# Chapter 1: A Review of Basic Concepts A

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# A Review of Basic Concepts

## Describing Quantitative Data Numerically

The mean of a sample of $n$ measurements $y_1,\ldots,y_n$ is

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

The mean of a population is $E(y)=\mu$.

The variance of a sample of $n$ measurements $y_1,\ldots,y_n$ is

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2\right]$$

The mean of a population is $E[(y-\mu)^2]=\sigma^2$.

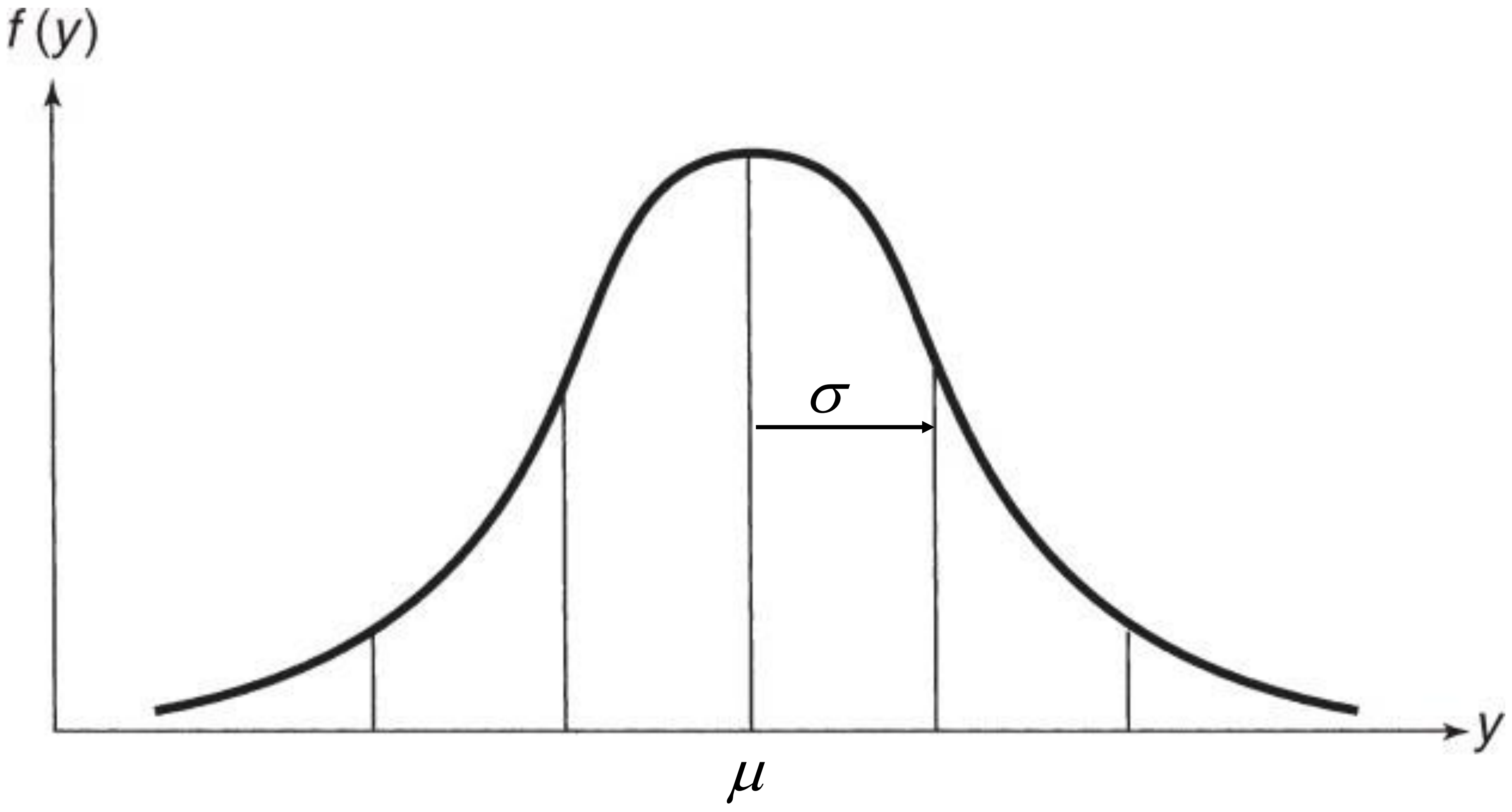
The sample standard deviation is $s$ and the population standard deviation is $\sigma$.

**R Code**

```
Y <- c(1,2,3,4,5)
n <- length(Y)
# sample mean
sumY <- sum(Y)
Ybar <- sumY/n
Ybar
mean(Y)
# sample standard deviation
s2<- sum((Y-Ybar)**2)/(n-1)
s <- sqrt(s2)
sd(Y)
```

