Chapter 6: Confidence Interval Estimates

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6.1 Introduction to Estimation

A **Point Estimate** for a population parameter is a single-valued estimate of that parameter. i.e. $\bar{X}$ for μ or s^2 for σ^2.

A **Confidence Interval** (CI) estimate is a range of values for a population parameter with a confidence attached (i.e., 95%).

A CI starts with the point estimate and builds in what is called the **Margin of Error**. The margin of error incorporates probabilities.

$$\underbrace{\bar{X}}_{\text{PE}} \pm \underbrace{\text{that depends on a probability}}_{\text{ME}}$$

6.1 Introduction to Estimation

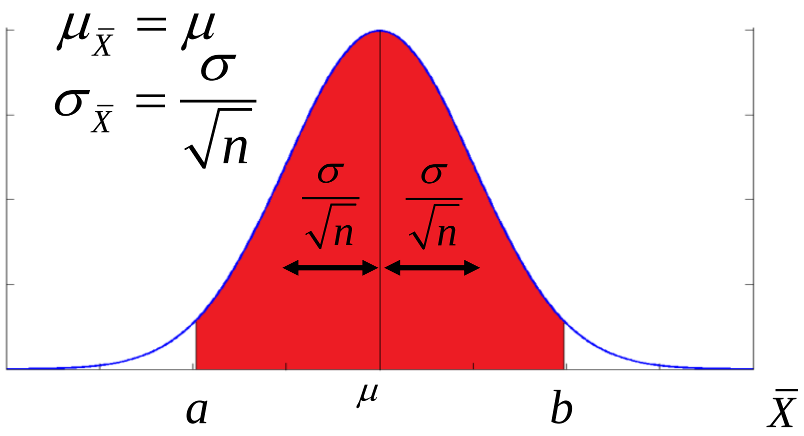
We will have confidence intervals for several scenarios.

Number of Samples	Outcome Variable	Parameter to be Estimated
One sample	Continuous	Mean: μ
Two independent samples	Continuous	Difference in means: $\mu_1 - \mu_2$
Two dependent, matched samples	Continuous	Mean difference: $\mu_d = \mu_1 - \mu_2$
One sample	Dichotomous	Proportion (e.g., prevalence, cumulative incidence): p
Two independent samples	Dichotomous	Difference or ratio of proportions (e.g., attributable risk, relative risk, odds ratio): $p_1 - p_2$, p_1 / p_2

6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is "large" then $\bar{X}$ has a normal distribution.

We can make probability statements. First we assume σ known.



σ known

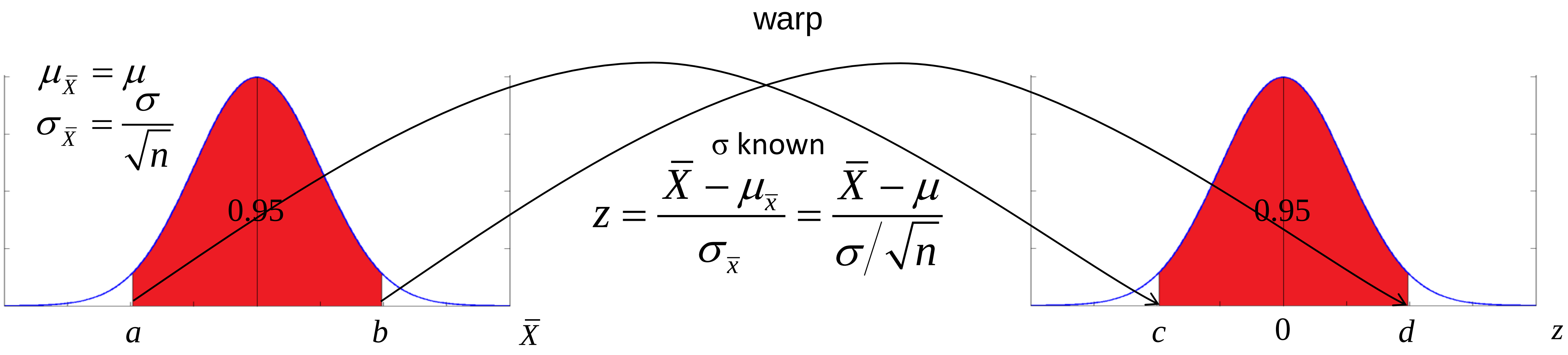
$$P(a < \bar{X} < b) = 0.95$$

Want to find a and b.

