Chapter 10: Introduction to Time Series Modeling and Forecasting B

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Intro to Time Series Modeling & Forecasting

Constructing Time Series Models

The time series model is $y_t=E(y_t)+R_t$ where $E(y_t)=\beta_0+\beta_1 x_1+\ldots+\beta_k x_k$ and R_t represents the random residual.

We assume that the residual R_t has mean 0 and constant variance σ^2, but that residuals are autocorrelated, $Cor(R_t,R_{t+1})\neq 0$.

The time series model consists of a pair of (sub) models,
one for the deterministic component $E(y_t)$ and
one for the autocorrelated residuals R_t.

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Constructing Time Series Models

The deterministic component selected in the same way as for previous regression models, i.e.

$$E(y_t) = \beta_0 + \beta_1 x_t$$

$$E(y_t) = \beta_0 + \beta_1 x_t + \beta_2 x_t^2$$

$$E(y_t) = \beta_0 + \beta_1 x_t + \beta_2 t + \beta_3 x_t t$$

Lagged Independent Variables Model

$$E(y_t) = \beta_0 + \beta_1 x_{t-1}$$

$$E(y_t) = \beta_0 + \beta_1 x_{t-1} + \beta_2 x_{t-1}^2$$

Delayed effect from t-1

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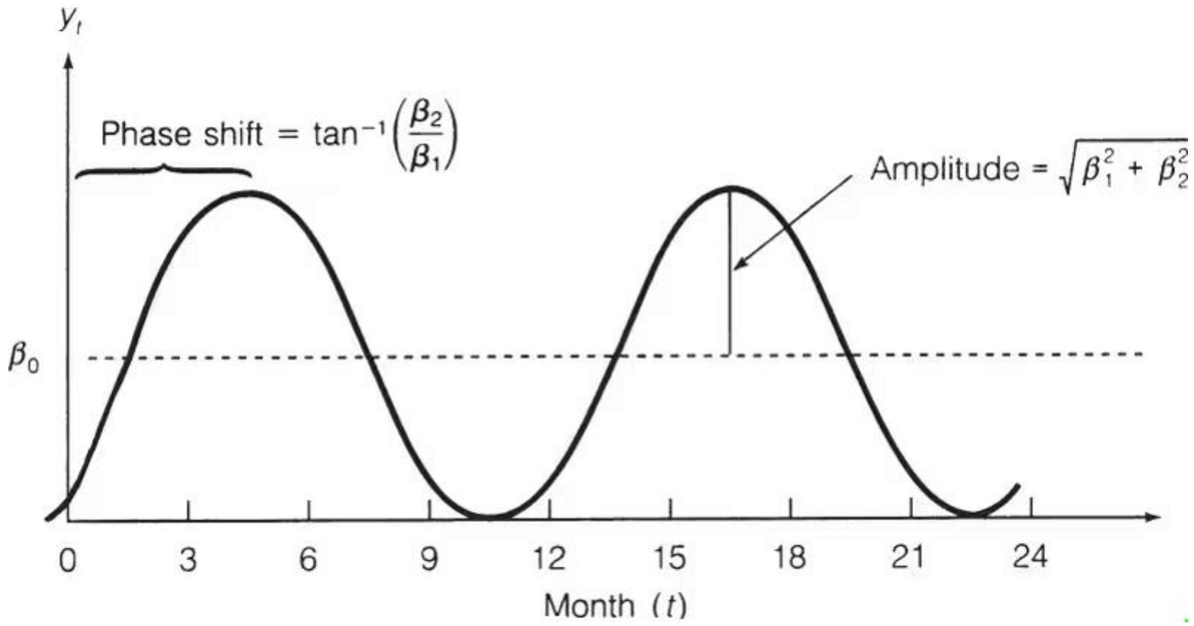
Constructing Time Series Models

Quite often, the deterministic component has distinct seasonal patterns, which can be modeled with

