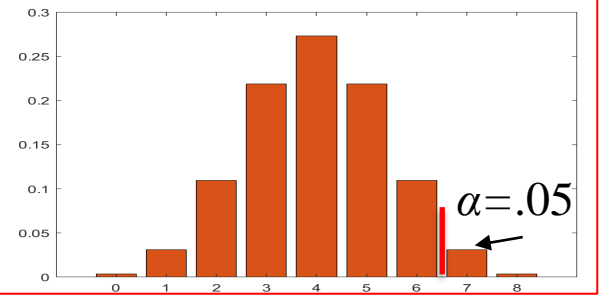
# 10.3 Tests with Matched Samples – Sign Test

**Sign Test** for median difference ( $\delta$ ), nonparametric version of  $t$ -test.

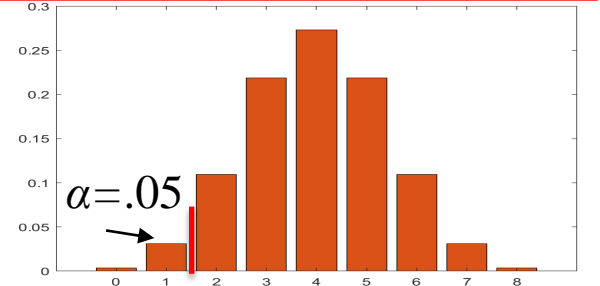
**Step 1:** Set up the hypotheses and determine the level of significance. There are three possible pairs.

For our example.

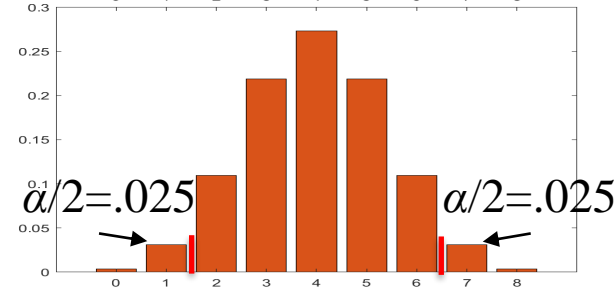
$H_0: \delta=0$  vs.  $H_1: \delta > 0$  (prove greater than, **upper tailed test**)  
 $\leq$  reject for “large” median differences



$H_0: \delta = 0$  vs.  $H_1: \delta < 0$  (prove less than, **lower tailed test**)  
 $\geq$  reject for “small” median differences



$H_0: \delta = 0$  vs.  $H_1: \delta \neq 0$  (prove not equal to, **two tailed test**)  
 reject for “large” or “small” median differences



## 10.3 Tests with Matched Samples – Sign Test

$$H_0: \delta \leq 0 \text{ vs. } H_1: \delta > 0$$

We compare each difference with the conjectured median difference 0.

If a difference is larger than the hypothesized difference 0, replace with a +.

If a difference is smaller than the hypothesized difference 0, replace with a –.

If the difference is equal to the hypothesized difference 0, replace with a 0.

The test statistic is  $x$  the number of +'s.

Reject  $H_0$  based on binomial probabilities,  $p=1/2$ .



## 10.3 Tests with Matched Samples – Sign Test

The hypothesis testing process consists of 5 Steps.

**Step 2:** Select the appropriate test statistic.

The test statistic is a single (decision) number.

$$H_0: \delta \leq 0 \text{ vs. } H_1: \delta > 0$$

$$x = (\text{the number of differences} > 0)$$

When using Table 6, “the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller.**”  
Page 240 of Sullivan.

So it would be # d's < 0 using Table 6.