6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is “large” then $\bar{X}$ has a normal distribution.

We convert from $\bar{X}$ to z and find our values for area (probability).



σ known

$P(a < \bar{X} < b) = 0.95$

$P(c < z < d) = 0.95$

We get that c=-1.96 and d=+1.96.

6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is "large" then $\bar{X}$ has a normal distribution.

Performing a little algebra.

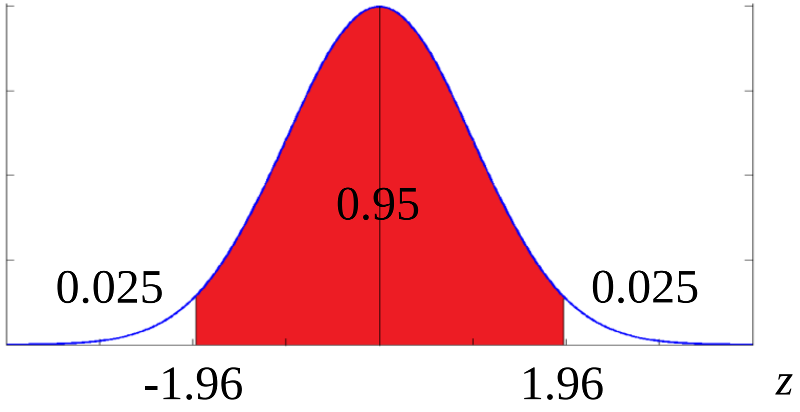
$$1.96 > z \qquad\qquad -1.96 < z$$

$$1.96 > \frac{\bar{X}-\mu_{\bar{x}}}{\sigma_{\bar{x}}} \qquad\qquad -1.96 < \frac{\bar{X}-\mu_{\bar{x}}}{\sigma_{\bar{x}}}$$

$$1.96\frac{\sigma}{\sqrt{n}} > \bar{X}-\mu \qquad\qquad -1.96\frac{\sigma}{\sqrt{n}} < \bar{X}-\mu$$

$$1.96\frac{\sigma}{\sqrt{n}}-\bar{X} > -\mu \qquad\qquad -1.96\frac{\sigma}{\sqrt{n}}-\bar{X} < -\mu$$

$$\bar{X}-1.96\frac{\sigma}{\sqrt{n}} < \mu \qquad\qquad \bar{X}+1.96\frac{\sigma}{\sqrt{n}} > \mu$$



$$P(c<z<d)=0.95$$

from table

$$c=-1.96 \qquad d=1.96$$

σ known

$$\bar{X}-1.95\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}+1.95\frac{\sigma}{\sqrt{n}}$$

Look for 0.975 in the table and note row 1.9 and col 0.06. The z value that has an area of 0.975 less than it is 1.96.

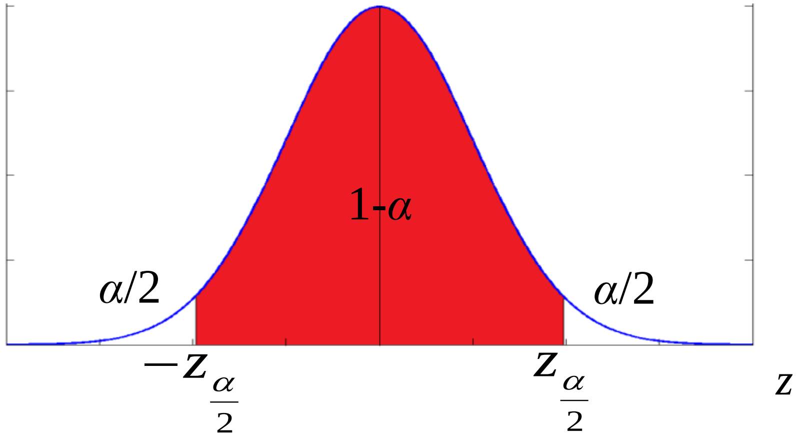
6.1 Introduction to Estimation – One Sample, Continuous, Mean

We know that if n is "large" then $\bar{X}$ has a normal distribution.

