Chapter 7: Hypothesis Testing Procedures II

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Hypothesis Testing

We make decisions every day in our lives.

Should I believe A or should I believe B (not A)?

Two Competing Hypotheses. A and B.

Null Hypothesis ($\mathbf{H_0}$): No difference, no association, or no effect.

Alternative Hypothesis ($\mathbf{H_1}$): Investigators belief.

The Alternative Hypothesis is always set up to be what you want to build up evidence to prove.

7.1 Introduction to Hypothesis Testing

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance.

Step 2: Select the appropriate test statistic.

Step 3: Set-up the decision rule.

Step 4: Compute the test statistic.

Step 5: Conclusion.

7.5 Tests with Two Independent Samples, Continuous Outcome

We often have two populations that we are studying.

We may be interested in knowing if the mean μ_1 of population 1
is different (while accounting for random statistical variation)
from the mean μ_1 of population 2.

When we have independent random sample from each population

7.5 Tests with Two Independent Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

There are three possible pairs.

H_0: $\mu_1 = \mu_2$ *vs.* H_1: $\mu_1 > \mu_2$ (prove greater than)

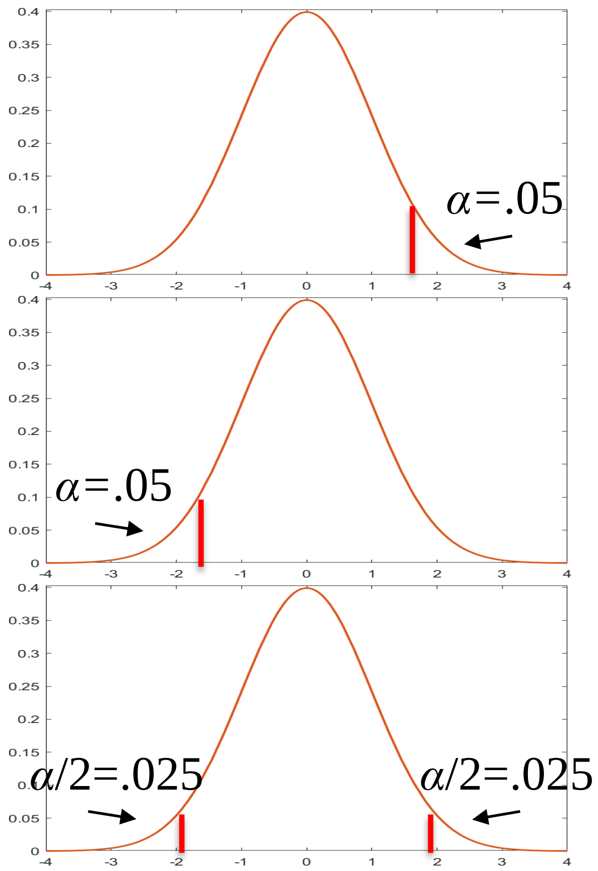
$\leq$ reject for "large" $\bar{X}_1 - \bar{X}_2$ or z's

H_0: $\mu_1 = \mu_2$ *vs.* H_1: $\mu_1 < \mu_2$ (prove less than)

$\geq$ reject for "small" $\bar{X}_1 - \bar{X}_2$ or z's

H_0: $\mu_1 = \mu_2$ *vs.* H_1: $\mu_1 \neq \mu_2$ (prove not equal to)

reject for "large" or "small" $\bar{X}_1 - \bar{X}_2$ or z's



7.5 Tests with Two Independent Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

$$\bar{X}_i = \frac{1}{n}\sum X \qquad i=1,2$$

The test statistic is a single (decision) number.

n large

$$z = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{S_P\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

n small

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{S_P\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad df=n_1+n_2-2$$

$$S_P = \sqrt{\frac{(n_1-1)(s_1)^2 + (n_2-1)(s_2)^2}{n_1+n_2-2}}$$

Use the test statistic that depends on data and null hypothesis with a critical value z_a (or $t_{a,df}$) that depends on significance level α to make decision.

$a = \alpha$ or $\alpha/2$

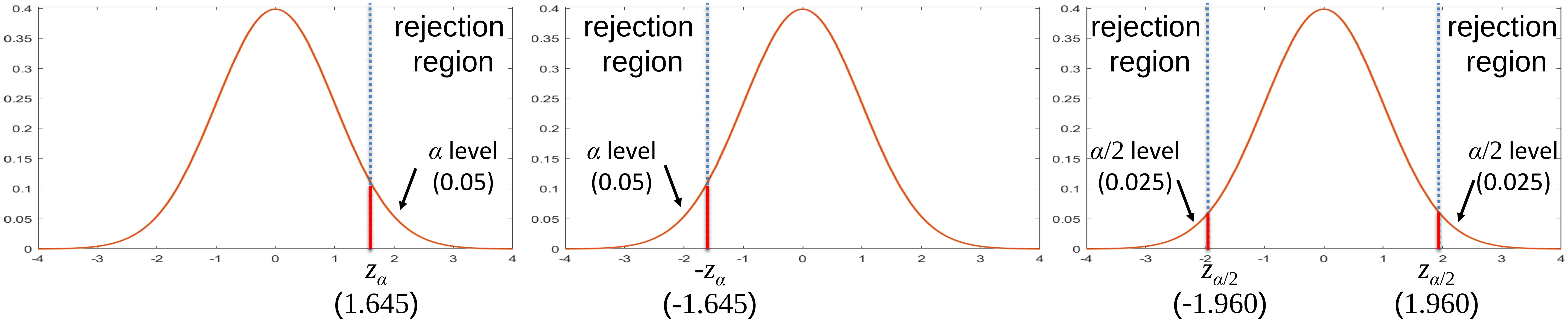
We will test hypotheses on various parameters with various test statistics.