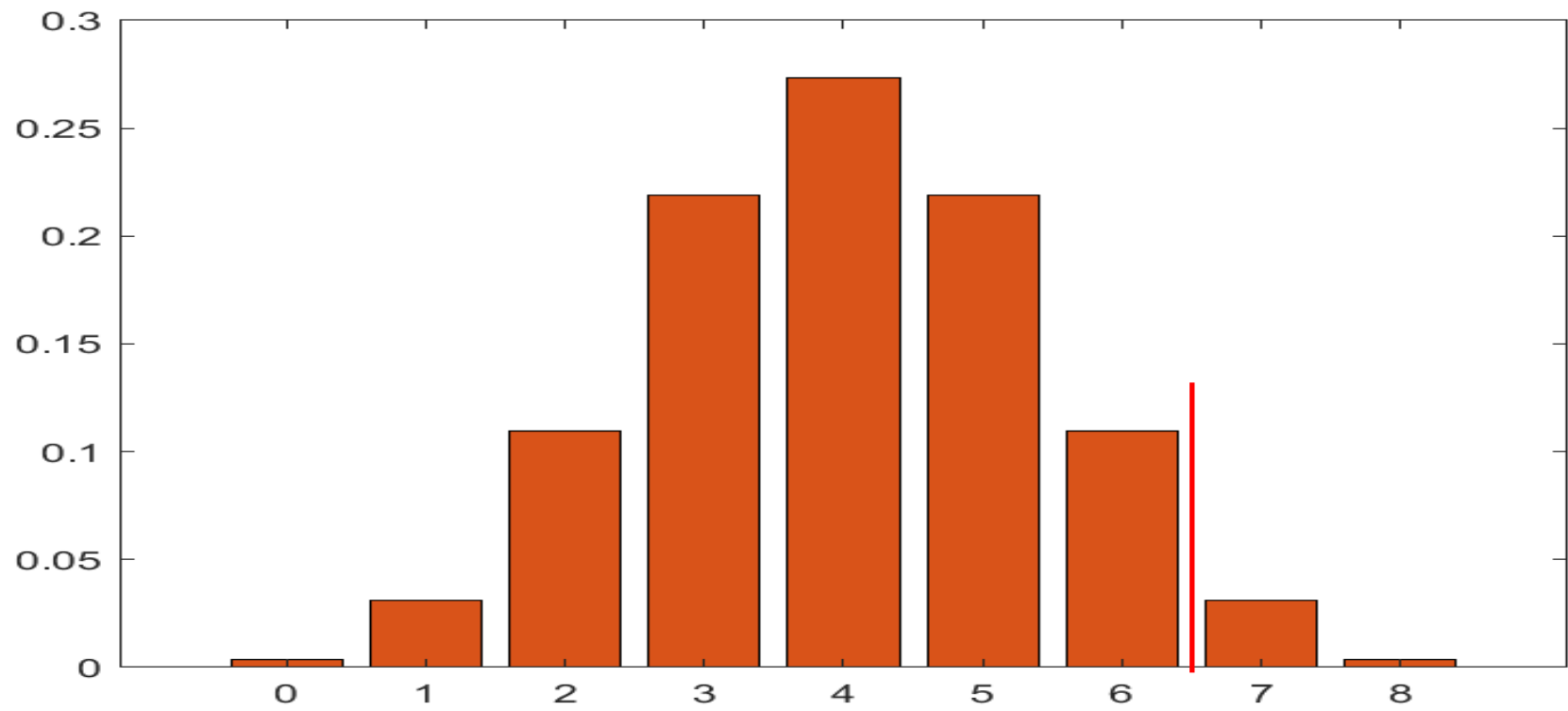
Use the test statistic  $x$  that depends on data and null hypothesis with a critical  $x_a$  value from a cumulative binomial distribution  $p=1/2$ .

$x_a = x$  value with cumulative probability  $a$  larger than it from binomial,  $n$ ,  $p=1/2$ .  $a = \alpha$  or  $\alpha/2$

# 10.3 Tests with Matched Samples – Sign Test

$H_0: \delta \leq 0$  vs.  $H_1: \delta > 0$

Binomial Distribution,  $n=8, p=0.5$



x	P(X=x)	CumSum	CumSumR
0	0.004	0.004	1.000
1	0.031	0.035	0.996
2	0.109	0.145	0.965
3	0.219	0.363	0.856
4	0.273	0.637	0.637
5	0.219	0.856	0.363
6	0.109	0.965	0.145
7	0.031	0.996	0.035
8	0.004	1.000	0.004

Value with 0.05 less than. Compliment.

p-value

Value with 0.05 larger than.

See also Table 6

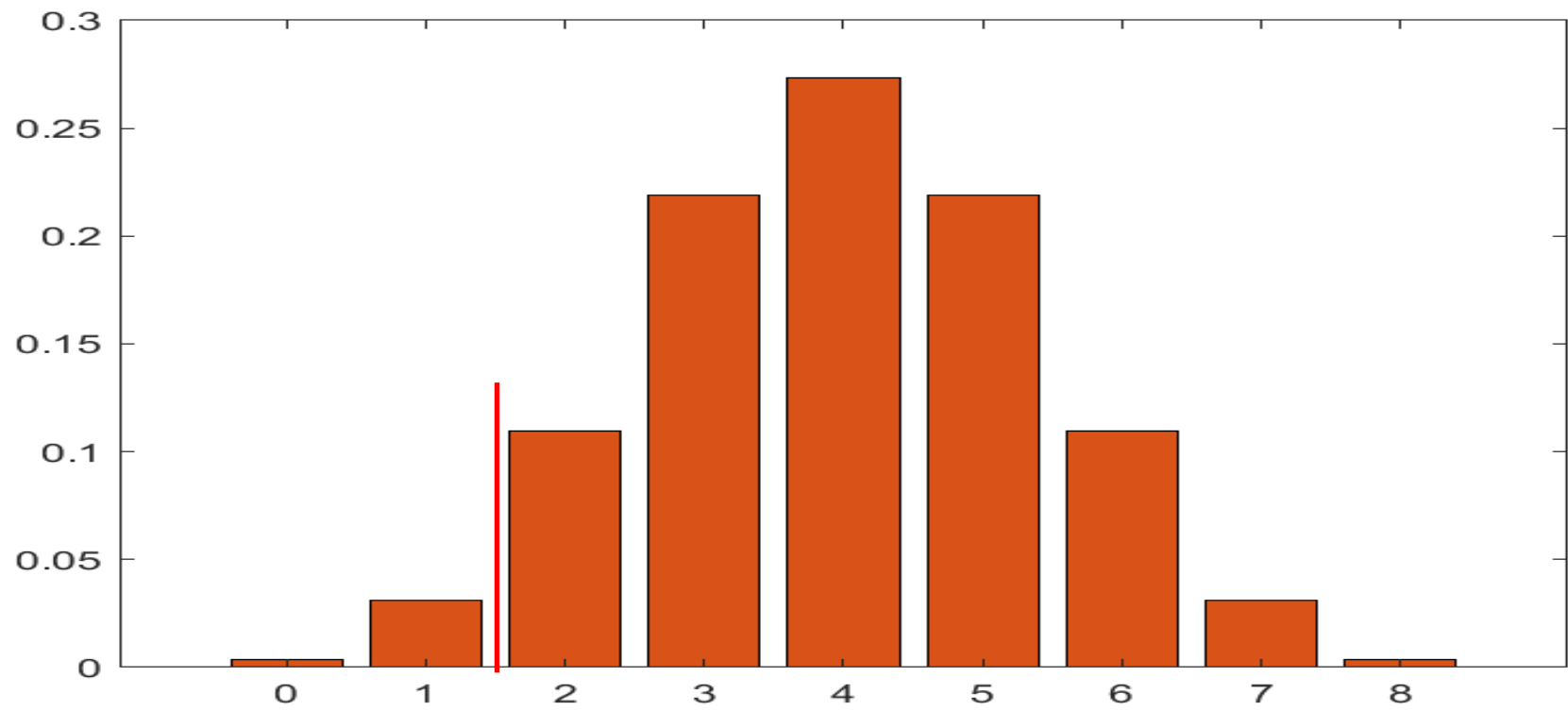
$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Here  $H_1: \delta > 0$  is the problem, reject for large #  $d$ 's.