# A Review of Basic Concepts

## The Normal Probability Distribution




$$f(y\mid\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

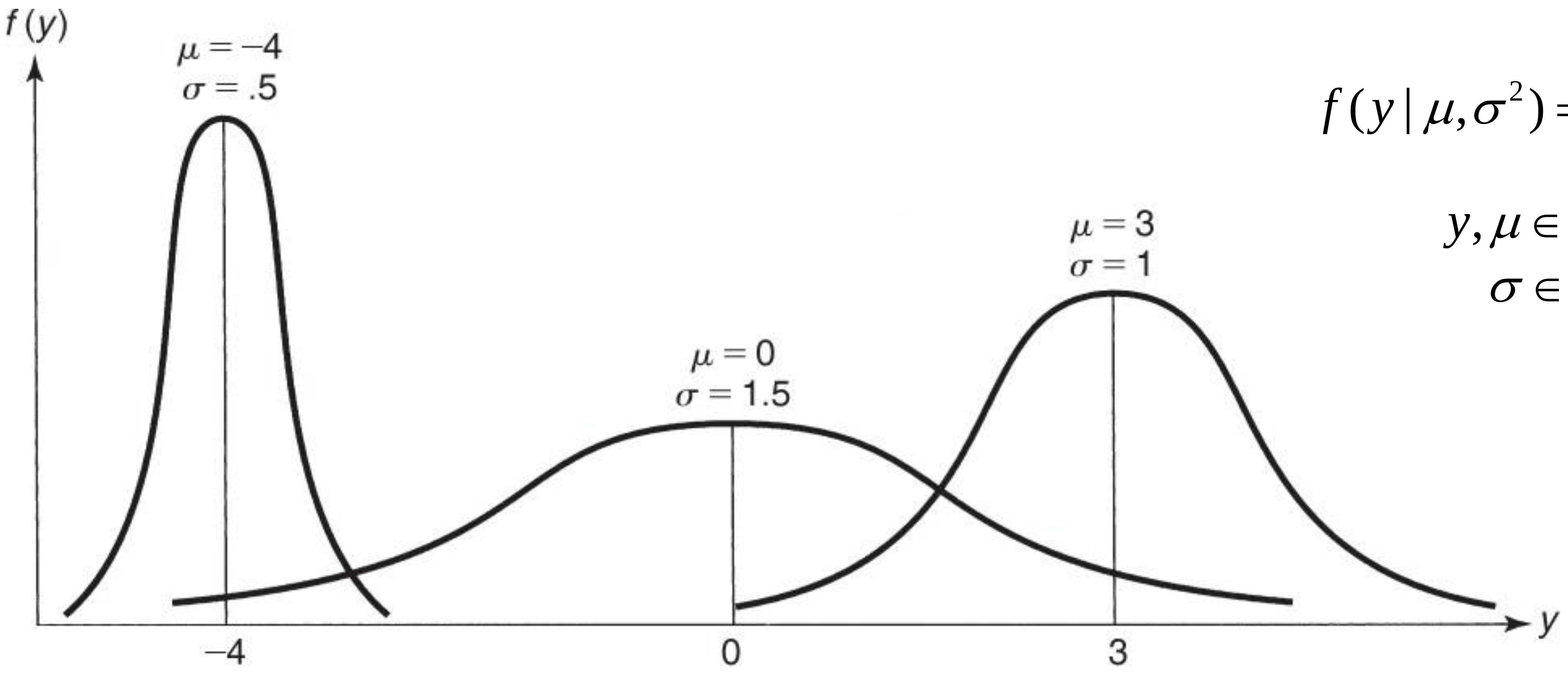
$$y,\mu\in\mathbb{R}$$

$$\sigma\in\mathbb{R}^+$$

$e$ = 2.718281828459046…
$\pi$ = 3.141592653589793…
$\mu$ = population mean
σ = population std. deviation

# A Review of Basic Concepts

## The Normal Probability Distribution




$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$y, \mu \in \mathbb{R}$$

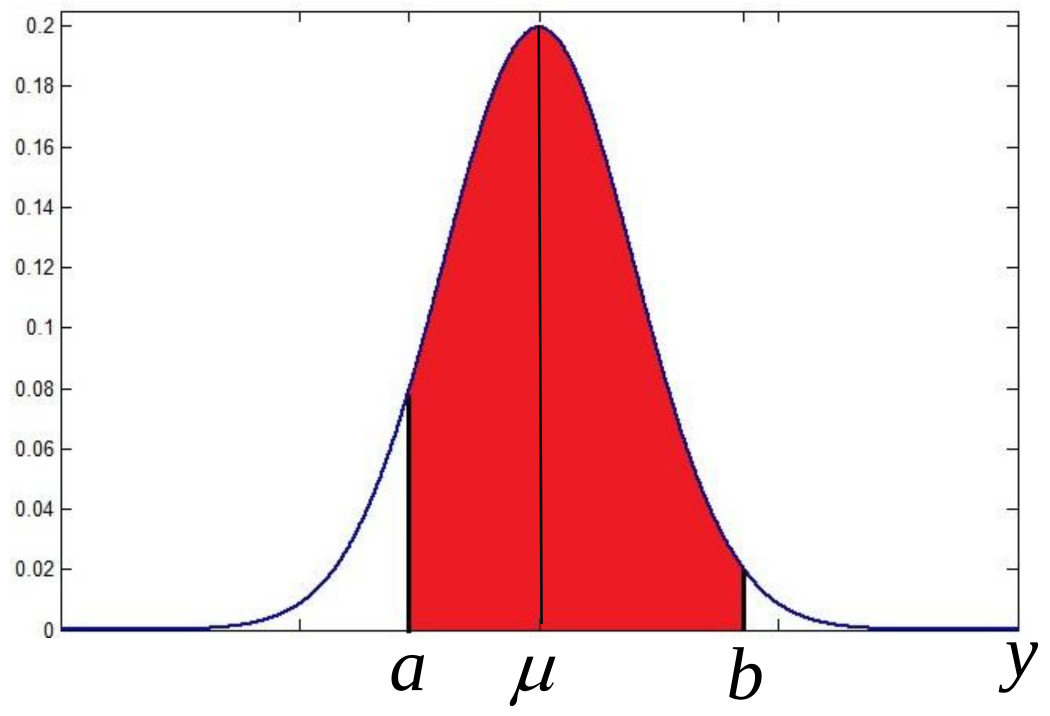
$$\sigma \in \mathbb{R}^+$$

# A Review of Basic Concepts

**The Normal Probability Distribution**

Areas of continuous functions are found with Calculus.

$$A = \int_a^b \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}_{f(y)} dy = P(a < y < b)$$




But we can't integrate the normal distribution. So, we transform to standard normal.

$$z = \frac{y-\mu}{\sigma}$$

And look up the areas in a table.