7.5 Tests with Two Independent Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

H_0: $\mu_1=\mu_2$ vs. H_1: $\mu_1>\mu_2$ | H_0: $\mu_1=\mu_2$ vs. H_1: $\mu_1<\mu_2$ | H_0: $\mu_1=\mu_2$ vs. H_1: $\mu_1\neq\mu_2$



Reject H_0 if $z\geq z_\alpha$ (or $t\geq t_{\alpha,df}$) | Reject H_0 if $z\leq z_\alpha$ (or $t\leq t_{\alpha,df}$) | Reject H_0 $z\leq z_{\alpha/2}$ or $z\geq z_{\alpha/2}$ (or $t\leq t_{\alpha/2,df}$ or $t\geq t_{\alpha/2,df}$)

7.5 Tests with Two Independent Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.
Use sample data n_1 from population 1 and n_2 from population 2
to compute z (or t).
Compare test statistic z (or t) to critical value(s) $z_{\alpha/2}$ (or $t_{\alpha/2,df}$) with rule.

Step 5: Conclusion.
Make a decision, reject H_0 or not to reject H_0.
Interpret the results.

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{S_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$df=n_1+n_2-2$

7.3 Tests with One Sample, Dichotomous Outcome

	Sample Size	Mean	Standard Deviation
New drug	15	195.9	28.7
Placebo	15	227.4	30.3

Example: Is the mean cholesterol of new drug < mean of placebo?

Step 1: Null and Alternative Hypotheses.

$H_0: \mu_1 \geq \mu_2$ vs. $H_1: \mu_1 < \mu_2$

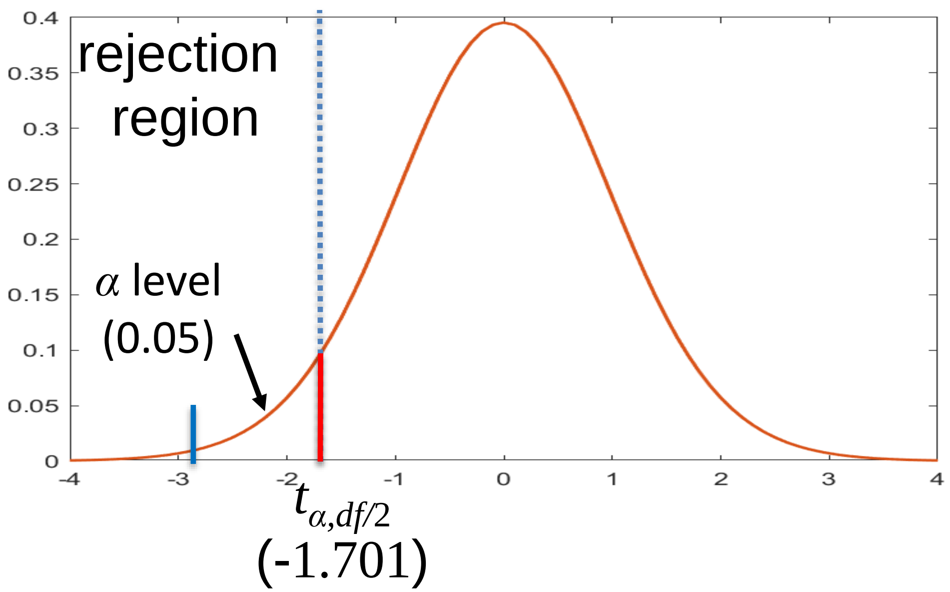
Step 2: Test Statistic.

$$t = \frac{(\bar{X}_1 - \bar{X}_2)}{S_P\sqrt{1/n_1 + 1/n_2}} \qquad df=n_1+n_2-2$$

Step 3: Decision Rule. $\alpha=0.05$, $df=15+15-2=28$

Reject H_0 if $t \leq -1.701$.

$df=28$



Step 4: Compute test statistic.

$$t = (195.9 - 227.4)\big/\left(29.5\sqrt{1/15+1/15}\right) = -2.92$$

$$\bar{X}_i = \frac{1}{n}\sum X \qquad i=1,2$$

$$S_P = \sqrt{\frac{(15-1)(28.7)^2 + (15-1)(30.3)^2}{15+15-2}} = 29.5$$

Step 5: Conclusion

Because $-2.92 \leq -1.701$, reject and conclude mean of drug less than placebo.

7.6 Tests with Matched Samples, Continuous Outcome

We often encounter two samples where there are matched pairs.
This is often the case for before vs. after, twins, couples, etc.
We subtract x_1 from sample 1 and x_2 from sample 2 for each pair.