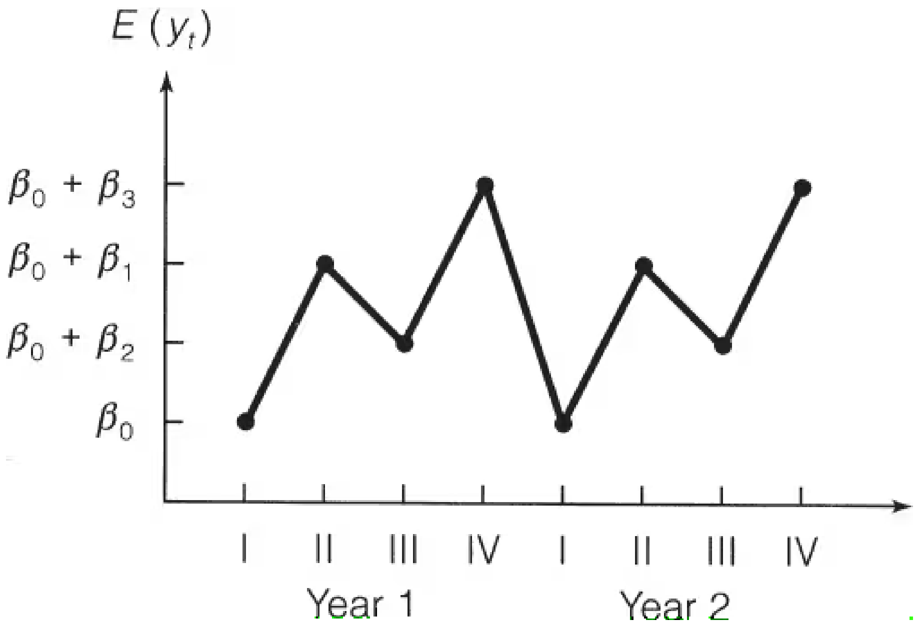
cosines and sines, T=12 months

$$E(y_t) = \beta_0 + \beta_1 \cos 2\pi\left(\frac{1}{T}\right)t + \beta_2 \sin 2\pi\left(\frac{1}{T}\right)t$$



Annual four seasons

$$E(y_t) = \beta_0 + \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3$$



$$Q_1 = \begin{cases} 1 \text{ if spring} \\ 0 \text{ if not} \end{cases}$$

$$Q_2 = \begin{cases} 1 \text{ if summer} \\ 0 \text{ if not} \end{cases}$$

$$Q_3 = \begin{cases} 1 \text{ if fall} \\ 0 \text{ if not} \end{cases}$$

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Constructing Time Series Models

Choosing the residual component R_t depends on the pattern of autocorrelation. The general autoregressive model is

$$R_t = \phi_1 R_{t-1} + \phi_2 R_{t-2} + \cdots + \phi_1 R_{t-p} + \varepsilon_t \ , \quad E(\varepsilon_t)=0, \ Var(\varepsilon_t)=\sigma^2, \ Cor(\varepsilon_t,\varepsilon_{t-1})=0$$

and the simplest of which is the first order AR model

$$R_t = \phi R_{t-1} + \varepsilon_t$$

In practice, the autocorrelation generally decreases as time increases.

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Constructing Time Series Models

We can “whiten” or uncorrelated our residuals by

$$R_t = \phi R_{t-1} + \varepsilon_t$$

$$y_t = \beta_0 + \beta_1 t + R_t$$

$$y_{t-1} = \beta_0 + \beta_1 (t-1) + R_{t-1}$$

$$\phi y_{t-1} = \phi\beta_0 + \phi\beta_1 (t-1) + \phi R_{t-1}$$

$$y_t - \phi y_{t-1} = \beta_0 - \phi\beta_0 + \beta_1 [t - \phi(t-1)] + R_t - \phi R_{t-1}$$

$y_t^* = \beta_0^* + \beta_1^* t^* + \varepsilon_t$, $E(\varepsilon_t)=0$, $Var(\varepsilon_t)=\sigma^2$, $Cor(\varepsilon_t,\varepsilon_{t-1})=0$.

$\hat{\beta}_0 = \hat{\beta}_0^* / (1-\hat{\phi})$ and $\hat{\beta}_1 = \hat{\beta}_1^*$

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Fitting Time Series Models with Autoregressive Errors

SALES35.txt

Example: Data on annual sales y for 35 years.
The model is $E(y) = \beta_0 + \beta_1 t$.