σ known

Therefore a 95% CI for μ is $\bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$.

$$P(c < z < d) = 0.95$$



More generally, a $100(1\text{-}\alpha)$% CI for μ is:

σ known

$$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$P(-z_{\frac{\alpha}{2}} < z < z_{\frac{\alpha}{2}}) = 1 - \alpha$$

from table

where $z_{\frac{\alpha}{2}}$ is the z value that has area of $\alpha/2$ larger.

May see $z(\alpha/2)$.

6.2 Confidence Intervals for One Sample, Continuous Outcome

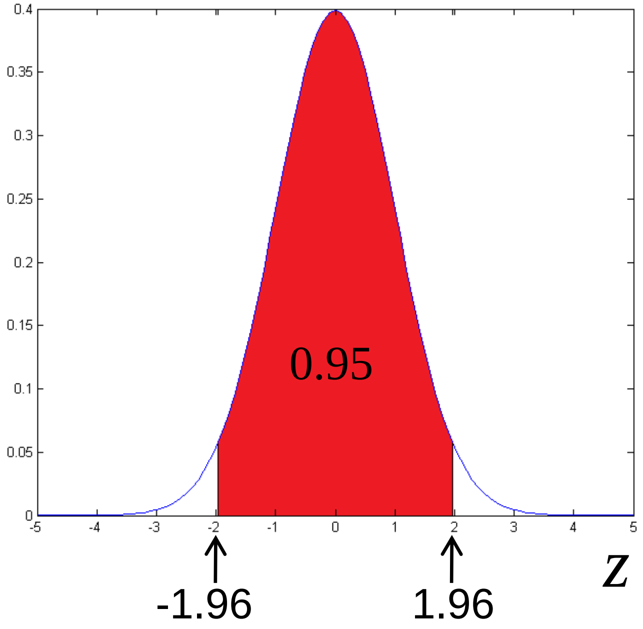
However, we never truly know what σ is, so we estimate it by s.

But when we do this our distribution changes from normal-z to Student-t.

ignore

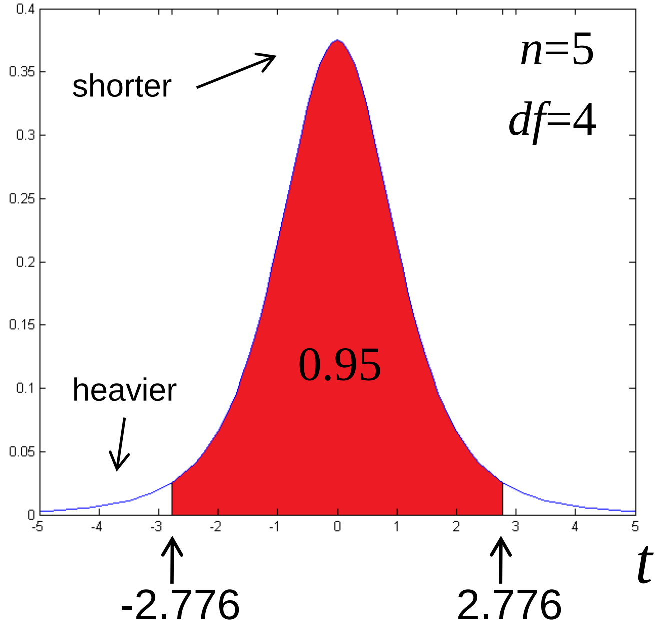
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$



σ, z ←

s, t →



ignore

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1 + \frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$

$\nu = df = n - 1$

$$t = \frac{\bar{X} - \mu}{s / \sqrt{n}}$$

σ known

$$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \longrightarrow \bar{X} \pm t_{\frac{\alpha}{2},df} \frac{s}{\sqrt{n}}$$