The differences are labeled generically $d=x_1-x_2$ and so the sample of differences is $d_1,\ldots,d_n$. $\bar{X}_d$

Once we have these differences we treat them exactly the same as we did in Section 7.2 Tests with One Sample, Continuous Outcome.

7.6 Tests with Matched Samples, Continuous Outcome

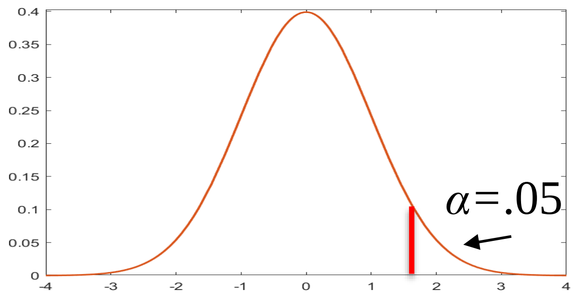
The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

There are three possible pairs.

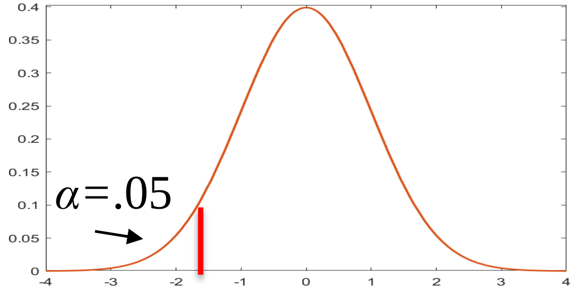
H_0: μ_d=0 *vs*. H_1: μ_d>0 (prove greater than)

$\leq$ reject for “large" $\bar{X}_d$ or z’s



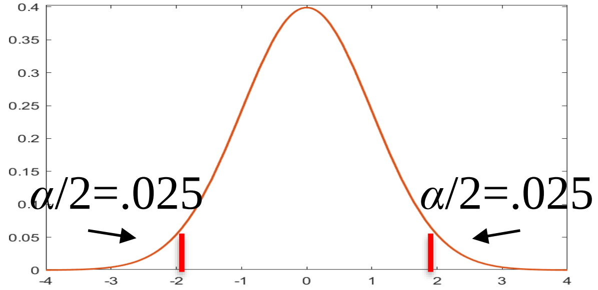
H_0: μ_d=0 *vs*. H_1: μ_d<0 (prove less than)

$\geq$ reject for “small” $\bar{X}_d$ or z’s



H_0: μ_d=0 *vs*. H_1: $\mu_d\neq$0 (prove not equal to)

reject for “large” or “small” $\bar{X}_d$ or z’s



7.6 Tests with Matched Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number.

$$\bar{X}_d = \frac{1}{n}\sum d$$

$$s_d = \sqrt{\frac{1}{n-1}\left[\sum X^2 - \frac{1}{n}\left(\sum X\right)^2\right]}$$

n large

$$z = \frac{\bar{X}_d - 0}{s_d / \sqrt{n}}$$

n small

$$t = \frac{\bar{X}_d - 0}{s_d / \sqrt{n}}$$

$df = n$-1

Use the test statistic that depends on data and null hypothesis with a critical value z_a (or $t_{a,df}$) that depends on significance level α to make decision.

$a = \alpha$ or $\alpha/2$

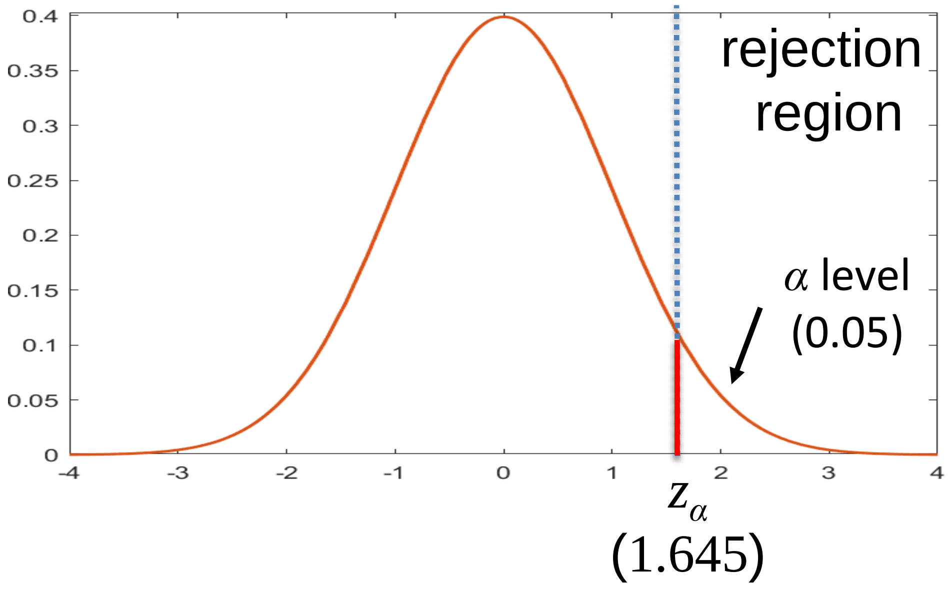
We will test hypotheses on various parameters with various test statistics.