When using Table 6, the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller**. Page 240 of Sullivan.

# 10.3 Tests with Matched Samples – Sign Test

Binomial Distribution,  $n=8, p=0.5$



Value with 0.05 less than. Compliment.

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Symmetry

Book turn it into  $H_1: \delta < 0$  problem, reject for small #  $d$ 's.

Table 6

Two-Sided Test $\alpha$	.10	.05	.02	.01
One-Sided Test $\alpha$	.05	.025	.01	.005

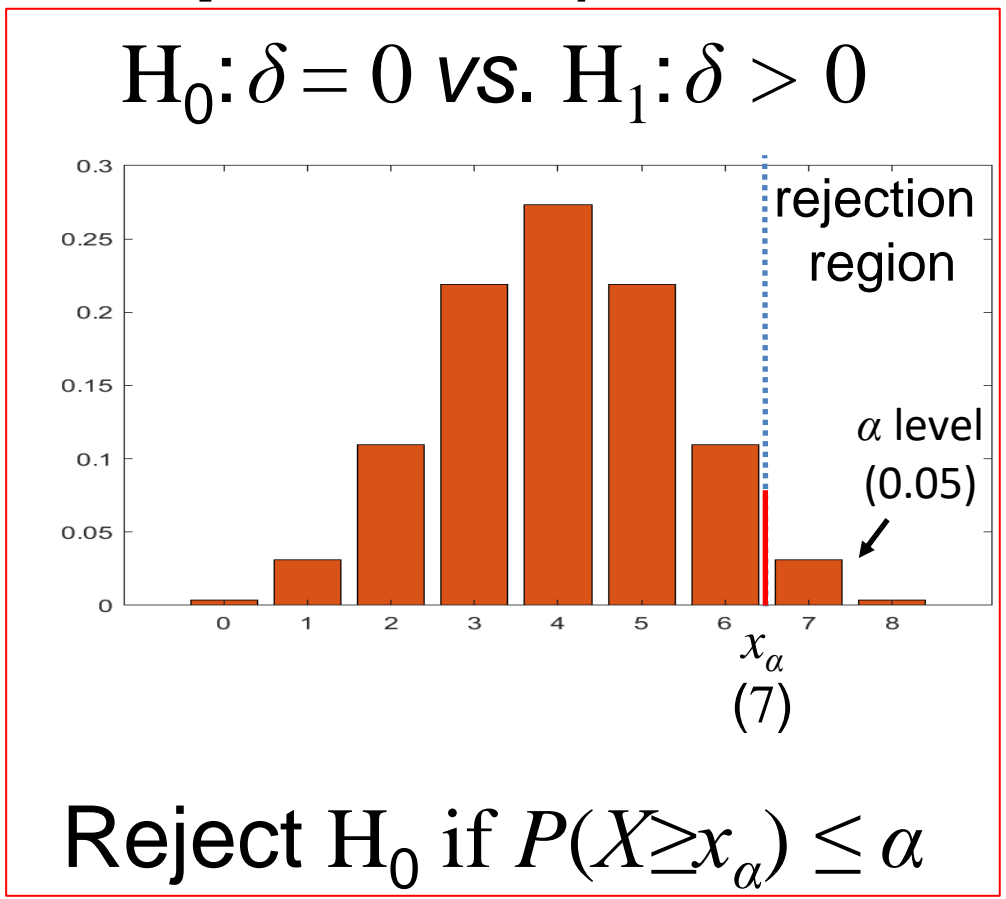
n	x			CumSum
	x	P(X=x)	CumSum	
1	0	0.004	0.004	
2	1	0.031	0.035	
3				
4				
5	0			
6	0	0		
7	0	0	0	
8	1	0	0	
9	1	1	0	
10	1	1	0	

When using Table 6, the test statistic for the Sign test is the number of positive signs or the number of negative signs, **whichever is smaller**. Page 240 of Sullivan.