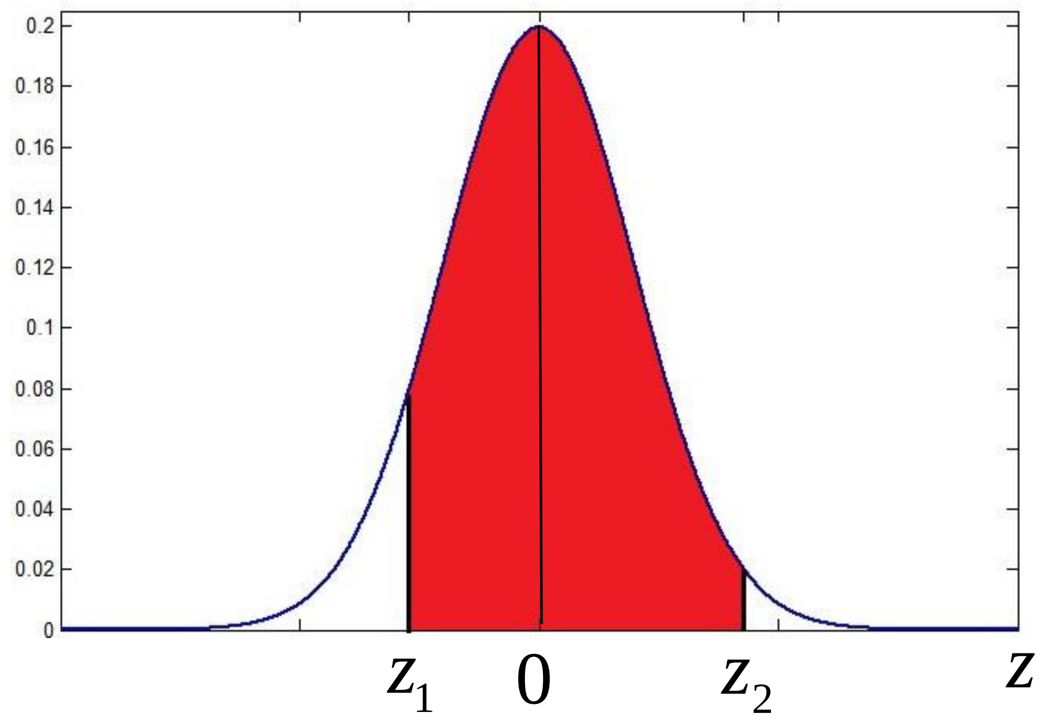
# A Review of Basic Concepts

**The Normal Probability Distribution**

Convert $a$ and $b$ to $z_1$ and $z_2$.

$$z_1 = \frac{a-\mu}{\sigma}$$

$$z_2 = \frac{b-\mu}{\sigma}$$



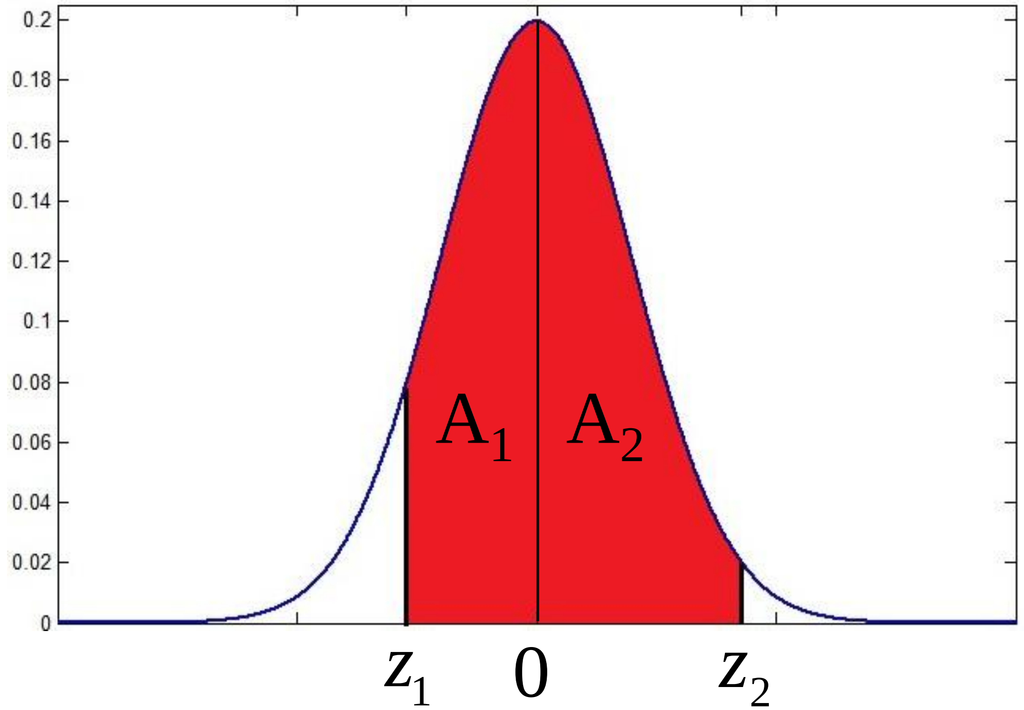

Look up area between $z_1$ and $0$ as $0$ to $z_1$.