7.6 Tests with Matched Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

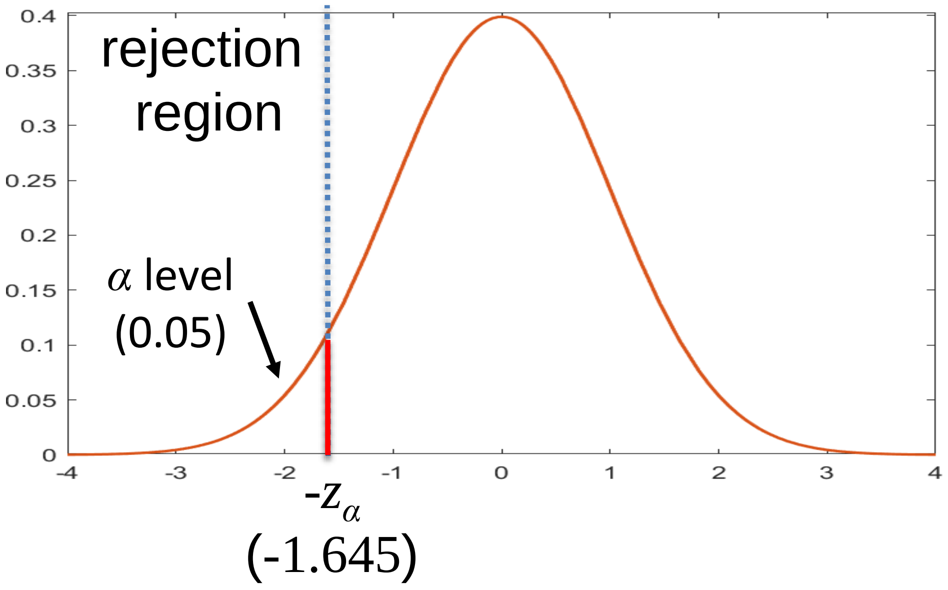
Step 3: Set-up the decision rule.

$H_0: \mu_d = 0$ *vs.* $H_1: \mu_d > 0$



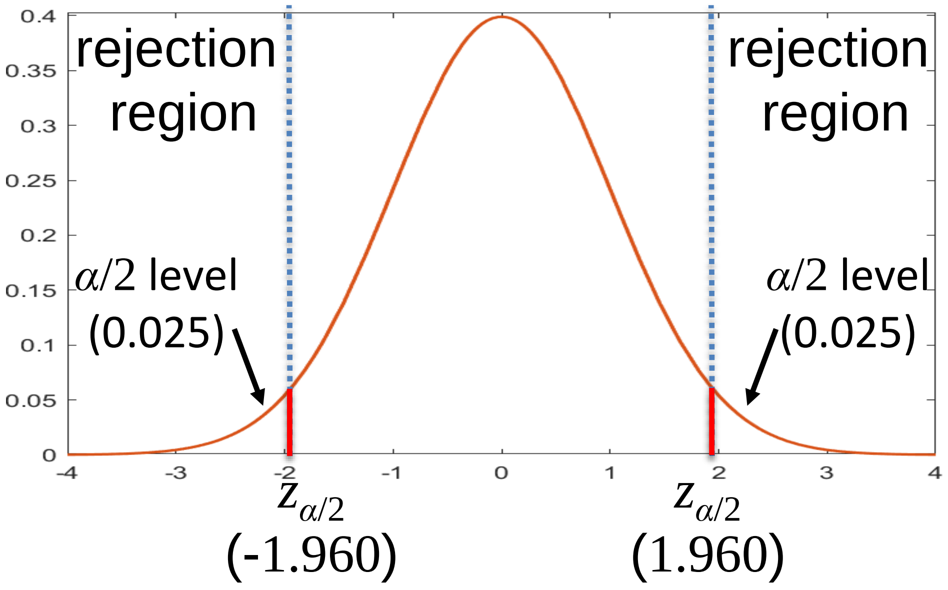
Reject H_0 if $z \geq z_\alpha$ (or $t \geq t_{\alpha,df}$)

$H_0: \mu_d = 0$ *vs.* $H_1: \mu_d < 0$



Reject H_0 if $z \leq z_\alpha$ (or $t \leq t_{\alpha,df}$)

$H_0: \mu_d = 0$ *vs.* $H_1: \mu_d \neq 0$



Reject H_0 $z \leq z_{\alpha/2}$ or $z \geq z_{\alpha/2}$ (or $t \leq t_{\alpha/2,df}$ or $t \geq t_{\alpha/2,df}$)

7.6 Tests with Matched Samples, Continuous Outcome

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.

Use sample data n_1 from population 1 and n_2 from population 2

to compute z (or t).

Compare test statistic z (or t) to critical value(s) $z_{\alpha/2}$ (or $t_{\alpha/2,df}$) with rule.

Step 5: Conclusion.

Make a decision, reject H_0 or not to reject H_0.

Interpret the results.

$$t = \frac{\bar{X}_d - 0}{s_d / \sqrt{n}}$$

7.6 Tests with Matched Samples, Continuous Outcome

Example: Is there a difference in mean of new drug from baseline?

Step 1: Null and Alternative Hypotheses.