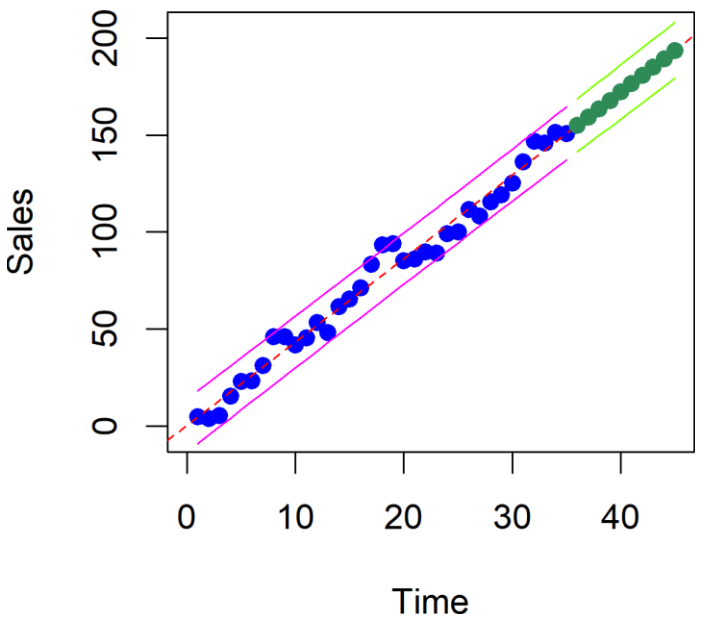
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 = 0.4015 + 4.2956t$$

```
             Estimate Std. Error    t value     Pr(>|t|)
(Intercept) 0.4015126  2.2057083  0.1820334 8.566701e-01
t           4.2956303  0.1068669 40.1960721 1.306377e-29
```

```
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
Model      1  65875   65875  1615.7 < 2.2e-16 ***
Residuals 33   1345      41
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