Look up area between $0$ and $z_2$.

Add together.

# A Review of Basic Concepts

## The Normal Probability Distribution




Look up area between $z_1$ and 0 as 0 to $z_1$.

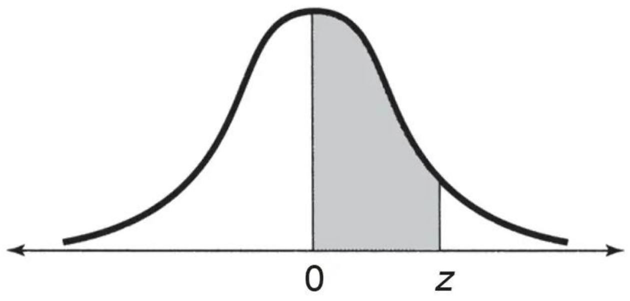
$A_1 = P(a<y<0)$

Look up area between 0 and $z_2$.

$A_2 = P(0<y<b)$

Add together.

$A_1 + A_2$




**R Code**

```
df <- 4
pval <- 0.975
qt(pval, df = df,
lower.tail = FALSE)
```

```
mu <- 0
sd <- 1
y <- 1.96
1-pnorm(y, mu, sd)
```

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| .1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| .2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| .3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| .4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| .5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| .6 | .2257 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2517 | .2549 |
| .7 | .2580 | .2611 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| .8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| .9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |

# A Review of Basic Concepts

## Sampling Distribution and the Central Limit Theorem

**Theorem 1.1: Sampling distribution of the Sampling Mean**

If $y_1,\ldots,y_n$ represent a random sample of $n$ measurements ~~from a large (or infinite) population~~ with mean $\mu$ and standard deviation σ then, regardless of the form of the population relative distribution, the mean and standard error of estimate of the sampling distribution of $\bar{y}$ will be

Mean: $\mu_{\bar{y}} = E(y) = \mu$

Standard error of estimate: $\sigma_{\bar{y}} = \dfrac{\sqrt{E[(y-\mu)^2]}}{\sqrt{n}} = \dfrac{\sigma}{\sqrt{n}}$

$$\mu = \int y f(y) dy < \infty$$

$$\sigma^2 = \int (y-\mu)^2 f(y) dy < \infty$$

# A Review of Basic Concepts

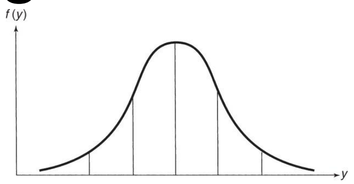
## Sampling Distribution and the Central Limit Theorem

**Theorem 1.2: Central Limit Theorem**

$E(y)$

For large sample sizes, the mean $\bar{y}$ of a sample from a population with mean $\mu$ and standard deviation σ ← $\sqrt{E[(y-\mu)^2]}$

has a sampling distribution that is approximately normal, regardless of the probability distribution of the sampled population.

$n$

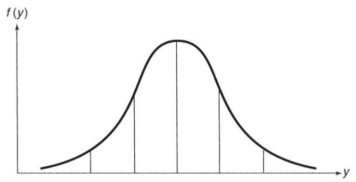
The larger the sample size, the better will be the normal approximation to the sampling distribution of $\bar{y}$ .

# A Review of Basic Concepts

**Sampling Distribution and the CLT**

When $n$ is large,

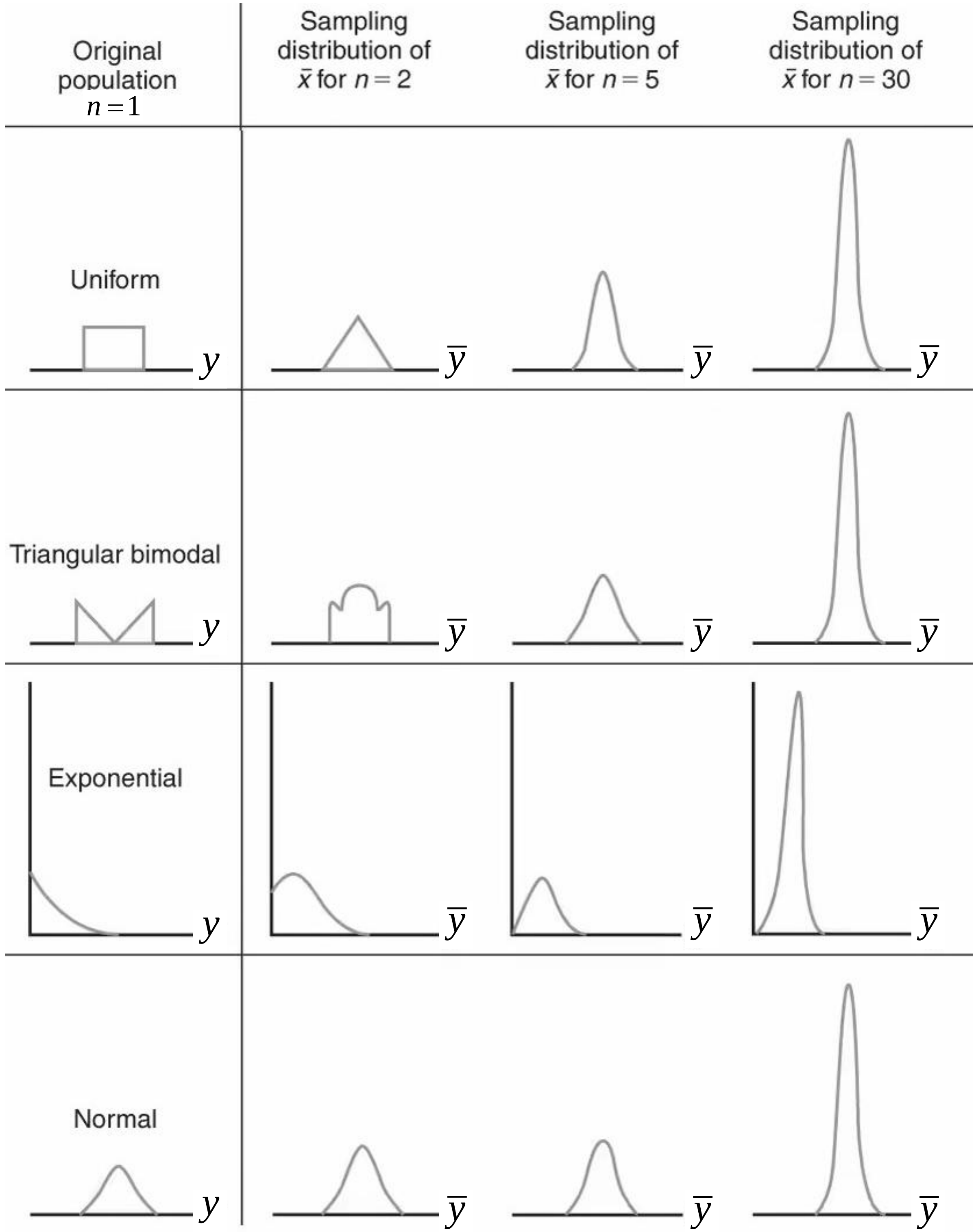
$$\bar{y} \sim N\left(\mu_{\bar{y}} = \mu, \sigma^2_{\bar{y}} = \frac{\sigma^2}{n}\right) \quad ,$$