H_0: $\mu_d = 0$ *vs.* H_1: $\mu_d \neq 0$

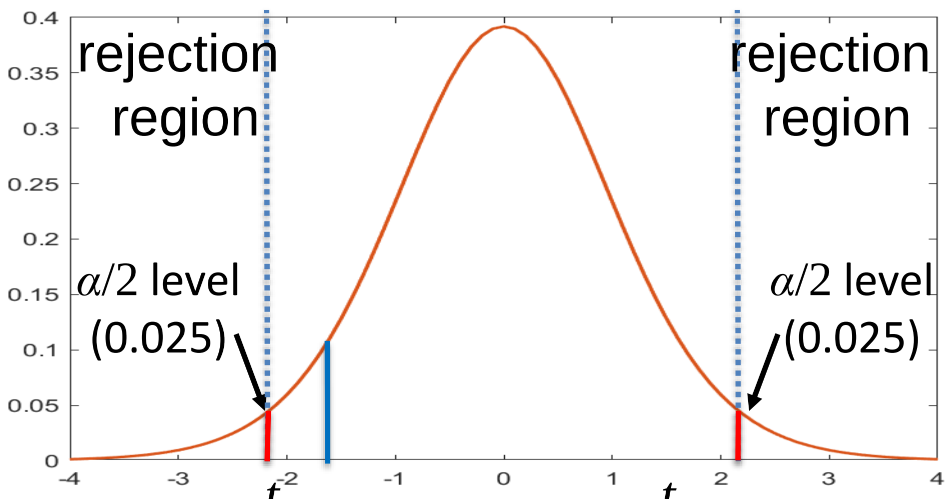
Step 2: Test Statistic.

$$t = \frac{\bar{X}_d}{s_d / \sqrt{n}} \qquad df=n\text{-}1$$

Step 3: Decision Rule. α=0.05 , df=15-1=14

Reject H_0 if $t \leq$ -2.145 or $t \geq 2.145$.

df=14



$t_{\alpha,df/2}$ (-2.145) $t_{\alpha,df/2}$ (2.145)

Number	Baseline	6 Weeks	Difference
1	215	205	10
2	190	156	34
3	230	190	40
4	220	180	40
5	214	201	13
6	240	227	13
7	210	197	13
8	193	173	20
9	210	204	6
10	230	217	13
11	180	142	38
12	260	262	-2
13	210	207	3
14	190	184	6
15	200	193	7

$\bar{X}_d = -5.30$

Step 4: Compute test statistic.

$$\bar{X}_d = \frac{1}{n}\sum d$$

$$s_d^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{1}{n}\left(\sum d\right)^2\right]$$

$$t = -5.3 \big/ (12.8 / \sqrt{15}) = -1.60$$

Step 5: Conclusion

Because -2.145 $\leq$ -1.60, do not reject H_0 and conclude no reduction.

7.7 Tests with Two Independent Samples, Dichotomous Outcome

We often have two populations that we are studying.

We may be interested in knowing if the proportion p_1 of population 1
is different (while accounting for random statistical variation)
from the proportion p_1 of population 2.

When we have independent random sample from each population
and the sample sizes are large.

7.7 Tests with Two Independent Samples, Dichotomous Outcome

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

There are three possible pairs.

H_0: $p_1 = p_2$ *vs*. H_1: $p_1 > p_2$ (prove greater than)

risk difference *RD*

$\leq$ reject for “large" $\hat{p}_1 - \hat{p}_2$ or z’s

H_0: $p_1 = p_2$ *vs*. H_1: $p_1 < p_2$ (prove less than)

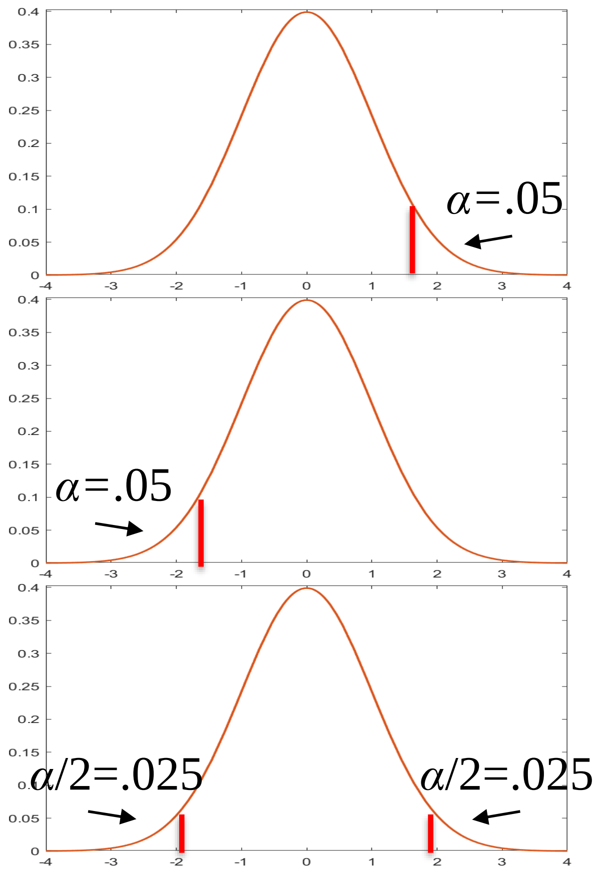
risk difference *RD*

$\geq$ reject for “small” $\hat{p}_1 - \hat{p}_2$ or z’s

H_0: $p_1 = p_2$ *vs*. H_1: $p_1 \neq p_2$ (prove not equal to)

risk difference *RD*

reject for “large” or “small” $\hat{p}_1 - \hat{p}_2$ or z’s



7.7 Tests with Two Independent Samples, Dichotomous Outcome

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number.