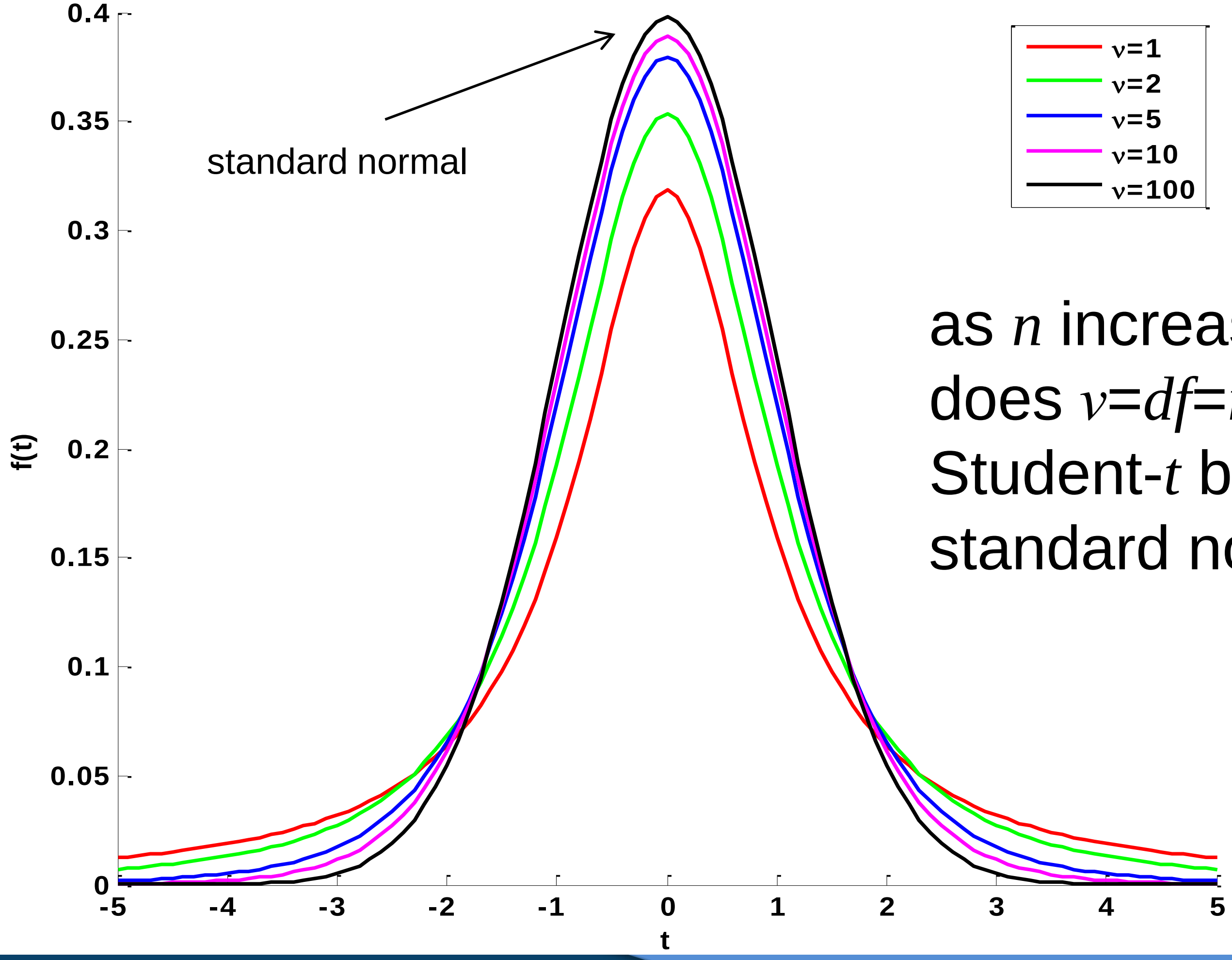
σ unknown

By using s instead of σ we have more variability.

Additional variability depends on n. $s \to \sigma$ as $n \nearrow$

6.2 Confidence Intervals for One Sample, Continuous Outcome

The Student-t distribution is like a superset of the normal distribution.



as n increases, so does v=df=n-1 and Student-t becomes standard normal

Confidence Level	80%	90%	95%	98%	99%
Two-Sided Test α	.20	.10	.05	.02	.01
One-Sided Test α	.10	.05	.025	.01	.005
df					
1	3.078	6.314	12.71	31.82	63.66
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
$Z \rightarrow$ ∞	1.282	1.645	1.960	2.326	2.576

6.2 Confidence Intervals for One Sample, Continuous Outcome

Example: Find the value of $t_{0.025,10}$.