no matter what distribution our original measurements come from, when $n$ is large, $\bar{y}$ has a normal distribution

This means that we can use the $z$ table to get areas!

$$z = \frac{\bar{y} - \mu}{\sigma / \sqrt{n}}$$



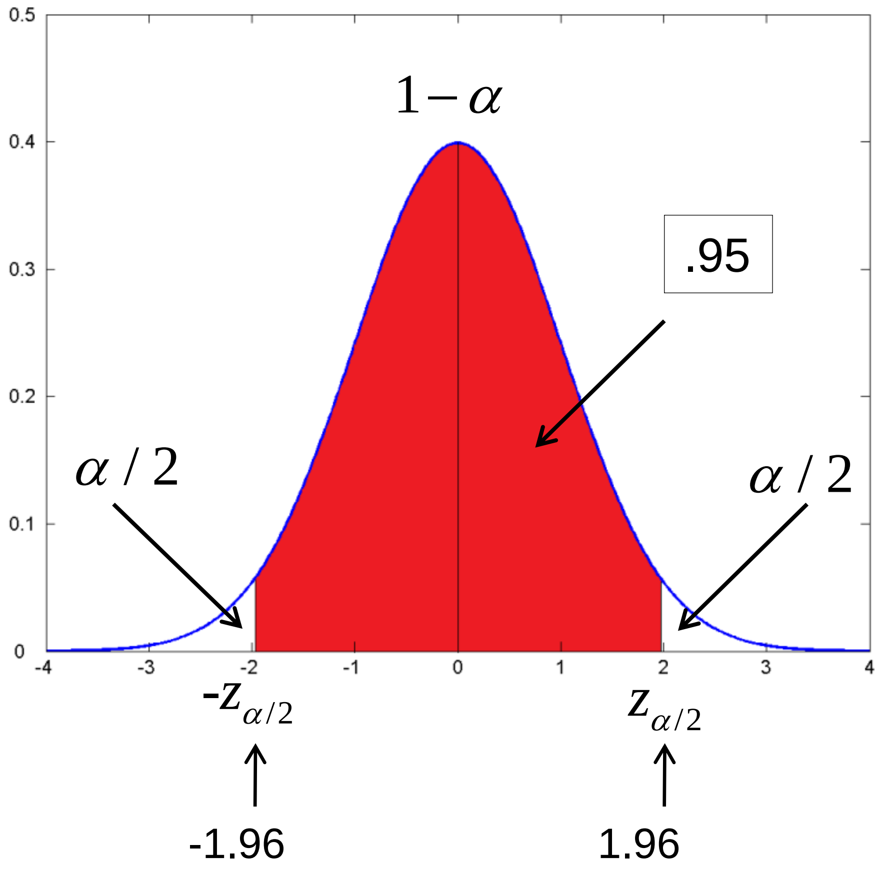
# A Review of Basic Concepts

## Estimating a Population Mean

When we estimate a parameter like $\mu$ with a single value like $\bar{y}$ , it is called a point estimator. We often are interested in a range of values within which we have a prespecified level of confidence that the interval contains $\mu$.

We know that $P(-1.96<z<1.96)=0.95$,
or more generally, $P(-z_{\alpha/2}<z< z_{\alpha/2})=1-\alpha$.

Where $z_{\alpha/2}$ is called the confidence coefficient.
$z_{\alpha/2}$ is the value of $z$ with an area $\alpha/2$ larger than it.