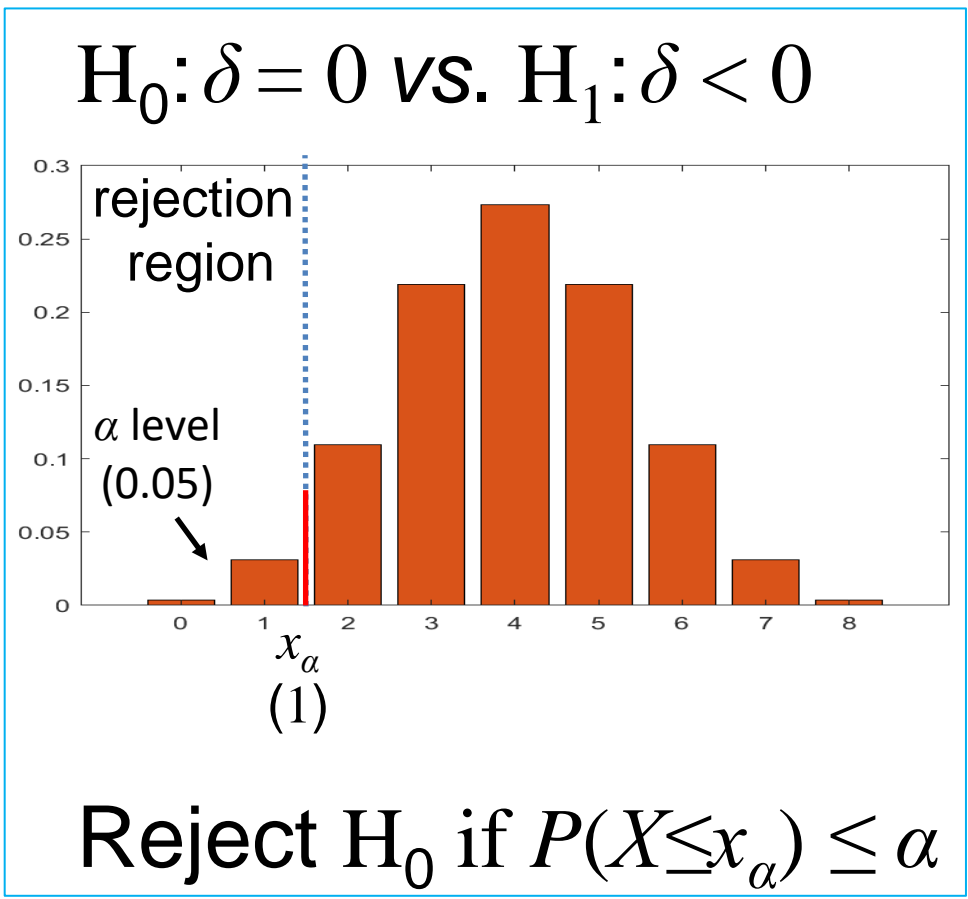
# 10.3 Tests with Matched Samples – Sign Test

The hypothesis testing process consists of 5 Steps.

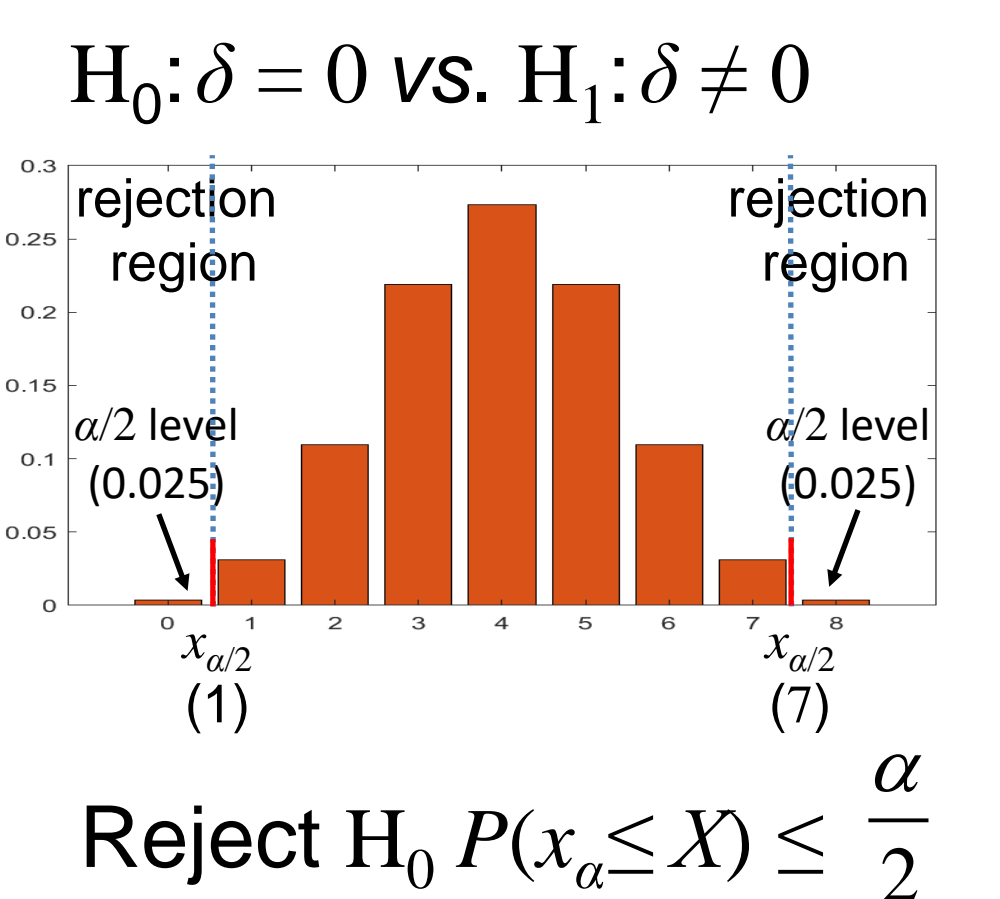
## Step 3: Set-up the decision rule.