$\alpha/2$ $df=n$-1

The (critical) value of t that has an area of 0.025 larger than it when we have 10 degrees of freedom is 2.228.

This is the value we use for a 95% CI when α=0.05 and n=11.

Book says $n\geq30$ use bottom z value.

Confidence Level	80%	90%	95%	98%	99%
Two-Sided Test α	.20	.10	.05	.02	.01
One-Sided Test α	.10	.05	.025	.01	.005
$df=n$-1					
1	3.078	6.314	12.71	31.82	63.66
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
z → ∞	1.282	1.645	1.960	2.326	2.576

6.2 Confidence Intervals for One Sample, Continuous Outcome

$100(1-\alpha)\%$ ↙

Example: Suppose we wish to compute a 95% CI for true systolic BP. A random sample of n=10 is take with sample mean $\bar{X}$=121.2 mm Hg and sample standard deviation s=11.1 mm Hg.

n=10, df=9 and α=0.05

6.2 Confidence Intervals for One Sample, Continuous Outcome

100(1-α)%

Example: Suppose we wish to compute a 95% CI for true systolic BP.

A random sample of n=10 is take with sample mean $\bar{X}$=121.2 mm Hg

and sample standard deviation s=11.1 mm Hg.

The equation (when σ unknown) is $\bar{X} \pm t_{\frac{\alpha}{2},df} \frac{s}{\sqrt{n}}$, df=n-1.

We find the critical t value in the table.

Down to row df=9 and over to column

CI=95% (or two side-test α=0.05,

or one side-test α=0.025).

$$121.2 \pm (2.262)\frac{11.1}{\sqrt{10}} \rightarrow 113.3 \ \ to\ 129.1$$

mm Hg

n=10, df=9 and α=0.05

Confidence Level	80%	90%	95%	98%	99%
Two-Sided Test α	.20	.10	.05	.02	.01
One-Sided Test α	.10	.05	.025	.01	.005
df					
1	3.078	6.314	12.71	31.82	63.66
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9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169

6.3 Confidence Intervals for One Sample, Dichotomous Outcome

Dichotomous when we have Yes/No or Heads/Tails or 1/0 outcomes.

Think of data

Yes	No	Yes	Yes	No	Yes	No
Heads	Tails	Heads	Heads	Tails	Heads	Tails
1	0	1	1	0	1	0

If we estimate the probability p of Yes or Heads or 1 as

$\hat{p} = \frac{x}{n}$ where x is the number of Yes or Heads or 1.

We think of this as an average $\hat{p} = \frac{1+0+1+1+0+1+0}{7}$.

If n is "large," then the central limit theorem applies.

The **Central Limit Theorem** (**CLT**) says, that if n is large, i.e. $n>30$, then the average has an approximately normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$ no matter what original distribution the data $x_1,\ldots,x_n$ came from.

6.3 Confidence Intervals for One Sample, Dichotomous Outcome

Going through the same process as for μ we will find that:

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \text{or} \quad \hat{p} - z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

This CI only applies when n is large and $\min[n\hat{p}, n(1-\hat{p})] \geq 5$.

We use the z table as we did for CI for μ when n large.

No small n CI here. Do not use Student-t.

6.3 Confidence Intervals for One Sample, Dichotomous Outcome

Example: A sample of n=3532 hypertensive patients were examined and it was found that x=1219 were treated. Generate a 95% CI for p.

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\hat{p} = \frac{x}{n} = \frac{1219}{3532} = 0.345$$

$$0.345 \pm 1.96 \sqrt{\frac{0.345(1-0.345)}{3532}}$$

from table

$$0.345 \pm 0.016$$

$$0.320 < p < 0.361$$

A 95% CI for the probability a randomly selected person is treated.