n large

risk difference RD

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \hat{p}_1 = \frac{x_1}{n_1} \qquad \hat{p}_2 = \frac{x_2}{n_2} \qquad \hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Use the test statistic that depends on data and null hypothesis with a critical value z_a that depends on significance level α to make decision.

$a = \alpha$ or $\alpha/2$

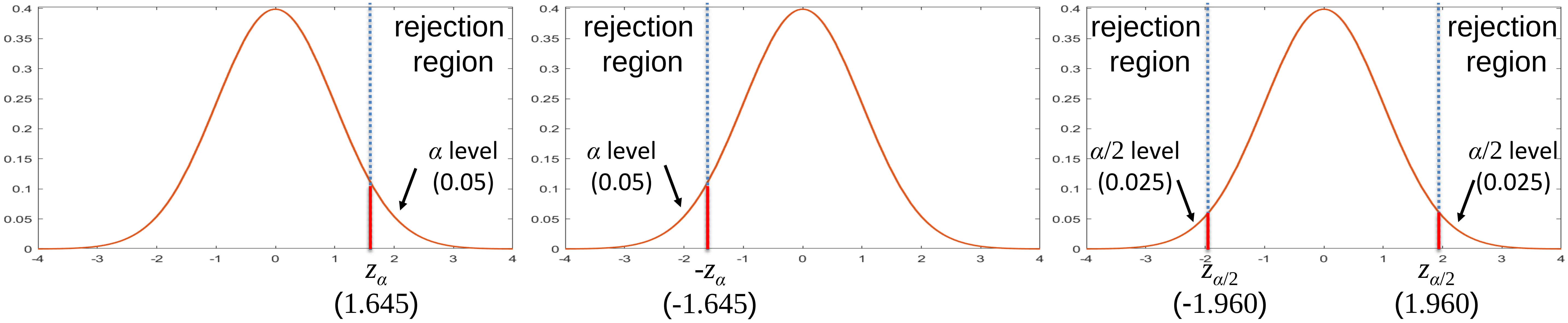
We will test hypotheses on various parameters with various test statistics.

7.7 Tests with Two Independent Samples, Dichotomous Outcome

The hypothesis testing process consists of 5 Steps.

Step 3: Set-up the decision rule.

$H_0: p_1=p_2$ vs. $H_1: p_1>p_2$ $\quad$ $H_0: p_1=p_2$ vs. $H_1: p_1<p_2$ $\quad$ $H_0: p_1=p_2$ vs. $H_1: p_1 \neq p_2$



Reject H_0 if $z \geq z_\alpha$ $\quad$ Reject H_0 if $z \leq z_\alpha$ $\quad$ Reject H_0 $z \leq z_{\alpha/2}$ or $z \geq z_{\alpha/2}$

7.7 Tests with Two Independent Samples, Dichotomous Outcome

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.

Use sample data n_1 from population 1 and n_2 from population 2 to compute z.

Compare test statistic z to critical value(s) $z_{\alpha/2}$ with rule.

Step 5: Conclusion.

Make a decision, reject H_0 or not to reject H_0.

Interpret the results.

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

7.7 Tests with Two Independent Samples, Dichotomous Outcome

The hypothesis test on risk difference

H_0: $p_1 = p_2$ *vs*. H_1: $p_1 \neq p_2$

H_0: $RD = 0$ *vs*. H_1: $RD \neq 0$

Is equivalent to the two hypothesis tests

Risk Ratio RR

H_0: $RR = 1$ *vs*. H_1: $RR \neq 1$

and

Odds Ratio OR

H_0: $OR = 1$ *vs*. H_1: $OR \neq 1$

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

We often have more than two populations that we are studying.

We may be interested in knowing if the mean μ_1 of population 1
is different (while accounting for random statistical variation)
from the mean μ_1 of population 2, and …. the mean μ_k of population k.

When we have independent random sample from each population

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

The hypothesis testing process consists of 5 Steps.

Step 1: Set up the hypotheses and determine the level of significance α.

H_0: $\mu_1=\mu_2 \ldots =\mu_k$ *vs*. H_1: at least two μ's different

reject for “large" disparities or $F=MSB/MSE$.

We will assume the means are equal and calculate two different variances.

If the means are truly equal, the two different variances will be the same.

If the means are noy equal, the two different variances will be different.

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

The hypothesis testing process consists of 5 Steps.

Step 2: Select the appropriate test statistic.

The test statistic is a single (decision) number.

$$F = \frac{MSB}{MSE} \qquad MSB = \frac{\sum n_j(\bar{X}_j - \bar{X})^2}{k-1} \qquad MSE = \frac{\sum\sum n_j(X - \bar{X}_j)^2}{N-k}$$

$df_1 = k\text{-}1$

$df_2 = N\text{-}k$

Use the test statistic that depends on data and null hypothesis with a critical value F_{α,df_1,df_2}) that depends on significance level α to make decision.