```
"s,R-squared,adj R-squared"
 6.3852423 0.9799845 0.9793780
```



Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	65875	65875	1615.72	<.0001
Error	33	1345.45355	40.77132		
Corrected Total	34	67221			

Root MSE	6.38524	R-Square	0.9800
Dependent Mean	77.72286	Adj R-Sq	0.9794
Coeff Var	8.21540		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.40151	2.20571	0.18	0.8567
T	1	4.29563	0.10687	40.20	<.0001

T	SALES	RESIDUAL
1	4.8	0.102857
2	4	-4.99277
3	5.5	-7.7884
4	15.6	-1.98403
5	23.1	1.220336
6	23.3	-2.87529
7	31.4	0.929076
8	46	11.23345
9	46.1	7.037815
10	41.9	-1.45782
11	45.5	-2.15345
12	53.5	1.550924
13	48.4	-7.84471
14	61.6	1.059664
15	65.6	0.764034
16	71.4	2.268403
17	83.4	9.972773
18	93.6	15.87714
19	94.2	12.18151
20	85.4	-0.91412
21	86.2	-4.40975
22	89.9	-5.00538
23	89.2	-10.001
24	99.1	-4.39664
25	100.3	-7.49227
26	111.7	-0.3879
27	108.2	-8.18353
28	115.5	-5.17916
29	119.2	-5.77479
30	125.2	-4.07042
31	136.3	2.73395
32	146.8	8.938319
33	146.1	3.942689
34	151.4	4.947059
35	150.9	0.151429

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Fitting Time Series Models with Autoregressive Errors

SALES35.txt

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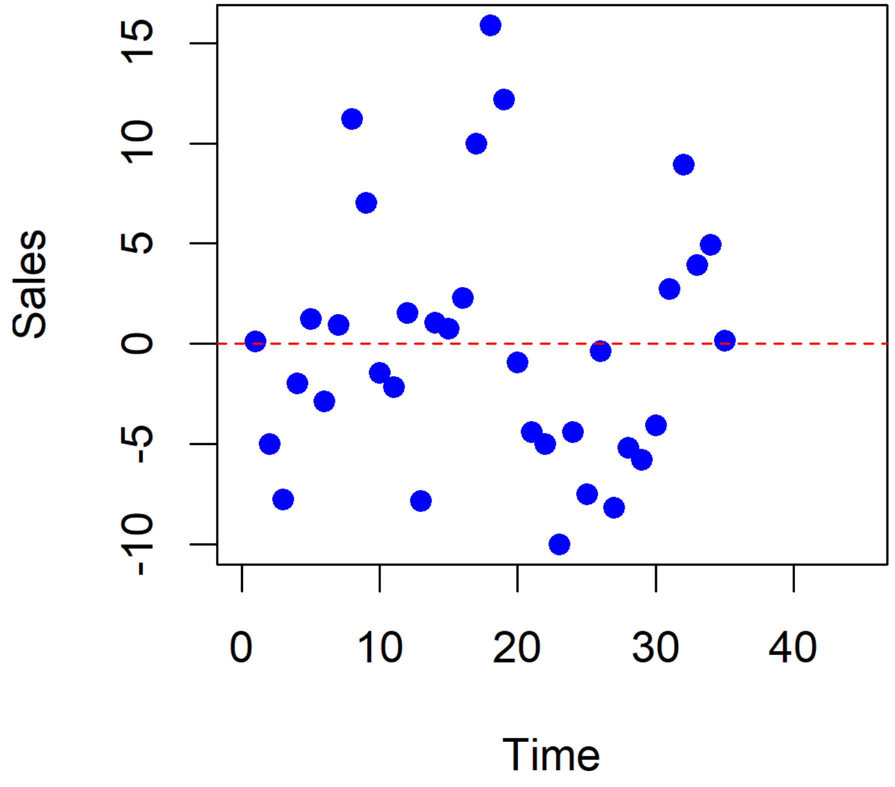
Example: Data on annual sales y for 35 years.
However, the Durbin-Watson statistic d=0.821<d_L=1.40 for α=0.05, n=35, k=1.
This means there is statistically significant AR(1) correlation.

Although the regression estimates are unbiased $E(\hat{\beta}_0)=\beta_0$ and $E(\hat{\beta}_1)=\beta_1$, when we assume no correlation and there is in fact correlation, the the standard errors are usually smaller than the true standard errors and, t-values computed by the methods will usually be inflated and will lead to a higher Type I error rate α.



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Example: Data on annual sales y for 35 years.

To account for autocorrelation, we postulate an AR(1) model,

$y_t = \beta_0 + \beta_1 t + R_t$, $R_t = \phi R_{t-1} + \varepsilon_t$ and estimate β_0, β_1, ϕ by least squares.

Ordinary Least Squares Estimates			
SSE	1345.45355	DFE	33
MSE	40.77132	Root MSE	6.38524
SBC	234.156237	AIC	231.045541
MAE	4.85208163	AICC	231.420541
MAPE	13.8946717	HQC	232.119353
Durbin-Watson	0.8207	Regress R-Square	0.9800
		Total R-Square	0.9800

Parameter Estimates

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.4015	2.2057	0.18	0.8567
T	1	4.2956	0.1069	40.20	<.0001

Estimates of Autocorrelations

Lag	Covariance	Correlation	−1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
0	38.4415	1.000000	| |********************|
1	22.6661	0.589624	| |************ |

Preliminary MSE	25.0771

Estimates of Autoregressive Parameters

Lag	Coefficient	Standard Error	t Value
1	−0.589624	0.142779	−4.13

Yule-Walker Estimates			
SSE	877.685377	DFE	32
MSE	27.42767	Root MSE	5.23714
SBC	223.18683	AIC	218.520786
MAE	4.07427092	AICC	219.29498
MAPE	12.142864	HQC	220.131504
Durbin-Watson	1.8217	Regress R-Square	0.9412
		Total R-Square	0.9869

Parameter Estimates

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.4058	3.9970	0.10	0.9198
T	1	4.2959	0.1898	22.63	<.0001

	Least Squares	Autoregressive
R^2	.980	.987
MSE	40.77	27.43
$\hat{\beta}_0$	.4015	.4058
$\hat{\beta}_1$	4.2956	4.2959
Standard error ($\hat{\beta}_0$)	2.2057	3.9970
Standard error ($\hat{\beta}_1$)	.1069	.1898
t statistic for H_0: $\beta_1 = 0$	40.20	22.63
	($p < .0001$)	($p < .0001$)
$\hat{\phi}$	—	.5896
t statistic for H_0: $\phi = 0$	—	4.13

$$\hat{y}_t = .4058 + 4.2959t + \hat{R}_t$$
$$\hat{R}_t = 0.5896\hat{R}_{t-1}$$

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Fitting Time Series Models with Autoregressive Errors

```
# read data
mydata <- read.delim("SALES35.txt",header=TRUE,sep="",dec=".")