6.4 Confidence Intervals for Two Independent Samples, Continuous Outcome

We often have two independent samples,

$x_1,\ldots,x_{n_1}$ from population 1 with mean μ_1 and standard deviation σ_1 and
$x_1,\ldots,x_{n_2}$ from population 2 with mean μ_2 and standard deviation σ_2.

Because of the central limit theorem, if n_1 and n_2 are "large," then $\bar{X}_1$ and $\bar{X}_2$ have normal distributions with means $\mu_{\bar{X}_1} = \mu_1$ and $\mu_{\bar{X}_2} = \mu_2$ and standard deviations $\sigma_{\bar{X}_1} = \sigma_1 / \sqrt{n_1}$ and $\sigma_{\bar{X}_2} = \sigma_2 / \sqrt{n_2}$.

Note variances add standard deviations do not.

6.4 Confidence Intervals for Two Independent Samples, Continuous Outcome

We can go through the same process as for one sample.

Therefore a 95% CI for μ_1-μ_2 is

$$(\bar{X}_1-\bar{X}_2)\pm t_{\frac{\alpha}{2},df}S_P\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \quad \text{for } n_1,n_2\leq 30$$

from table

$$(\bar{X}_1-\bar{X}_2)\pm z_{\frac{\alpha}{2}}S_P\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \quad \text{for } n_1,n_2\geq 30.$$

from table

The **Central Limit Theorem** (**CLT**) says, that if n is large, i.e. $n>30$, then the average has an approximately normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$ no matter what original distribution the data $x_1,\ldots,x_n$ came from.

6.4 Confidence Intervals for Two Independent Samples, Continuous Outcome

$$S_P^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$$

Example: A sample of n=10 males and females had systolic blood pressure measured. The data are: males: n_1=6, $\bar{X}_1$=117.5 mm Hg s_1=9.7 mm Hg and females n_2=4, $\bar{X}_2$=126.8 mm Hg, s_2=12 mm Hg.

Generate a 95% CI for μ_1-μ_2.

$$\bar{X}_1-\bar{X}_2 \pm t_{\frac{\alpha}{2},df} S_P\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \quad , \quad df = n_1+n_2-2 \quad , \quad S_P = \sqrt{\frac{(n_1-1)(s_1)^2+(n_2-1)(s_2)^2}{n_1+n_2-2}}$$

$$df = 6+4-2 = 8 \qquad S_P = \sqrt{\frac{(6-1)(9.7)^2+(4-1)(12.1)^2}{6+4-2}} = 10.6 \text{ mm Hg}$$

$$(117.5_1-126.8)\pm(2.306)(10.6)\sqrt{\frac{1}{6}+\frac{1}{4}} \longrightarrow -9.3\pm 15.78 \text{ mm Hg} \longrightarrow -25.08 \;\; to \;\; 6.48 \text{ mm Hg}$$

from table (2.306)

6.4 Confidence Intervals for Two Dependent Samples, Continuous Outcome

We often encounter two samples where there are matched pairs.
This is often the case for before vs. after, twins, couples, etc.
We subtract x_1 from sample 1 and x_2 from sample 2 for each pair.

The differences are labeled generically d=x_1-x_2 and so the sample of differences is $d_1,\ldots,d_n$. Because of the central limit theorem, $\bar{X}_d$ has a mean of μ_d and standard deviation of $\sigma_d/\sqrt{n}$.

Once we have these differences we treat them exactly the same as we did in Section 6.2 CIs for One Sample, Continuous Outcome.

$$\bar{X}_d \pm t_{\frac{\alpha}{2},df} \frac{s_d}{\sqrt{n}}$$

6.4 Confidence Intervals for Two Independent Samples, Continuous Outcome

Example: Difference in Systolic blood pressure between two visits.

$$\bar{X}_d \pm t_{\frac{\alpha}{2},df} \frac{s_d}{\sqrt{n}}$$

Compute a 95% CI.

$$\bar{X}_d = \frac{-79.0}{15} - 5.3 \text{ mm Hg}$$

$$s_d = \sqrt{\frac{2296.95}{15-1}} = \sqrt{164.07} = 12.8 \text{ mm Hg}$$

$$-5.3 \pm (2.145)\frac{12.8}{\sqrt{15}} \longrightarrow -5.3 \pm 7.1 \longrightarrow -12.4 \; to \; 1.8 \text{ mm Hg}$$