Reject for large  $x$ ,



Reject for small  $x$ .



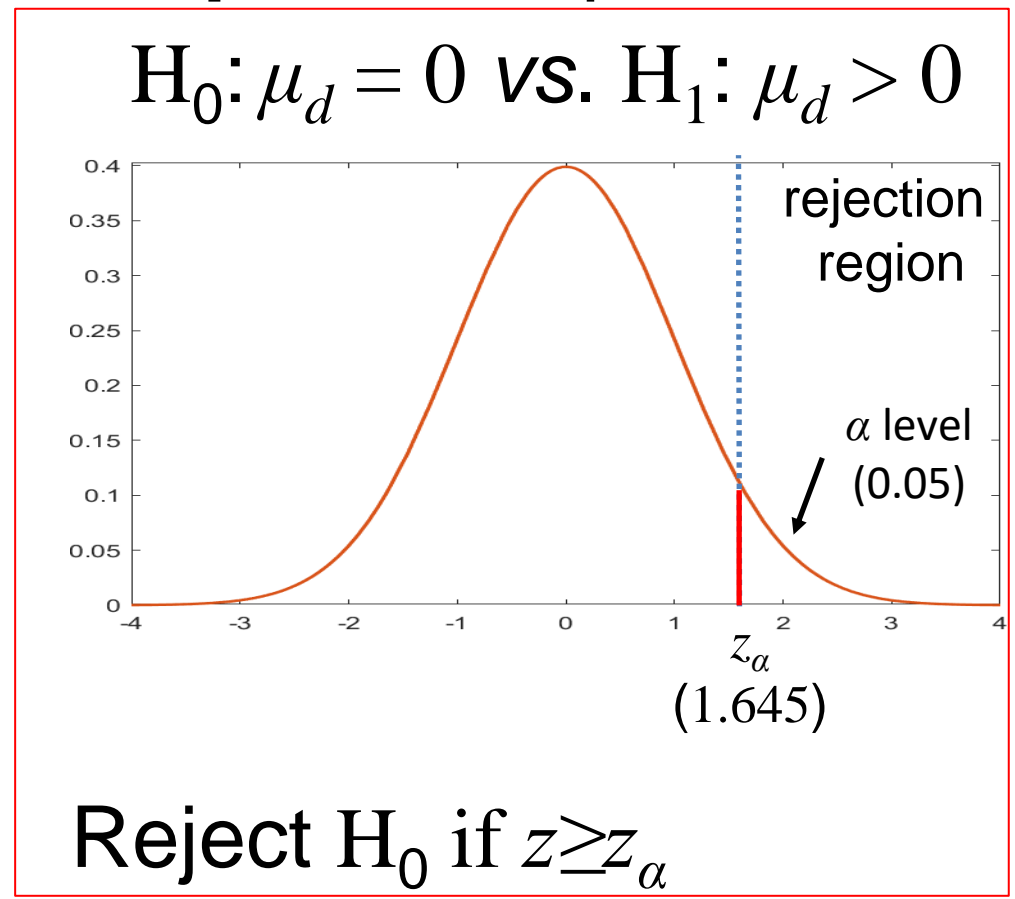
Reject for large or small.

# 7.1 Introduction to Hypothesis Testing

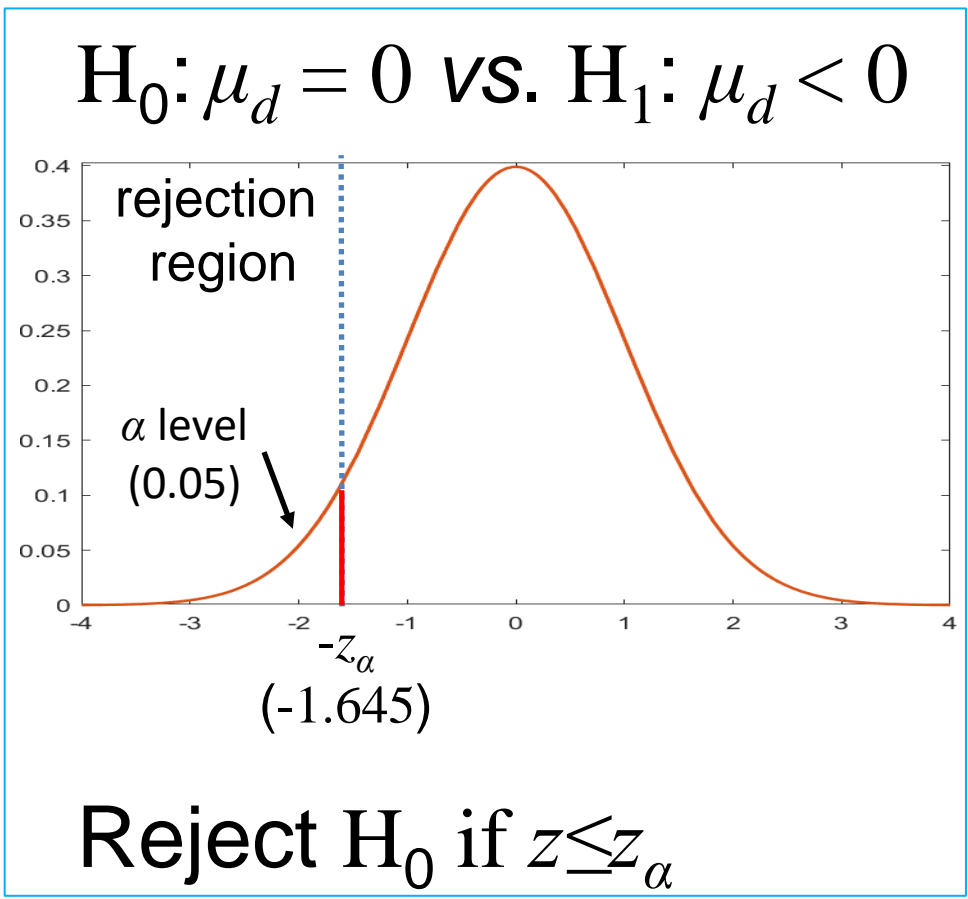
## RECALL

The hypothesis testing process consists of 5 Steps.