# parse out variables
n <- nrow(mydata)
t <- c(mydata[,1]) #T time
y <- c(mydata[,2]) #y sales
r <- c(mydata[,3]) #r residual

# fit uncorrelated line model and get statistics
mymodel1 <- lm(y~t)

# get uncorrelated error regression coefficients
b0<-mymodel1$coefficients[1]
b1<-mymodel1$coefficients[2]

# scatter plot with uncorrelated error line
plot(t,y,pch=19,col="blue",xlab='Time',ylab='Sales',xlim=c(0,n))
abline(mymodel1,lty=2,col='red')

# uncorrelated error residual plot with 0 line
plot(t,r,pch=19,col="blue",xlab='Time',ylab='Sales',xlim=c(0,n))
abline(lm(r~t),r,lty=2,col='red')
```

```
#AR 1 correlation from uncorrelated error model
phihat<-cor(r[1:(n-1)],r[(1+1):n])

# fit correlated line model and get statistics
yast<-y[2:n]-phihat*y[1:n-1]
tast<-t[2:n]-phihat*t[1:n-1]
mymodel2 <- lm(yast~tast)

# get correlated error regression coefficients
b0ast<-mymodel2$coefficients[1]/(1-phihat)
b1ast<-mymodel2$coefficients[2]
bast<-c(b0ast,b1ast)