Subject Identification Number	Examination 6	Examination 7	Difference	Difference – $\bar{X}_d$	(Difference – $\bar{X}_d$)^2
1	168	141	–27	–21.7	470.89
2	111	119	8	13.3	176.89
3	139	122	–17	–11.7	136.89
4	127	127	0	5.3	28.09
5	155	125	–30	–24.7	610.09
6	115	123	8	13.3	176.89
7	125	113	–12	–6.7	44.89
8	123	106	–17	–11.7	136.89
9	130	131	1	6.3	39.69
10	137	142	5	10.3	106.09
11	130	131	1	6.3	39.69
12	129	135	6	11.3	127.69
13	112	119	7	12.3	151.29
14	141	130	–11	–5.7	32.49
15	122	121	–1	4.3	18.49
			-79.6	0.5	2296.95

6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

The CI for a difference in proportions, risk difference is:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

We go through the same process.

6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

$$RR = \frac{\hat{p}_1}{\hat{p}_2}$$

The CI for the natural log of relative risk, $ln(RR)$ is:

$$ln(RR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{(n_1 - X_1)/X_1}{n_1} + \frac{(n_2 - X_2)/X_2}{n_2}}$$

CI for relative risk (RR) is:

$$\exp(Lower\ Limit), \exp(Upper\ \ Limit)$$

We go through the same process.

6.6 Confidence Intervals for Two Independent Samples, Dichotomous Outcome

$$OR = \frac{\hat{p}_1 / (1 - \hat{p}_1)}{\hat{p}_2 / (1 - \hat{p}_2)}$$

CI for the natural log of odds ratio, $ln(OR)$ is:

$$ln(OR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{X_1} + \frac{1}{n_1 - X_1} + \frac{1}{X_2} + \frac{1}{n_2 - X_2}}$$

CI for odds ratio, OR is:

$$\exp(Lower\ Limit), \exp(Upper\ \ Limit)$$

We go through the same process.

6.7 Summary

Number of Groups, Outcome: Parameter	Confidence Interval, $n<30$	Confidence Interval, $n\geq30$
One sample, continuous: CI for μ	$\bar{X} \pm t_{\frac{\alpha}{2},df} \frac{s}{\sqrt{n}}$	$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$
One sample, dichotomous: CI for p	(Not taught in this class.)	$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Two independent samples, continuous: CI for μ_1-μ_2	$(\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2},df} S_P \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$ $S_P = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$ $df = n_1+n_2-2$	$(\bar{X}_1 - \bar{X}_2) \pm z_{\frac{\alpha}{2}} S_P \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$ $S_P = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$
Two matched samples, continuous: CI for μ_d=μ_1-μ_2	$\bar{X}_d \pm t_{\frac{\alpha}{2},df} \frac{s_d}{\sqrt{n}}$	$\bar{X}_d \pm z_{\frac{\alpha}{2}} \frac{s_d}{\sqrt{n}}$
Two independent samples, dichotomous: CI for $RD=(p_1-p_2)$	(Not taught in this class.)	$(\hat{p}_1-\hat{p}_2) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$
CI for $ln(RR)=ln(p_1/p_2)$	(Not taught in this class.)	$\ln(RR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{(n_1-X_1)/X_1}{n_1}+\frac{(n_2-X_2)/X_2}{n_2}}$
CI for $RR=p_1/p_2$	(Not taught in this class.)	$\exp(Lower\ Limit), \exp(Upper\ \ Limit)$
CI for $ln(OR)$= $ln([p_1/(1-p_1)]/[p_2/(1-p_2)])$	(Not taught in this class.)	$ln(OR) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{X_1}+\frac{1}{n_1-X_1}+\frac{1}{X_2}+\frac{1}{n_2-X_2}}$
CI for $OR=[p_1/(1-p_1)]/[p_2/(1-p_2)]$	(Not taught in this class.)	$\exp(Lower\ Limit), \exp(Upper\ \ Limit)$

Questions?

Homework 6

Read Chapter 6.

Problems # 1, 3, 5, 7