Table 4 in book

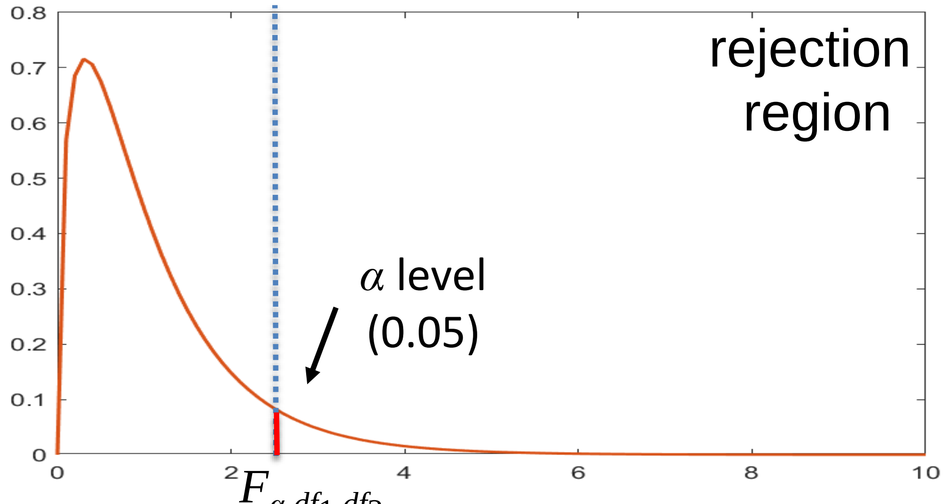
We will test a single hypotheses on means with the test statistic.

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

The hypothesis testing process consists of 5 Steps.

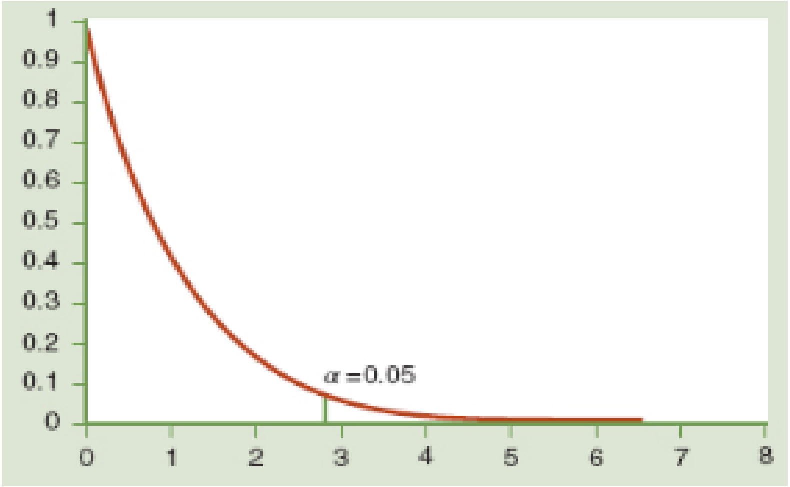
Step 3: Set-up the decision rule.

H_0: $\mu_1 = \mu_2 \ldots = \mu_k$ vs. H_1: at least two different



$df_1 = k\text{-}1 = 3$

$df_2 = N\text{-}k = 36$



book has wrong graph

Reject H_0 if $F \geq F_{\alpha,df_1,df_2}$.

Table 4 in book

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

The hypothesis testing process consists of 5 Steps.

Step 4: Compute the test statistic.

Use sample data n_1 from population 1 and n_2 from population 2 to compute test statistic F.

Compare test statistic F to critical value(s) F_{α,df_1,df_2} with rule.

Table 4 in book

$$F = \frac{MSB}{MSE}$$

Step 5: Conclusion.

Make a decision, reject H_0 or not to reject H_0.

Interpret the results.

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

Example: Find the value of $F_{0.05,3,16}$.

α $df_1=n_1$-1 $df_2=n_2$-1

The (critical) value of F that has an area of 0.05 larger than it when we have $df_1=3$ (numerator) and $df_2=16$ (denominator) degrees of freedom is 3.24.

$P(F_{df_1,df_2} > F) = 0.05$,
e.g., $P(F_{3,20} > 3.10) = 0.05$

	df_1													
df_2	1	2	3	4	5	6	7	8	9	10	20	30	40	50
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	248.0	250.1	251.1	251.8
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.46	19.47	19.48
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.62	8.59	8.58
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.75	5.72	5.70
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.50	4.46	4.44
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.81	3.77	3.75
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.38	3.34	3.32
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	3.08	3.04	3.02
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	2.94	2.86	2.83	2.80
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.77	2.70	2.66	2.64
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.65	2.57	2.53	2.51
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.54	2.47	2.43	2.40
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.46	2.38	2.34	2.31
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.39	2.31	2.27	2.24
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.33	2.25	2.20	2.18
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.28	2.19	2.15	2.12
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.23	2.15	2.10	2.08
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.19	2.11	2.06	2.04
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.16	2.07	2.03	2.00
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.12	2.04	1.99	1.97

This is the value we use for a 95% HT when $\alpha=0.05$, $n_1=6$, and $n_2=11$.