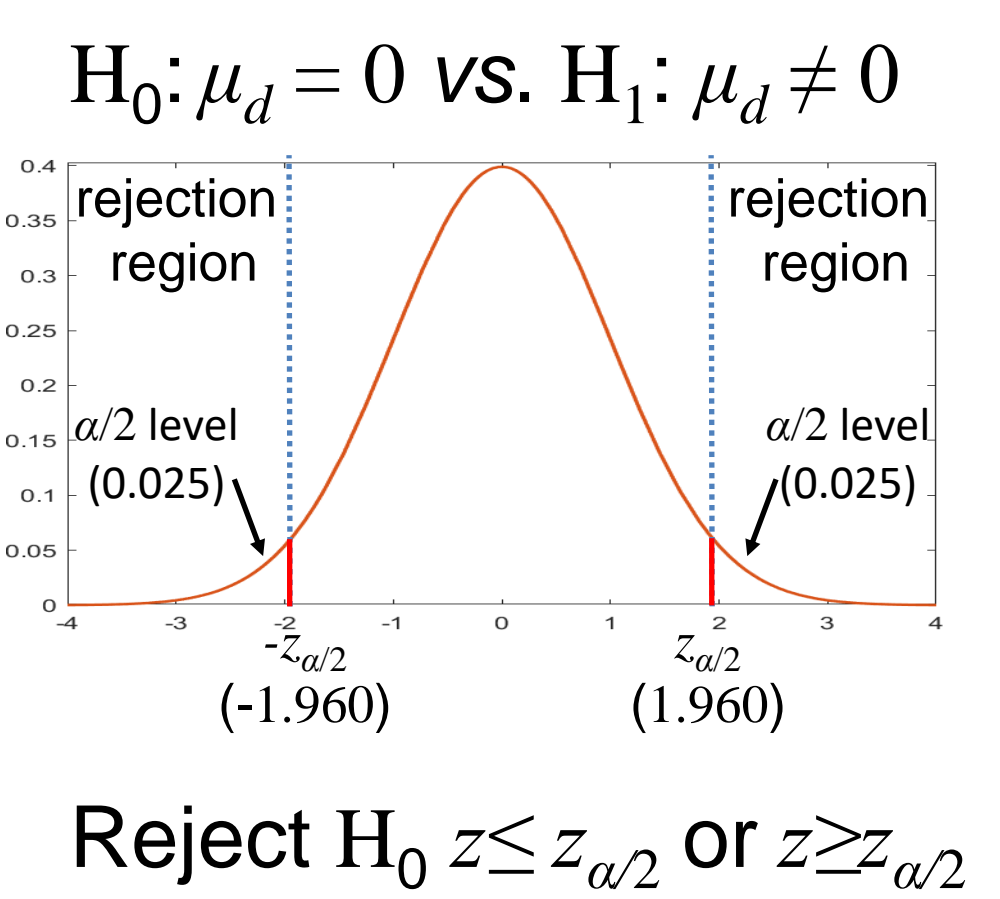
### Step 3: Set-up the decision rule.



Reject for large  $\bar{X}_d$ .



Reject for small  $\bar{X}_d$ .



Reject for large or small.

# 10.3 Tests with Matched Samples – Sign Test

**Step 4:** Compute the test statistic.

$x =$  (the number of differences  $> 0$ )  
 $x=6$  ( $8-6=2$  if we're going to use the table.)

If diff  $< MD_0$ , -.  
 If diff  $= MD_0$ , 0.  
 If diff  $> MD_0$ , +.

b	a	d	sorted	sign
85	75	10	-10	-1
70	50	20	-5	-1
40	50	-10	10	+1
65	40	25	10	+1
80	20	60	15	+1
75	65	10	20	+1
55	40	15	25	+1
20	25	-5	60	+1

**Step 5:** Because  $x=6 \leq x_{\alpha}=7$  or  $p\text{-value}=0.145 > \alpha=0.05$ .  
 ( $x=2 > x_{\alpha}=1$  when using Table 6)

do not reject  $H_0$

x	P(X=x)	CumSum	CumSumR
6	0.109	0.965	0.145
7	0.031	0.996	0.035
8	0.004	1.000	0.004

Two-Sided Test $\alpha$	.10	.05	.02	.01
One-Sided Test $\alpha$	.05	.025	.01	.005
7	0	0	0	
8	1	0	0	0
9	1	1	0	0
10	1	1	0	0

Table 6

x	P(X=x)	CumSum
0	0.004	0.004
1	0.031	0.035

See also Table 6 →

$x = \min(+\text{'s}, -\text{'s})$

**Note:**

If we used normal, we would not reject  $H_0$ ,  $t=1.533 < t_{0.05,7}=1.895$ .

$$t = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} \quad df=n-1 \quad \bar{X} = 15.6250 \quad s = 21.4539$$

## 10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

An alternative for the test for population median difference is the Wilcoxon Signed Rank test.