# A Review of Basic Concepts

$$z = \frac{\bar{y} - \mu}{\sigma / \sqrt{n}}$$

## Estimating a Population Mean

With some algebra on $P(-z_{\alpha/2} < z < z_{\alpha/2}) = 1 - \alpha$, we can see that …

$$z < z_{\alpha/2} \qquad\qquad -z_{\alpha/2} < z$$

$$\frac{\bar{y} - \mu_{\bar{y}}}{\sigma_{\bar{y}}} < z_{\alpha/2} \qquad\qquad -z_{\alpha/2} < \frac{\bar{y} - \mu_{\bar{y}}}{\sigma_{\bar{y}}}$$

$$\bar{y} - \mu < z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \qquad\qquad -z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \bar{y} - \mu$$

$$-\mu < z_{\alpha/2} \frac{\sigma}{\sqrt{n}} - \bar{y} \qquad\qquad -z_{\alpha/2} \frac{\sigma}{\sqrt{n}} - \bar{y} < -\mu$$

$$\bar{y} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu \qquad\qquad \mu < \bar{y} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

# A Review of Basic Concepts

## Estimating a Population Mean

Thus, a $(1\text{-}\alpha)\times 100\%$ confidence interval for $\mu$ is

$$\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{to} \quad \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

which if $\alpha$=0.05, a 95% confidence interval for $\mu$ is

$$\bar{y} - 1.96\frac{\sigma}{\sqrt{n}} \quad \text{to} \quad \bar{y} + 1.96\frac{\sigma}{\sqrt{n}} \quad .$$

# A Review of Basic Concepts

## Estimating a Population Mean

However, we never know the true value of σ, so we replace it by $s$

$$\bar{y} - z_{\alpha/2}\frac{s}{\sqrt{n}} \quad \text{to} \quad \bar{y} + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

but then we also need to replace $z$ by $t$, so our CI for $\mu$ is

$$\bar{y} - t_{\alpha/2,df}\frac{s}{\sqrt{n}} \quad \text{to} \quad \bar{y} + t_{\alpha/2,df}\frac{s}{\sqrt{n}} \; ,$$

where $df=n-1$ is our degrees of freedom.