The book only has $\alpha=0.05$, but would have another page for each α value.

7.8 Tests with More than Two Independent Samples, Continuous Outcome (ANOVA)

Low-Calorie	Low-Fat	Low-Carbohydrate	Control
8	2	3	2
9	4	5	2
6	3	4	-1
7	5	2	0
3	1	3	3
6.6	3.0	3.4	1.2

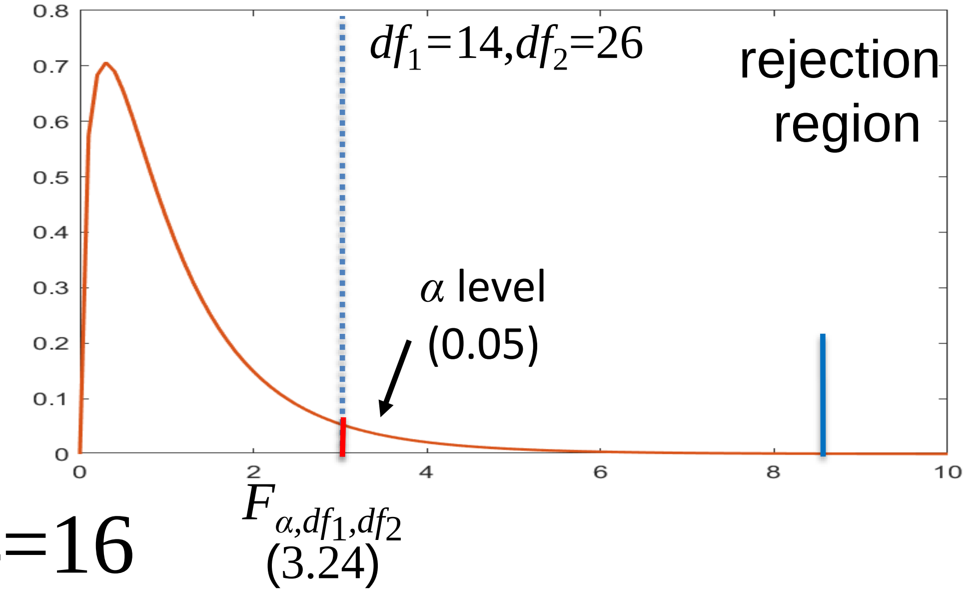
Example: Statistical difference in weight loss among 4 diets?

Step 1: Null and Alternative Hypotheses.

H_0: $\mu_1=\mu_2=\mu_3=\mu_4$ *vs.* H_1: at least two different

Step 2: Test Statistic.

$$F = MSB / MSE \qquad df_1=k\text{-}1 \qquad df_2=N\text{-}k$$



Step 3: Decision Rule. $\alpha=0.05$, $df_1=4\text{-}1=3$, $df_2=20\text{-}4=16$

Reject H_0 if $F \geq 3.24$.

$n_1=n_2=n_3=n_4=5$

Step 4: Compute test statistic.

to be calculated

$$MSB = \frac{\sum n_j(\bar{X}_j - \bar{X})^2}{k-1} = 25.3$$

$$F = 25.3 / 3.0 = 8.43$$

$$MSE = \frac{\sum\sum n_j(X - \bar{X}_j)^2}{N-k} = 3.0$$

Step 5: Conclusion

Because $8.43 > 3.24$, reject H_0 and conclude diets mean weight loss different.

7.9 Tests for Two or More Independent Samples, Categorical and Ordinal Outcomes

Hypothesis test follows similar to Section 7.4 one sample.

7.10 Summary

TABLE 7-50 Summary of Key Formulas for Tests of Hypothesis

Outcome Variable, Number of Groups: Null Hypothesis	Test Statistic*
Continuous outcome, two independent samples: $H_0: \mu_1 = \mu_2$	$z = \dfrac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{1/n_1 + 1/n_2}}$
Continuous outcome, two matched samples: $H_0: \mu_d = 0$	$z = \dfrac{\bar{X}_d - \mu_d}{s_d / \sqrt{n}}$
Continuous outcome, more than two independent samples: $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$	$F = \dfrac{\Sigma n_j(\bar{X}_j - \bar{X})^2 / (k-1)}{\Sigma\Sigma(X - \bar{X}_j)^2 / (N-K)}$
Dichotomous outcome, one sample: $H_0: p = p_0$	$z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$
Dichotomous outcome, two independent samples: $H_0: p_1 = p_2$, RD = 0, RR = 1, OR = 1	$z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}}$
Categorical or ordinal outcome, one sample: $H_0: p_1 = p_{10}, p_2 = p_{20}, \ldots, p_k = p_{k0}$	$\chi^2 = \Sigma\dfrac{(O-E)^2}{E}, df = k-1$
Categorical or ordinal outcome, two or more independent samples: H_0: Outcome and groups are independent	$\chi^2 = \Sigma\dfrac{(O-E)^2}{E}, df = (r-1)(c-1)$

Questions?

Homework 7 Part II

Read Chapter 7.

Problems # 12, 18, 10, 19, 11