### Step 1:

$H_0$ : The median difference is zero ( $H_0: \delta=0$ )

$H_1$ : The median difference is positive ( $H_1: \delta>0$ )

$H_0: \delta \leq 0$  vs.  $H_1: \delta > 0$   
 $\delta$  is population version of  $d$ .

We will calculate a test statistic  $W$  the smaller of  $W_+$  and  $W_-$ .

$W_+$  = sum of positive ranks

$W_-$  = sum of negative ranks  $\longrightarrow W = \min(W_+, W_-)$

If the median difference of the matched pairs is zero, then the sum of the positive ranks should be the same as the sum of the negative ranks.

# 10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

$H_0: \delta \leq 0$  vs.  $H_1: \delta > 0$

**Step 2:** Select the appropriate test statistic.

The test statistic is a single (decision) number summarizing information.

$W_+ = \text{sum of positive ranks} = 32$

$W_- = \text{sum of negative ranks} = 4$

$W = \min(W_+, W_-) = \min(32, 4) = 4$

b	a	d	sorted	sign	rank	SgnRnk
85	75	10	-10	-1	3	-3
70	50	20	-5	-1	1	-1
40	50	-10	10	+1	3	3
65	40	25	10	+1	3	3
80	20	60	15	+1	5	5
75	65	10	20	+1	6	6
55	40	15	25	+1	7	7
20	25	-5	60	+1	8	8

$n=8$

**IF**

Signed Ranks			
SgnRnk	SgnRnk	SgnRnk	SgnRnk
1	-4	-7	-8
2	-3	-5	-7
3	-2	-3	-6
4	-1	-1	-5
5	5	2	2
6	6	4	4
7	7	6	6
8	8	8	8

$W = 0$     $W = 10$     $W = 16$     $W = 26$

Reject  $H_0$  for small  $W$ .



# 10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

**Step 3:** Set-up the decision rule.

$n=8, \alpha=0.05$

If we did Two Sided Test

Reject  $H_0$  if  $W \leq W_{\alpha,n}$

Two-Sided Test $\alpha$	.10	.05	.02	.01
One-Sided Test $\alpha$	.05	.025	.01	.005
$n$				
5	1			
6	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3

Table 7

**Step 4:** Compute test statistic.

Already done,  $W=4$ .

**Step 5:** Conclusion.

Reject  $H_0$  because

$W=4 \leq W_{0.05,8}=4$ . Interpret.

# 10.3 Tests with Matched Samples – Wilcoxon Signed Rank Test

**Step 3:** Set-up the decision rule.

$n=8, \alpha=0.05$

If we did One Sided Test

Reject  $H_0$  if  $W \leq W_{\alpha,n}$

Two-Sided Test $\alpha$	.10	.05	.02	.01
One-Sided Test $\alpha$	.05	.025	.01	.005
$n$				
5	1			
6	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3

Table 7

**Step 4:** Compute test statistic.

Already done,  $W=4$ .

**Step 5:** Conclusion.

Reject  $H_0$  because

$W=4 \leq W_{0.05,8}=6$ . Interpret.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

## Step 1:

We can test if three or more population medians are different.

$H_0$ : The  $k$  population medians are equal

$H_1$ : The  $k$  population medians are not all equal

We will go through the same 5 hypothesis steps

## 7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

### RECALL

The hypothesis testing process consists of 5 Steps.

**Step 1:** Set up the hypotheses and determine the level of significance  $\alpha$ .

$H_0: \mu_1 = \mu_2 \dots = \mu_k$  vs.  $H_1$ : at least two  $\mu$ 's different  
reject for "large" disparities or  $F = MSB/MSE$ .

We will assume the means are equal and calculate two different variances.  
If the means are truly equal, the two different variances will be the same.  
If the means are not equal, the two different variances will be different.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

A clinical study is designed to assess differences in albumin levels.

5% Protein	10% Protein	15% Protein
3.1	3.8	4.0
2.6	4.1	5.5
2.9	2.9	5.0
	3.4	4.8
	4.2	

The question of interest is whether there is a difference in albumin levels among the three different diets.

The data is not normally distributed with same variance for ANOVA.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

**Step 1:** Set up the hypotheses and determine the level of significance  $\alpha$ .

$H_0: MD_1 = MD_2 \dots = MD_k$  vs.  $H_1$ : at least two  $MD$ 's different  
reject for "large" disparities  $H$ .

We will assume the medians are equal and see how different from equal.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

**Step 2:** Select the appropriate test statistic.

The test statistic is a single (decision) number.

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

$R_j$  is sum of ranks for sample  $j$ .

Use the test statistic that depends on data and null hypothesis with a critical value  $H_{\alpha, n_1, n_2, n_3}$  that depends on significance level  $\alpha$  to make decision.

Table 8 in book

We will test a single hypotheses on medians with the test statistic.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

Test

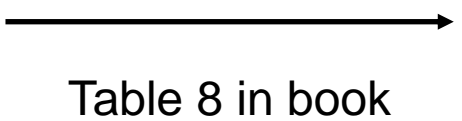
**Step 3:** Set-up the decision rule.

$$H_0: MD_1 = MD_2 = \dots = MD_k \text{ vs. } H_1: \text{at least two different}$$

Select a level of significance  $\alpha$ .