# A Review of Basic Concepts

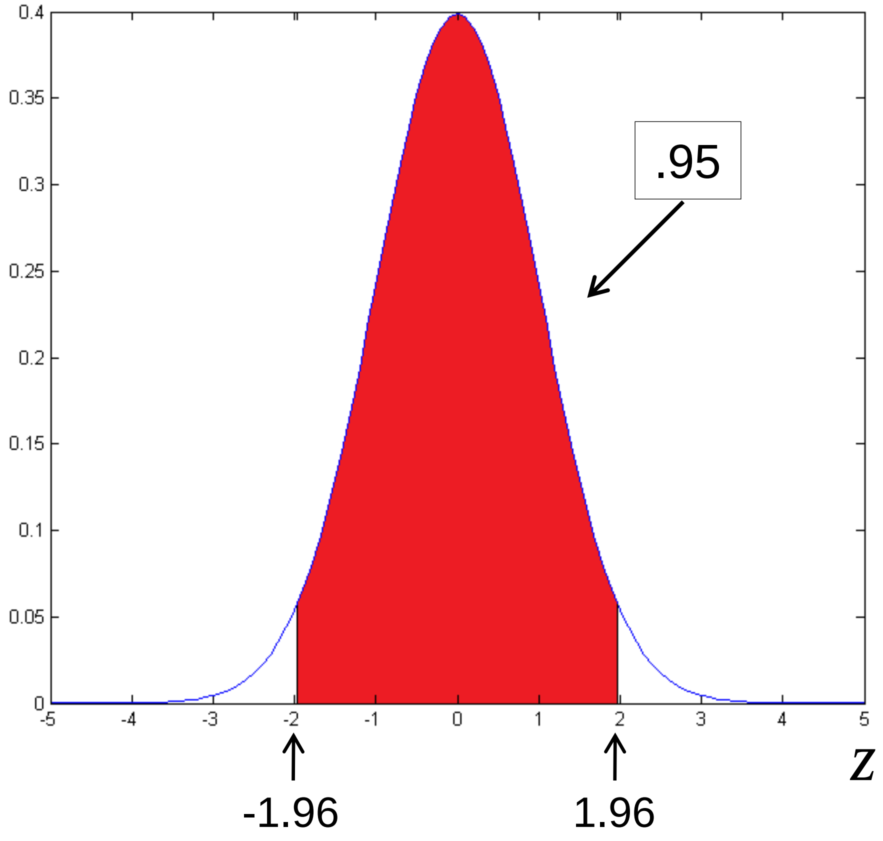
## Estimating a Population Mean

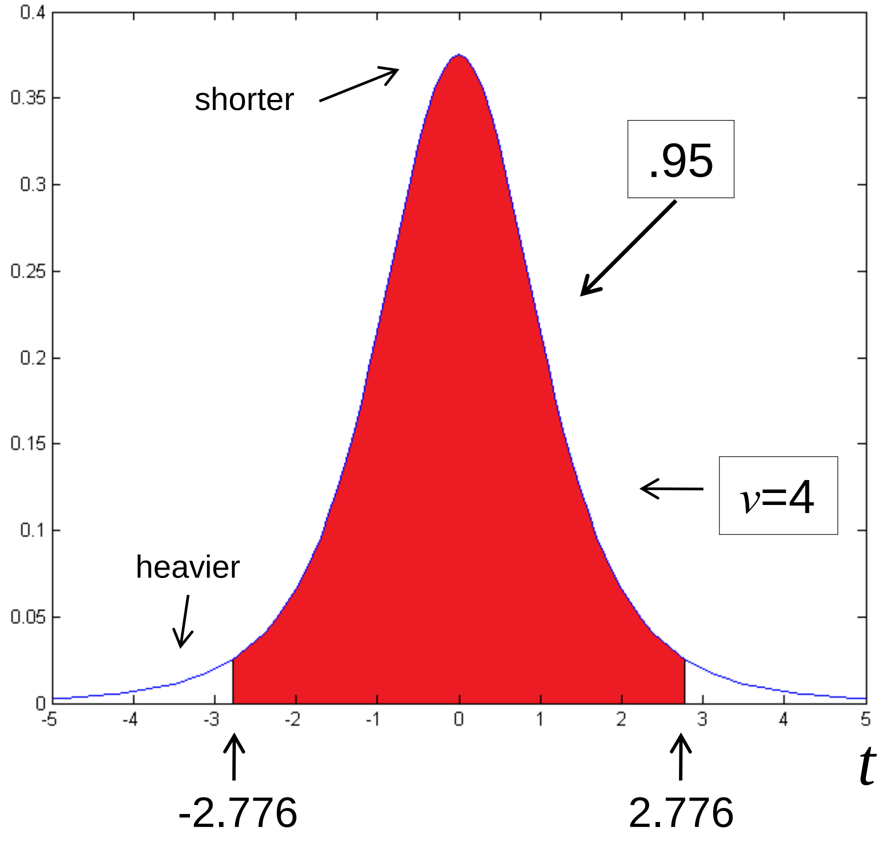
Since we estimated $\sigma$ by $s$ and changed $z$ to $t$, the distribution and areas have changed.

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

$$f(t\,|\,\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$

$\nu = df = n-1$



# A Review of Basic Concepts

## Estimating a Population Mean

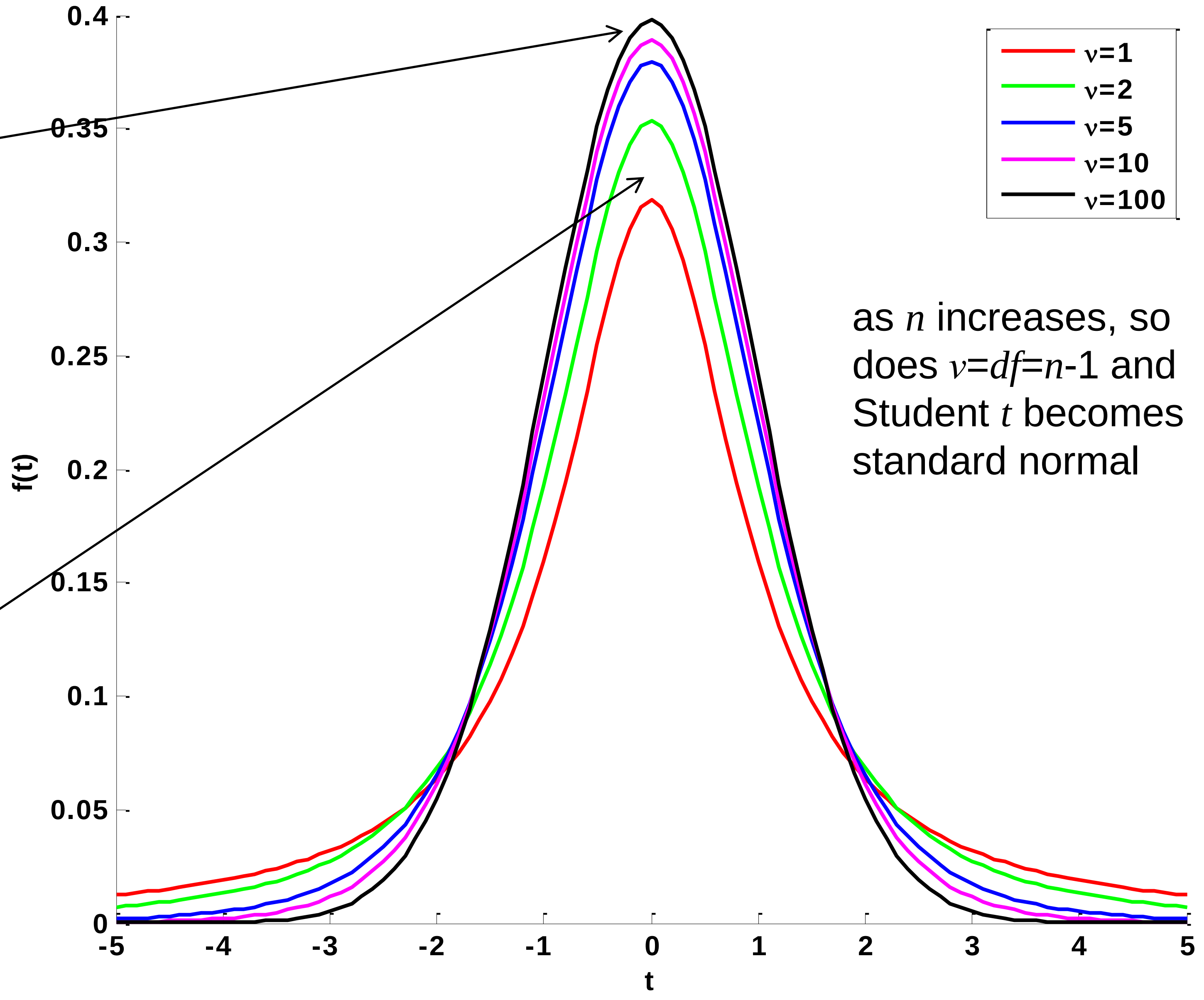
The standard normal dist. is:

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

The Student-t distribution is:

$$f(t|\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$

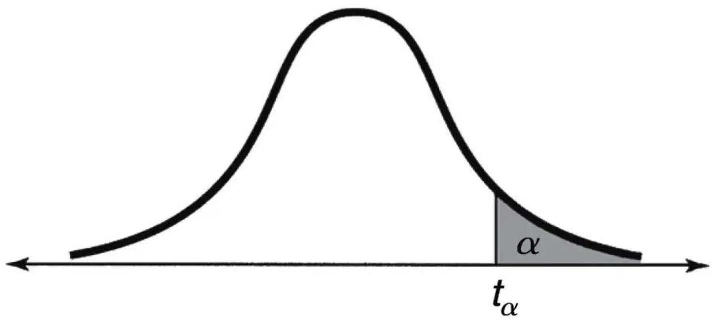
$$f(t|\nu)\rightarrow f(z)$$




as *n* increases, so does *ν*=*df*=*n*-1 and Student *t* becomes standard normal

# A Review of Basic Concepts

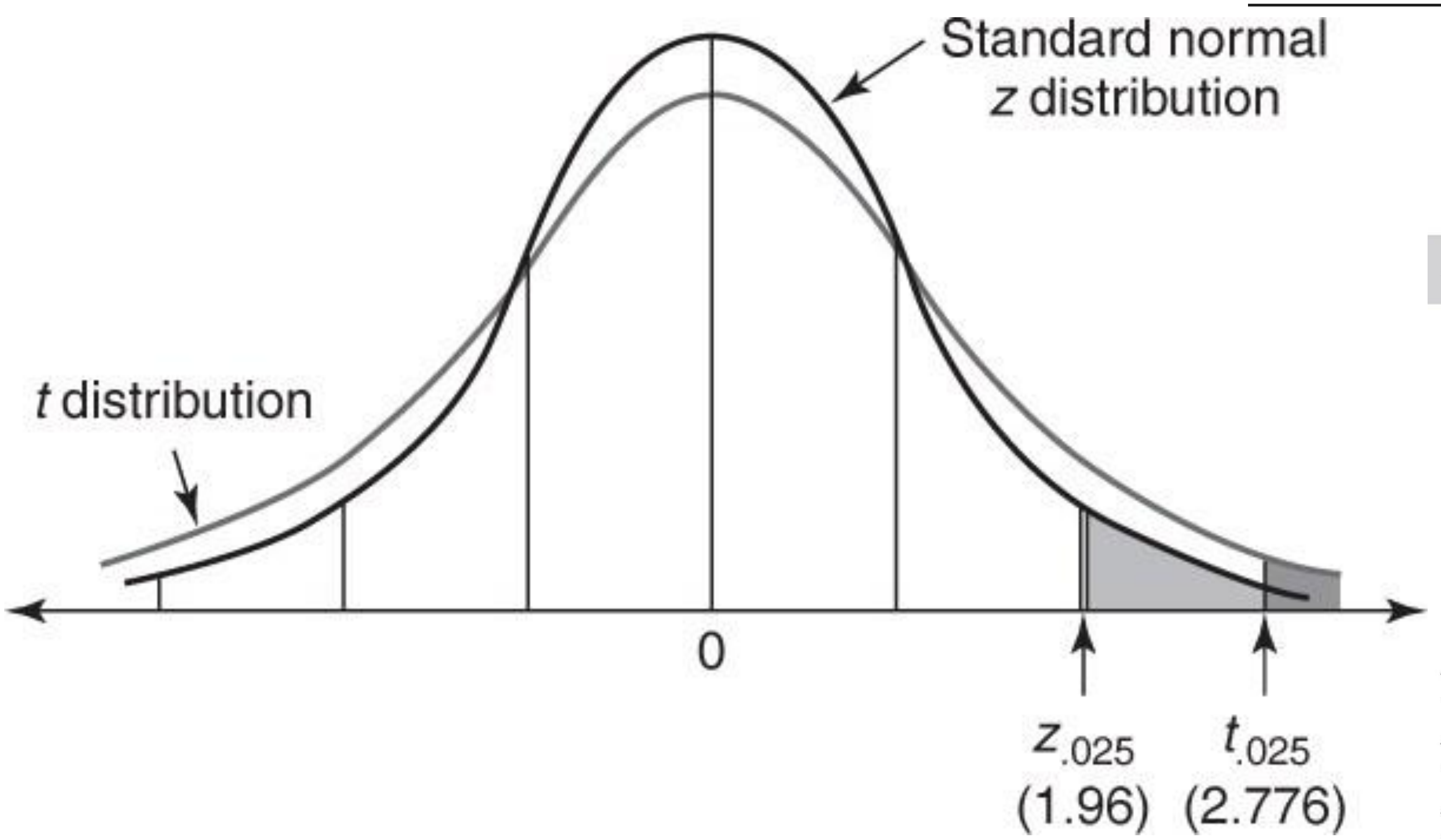
## Estimating a Population Mean




**R Code**

```
mean <- 0
sd <- 1
pval <- 0.975
qt(pval)
```

```
df <- 4
tval <- 2.776
pt(tval, df = df,
lower.tail = FALSE)
```




$t_{0.025,4}$

| Degrees of Freedom | $t_{.100}$ | $t_{.050}$ | $t_{.025}$ | $t_{.010}$ | $t_{.005}$ |
|---|---|---|---|---|---|
| 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| 4 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 |
| 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 |
| 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |

# A Review of Basic Concepts

**Homework:**

Read Chapter 1

## A Review of Basic Concepts

# Questions?