Table 8

Three groups			$\alpha = .05$	$\alpha = .01$
$n_1$	$n_2$	$n_3$		
2	2	2		
3	2	1		
3	2	2	4.714	
3	3	1	5.143	
3	3	2	5.361	
3	3	3	5.600	7.200
4	2	1		
4	2	2	5.333	
4	3	1	5.208	
4	3	2	5.444	6.444
4	3	3	5.791	6.745
4	4	1	4.967	6.667
4	4	2	5.455	7.036
4	4	3	5.598	7.144
4	4	4	5.692	7.654
5	2	1	5.000	
5	2	2	5.160	6.533
5	3	1	4.960	
5	3	2	5.251	6.909
5	3	3	5.648	7.079
5	4	1	4.985	6.955
5	4	2	5.273	7.205
5	4	3	5.656	7.445



$$\text{Reject } H_0 \text{ if } H \geq H_{\alpha, n_1, n_2, n_3}$$

Sample size order doesn't matter.



# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis Test

The hypothesis testing process consists of 5 Steps.

**Step 4:** Compute the test statistic.

Use sample data  $n_1$  from population 1 and  $n_2$  from population 2 and  $n_3$  from population 3 to compute test statistic  $H$ .

Compare test statistic  $H$  to critical value(s)  $H_{\alpha, n_1, n_2, n_3}$  with rule.

Table 8 in book

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

**Step 5:** Conclusion.

Make a decision, reject  $H_0$  or not to reject  $H_0$ .

Interpret the results.

$$N = n_1 + n_2 + n_3.$$

## 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

**Example:** Statistical difference in albumin for 3 diets?

**Step 1:** Null and Alternative Hypotheses.

$H_0: MD_1 = MD_2 = MD_3$  vs.  $H_1$ : at least two different

**Step 2:** Test Statistic.  $N = n_1 + n_2 + n_3$ .

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

**Step 3:** Decision Rule.  $\alpha = 0.05$ ,  $n_1 = 3$ ,  $n_2 = 5$ ,  $n_3 = 4$

Reject  $H_0$  if  $H \geq H_{\alpha, n_1, n_2, n_3} = 5.656$ .

Table 8

Three groups			$\alpha = .05$	$\alpha = .01$
$n_1$	$n_2$	$n_3$		
5	4	3	5.656	7.445

Sample size order doesn't matter.

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

**Example:** Statistical difference in albumin for 3 diets?

**Step 4:** Compute test statistic.

			Total Sample (Ordered Smallest to Largest)			Ranks		
5% Protein	10% Protein	15% Protein	5% Protein	10% Protein	15% Protein	5% Protein	10% Protein	15% Protein
3.1	3.8	4.0	2.6			1		
2.6	4.1	5.5	2.9	2.9		2.5	2.5	
2.9	2.9	5.0	3.1			4		
	3.4	4.8		3.4			5	
	4.2			3.8			6	
					4.0			7
				4.1			8	
				4.2			9	
					4.8			10
					5.0			11
					5.5			12

$$R_1 = 7.5 \quad R_2 = 30.5 \quad R_3 = 40$$

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

$$H = \frac{12}{12(12+1)} \left( \frac{7.5^2}{3} + \frac{30.5^2}{5} + \frac{40^2}{4} \right) - 3(12+1) = 7.52$$

# 10.4 Tests with More than Two Independent Samples – Kruskal-Wallis

**Example:** Statistical difference in albumin for 3 diets?

**Step 1:** Null and Alternative Hypotheses.

$H_0: MD_1=MD_2=MD_3$  vs.  $H_1$ : at least two different

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

**Step 2:** Test Statistic.

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

**Step 3:** Decision Rule.  $\alpha=0.05$ ,  $n_1=3$ ,  $n_2=5$ ,  $n_3=4$

Reject  $H_0$  if  $H \geq 5.656$ .

**Step 4:** Compute test statistic.

$$H = 7.52$$

**Step 5:** Conclusion

Reject  $H_0$  because  $7.52 > 5.656$ , and conclude difference in median albumin.

Table 8

Three groups			$\alpha = .05$	$\alpha = .01$
$n_1$	$n_2$	$n_3$		
5	4	3	5.656	7.445

Sample size order doesn't matter.

## 10.5 Summary

### Sign Test (one sample)

$x = \text{number of observations} > MD_0$   
 $<$

### Mann-Whitney U Test

$$U_1 = n_1 n_2 + \frac{n_1(n_1 + 1)}{2} - R_1$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2 + 1)}{2} - R_2$$

$$U = \min(U_1, U_2)$$

### Sign Test (two sample)

$x = \text{number of differences} > 0$   
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### Wilcoxon Signed Rank Test

(two sample)

$$W = \min(W_+, W_-)$$

$W_+$  = sum of positive ranks

$W_-$  = sum of negative ranks

### Kruskal-Wallis Test

(three or more samples)

$$H = \left( \frac{12}{N(N+1)} \sum_{j=1}^k \frac{R_j^2}{n_j} \right) - 3(N+1)$$

$R_j$  = sum of ranks for sample  $j$ .

# Questions?

# Homework 10

Read Chapter 10.

Problems # 6 (Sign Test), 7 (Wilcoxon Signed Rank Test),  
8 (Kruskal-Wallis Test) the  $n_1=n_2=n_3=n_4=5$  critical value is 7.377.