# fit and intervals
c   <-rep(1,n)      #Ones
X   <-matrix(cbind(c,t),nrow=n,ncol=2) #design matrix
yasthat<-X%*%bast

# scatter plot with uncorrelated error line
plot(t,y,pch=19,col="blue",xlab='Time',ylab='Sales',xlim=c(0,n))
abline(lm(y~t),r,lty=2,col='red')
lines(t,yasthat, type = "l", col = "green")
```

Intro to Time Series Modeling & Forecasting

Forecasting with Time Series Autoregressive Models

We can use the regression time series model to forecast future observations

$$y_t = \beta_0 + \beta_1 x_t + R_t \;, \quad R_t = \phi R_{t-1} + \varepsilon_t$$

once we have estimated β_0, β_1, ϕ by least squares.
Want to forecast at $t = n+1, n+2, \ldots$

$$y_{t+1} = \beta_0 + \beta_1 x_{t+1} + R_{t+1} \;, \quad R_{t+1} = \phi R_t + \varepsilon_{t+1}$$

$$y_{t+1} = \beta_0 + \beta_1 x_{t+1} + \phi R_t + \varepsilon_{t+1}$$

which is

$$F_{n+1} = \hat{\beta}_0 + \hat{\beta}_1 x_{n+1} + \hat{\phi}\hat{R}_n \;, \qquad \hat{R}_n = y_n - (\hat{\beta}_0 + \hat{\beta}_1 x_n)$$

$$F_{t+2} = \hat{\beta}_0 + \hat{\beta}_1 x_{n+2} + \hat{\phi}\hat{R}_{n+1} \;, \quad \hat{R}_{n+1} = \phi \hat{R}_n, \quad \ldots$$

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Forecasting with Time Series Autoregressive Models

Approximate 95% Forecasting Limits

$$\hat{y}_{n+1} \pm 1.96\sqrt{MSE}$$

$$\hat{y}_{n+2} \pm 1.96\sqrt{MSE(1+\hat{\phi}^2)}$$

$$\hat{y}_{n+3} \pm 1.96\sqrt{MSE(1+\hat{\phi}^2+\hat{\phi}^4)}$$

.

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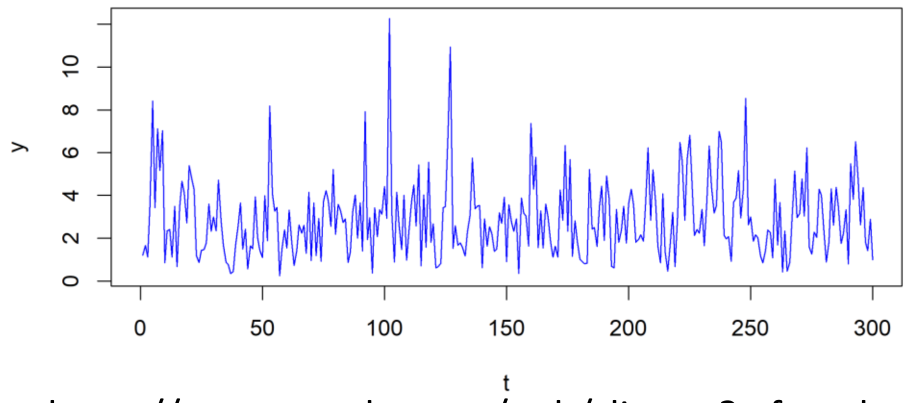
$$\hat{y}_{n+m} \pm 1.96\sqrt{MSE(1+\hat{\phi}^2+\hat{\phi}^4\cdots+\hat{\phi}^{2(m-1)})}$$

Intro to Time Series Modeling & Forecasting

Homework:
Read Chapter 10
Problems #: * (precipitationMKE), 27 (GOLDMON), 28

Submit at minimum one file with all your answers and another with your code.

* Fit a time series regression model to the MKE monthly precipitation data. Assume independent errors. Include all four components $y_t = T_t + C_t + S_t + R_t$. Examine residuals. Interpret.



https://www.weather.gov/wrh/climate?wfo=mkx

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2000	1.20	1.66	1.12	3.64	8.42	3.42	7.12	5.17	7.04	0.84	2.33	2.41
2001	1.11	3.48	0.67	3.45	4.68	4.13	2.70	5.41	4.76	4.29	1.19	0.86
2002	1.44	1.46	1.76	3.59	2.31	2.99	2.33	4.73	2.79	1.66	0.88	0.75
2003	0.35	0.43	1.64	2.61	3.65	1.49	2.43	0.57	1.65	1.51	3.94	2.03
2004	1.43	1.10	3.99	1.87	8.18	4.07	3.25	3.43	0.24	1.47	2.38	1.53
2005	3.31	1.79	0.72	1.41	2.62	2.23	2.60	1.29	4.17	0.95	3.65	1.18
2006	2.92	0.91	3.69	4.23	3.73	2.54	5.23	2.18	3.57	3.30	2.72	2.91
2007	0.86	1.36	3.21	4.02	1.99	3.64	1.40	7.92	1.93	2.96	0.36	3.41
2008	2.07	3.32	3.11	4.42	2.92	12.27	3.20	0.88	4.16	2.62	1.47	4.00
2009	0.97	2.29	3.68	4.50	2.56	5.44	0.71	4.04	1.57	5.57	1.80	2.68
2010	0.62	0.67	0.83	3.42	3.47	6.93	10.93	1.52	2.58	1.66	1.78	1.57
2011	1.16	2.30	3.08	5.75	3.37	3.48	3.53	0.62	2.91	1.63	2.53	2.23
2012	1.37	1.48	3.19	2.69	3.91	0.90	3.56	2.75	2.31	2.91	0.35	3.87
2013	3.17	3.03	1.63	7.38	4.30	5.80	1.55	3.27	1.54	3.59	2.97	1.79
2014	1.11	1.63	1.12	4.26	2.83	6.34	2.31	5.69	1.14	2.81	1.84	1.03
2015	0.91	0.81	0.83	5.22	2.43	2.49	1.60	3.46	4.44	1.90	4.93	3.82
2016	0.69	0.62	3.34	1.80	2.37	3.49	1.76	3.59	4.30	3.56	1.81	1.93
2017	2.16	1.85	3.72	6.23	2.83	5.21	3.69	1.63	0.85	4.09	1.37	0.46
2018	1.62	3.20	0.66	3.11	6.49	5.62	2.83	5.68	6.83	4.50	2.13	2.41
2019	2.23	3.35	1.64	3.77	6.32	4.42	3.17	3.53	7.00	6.48	2.14	1.96
2020	2.07	0.91	3.67	3.88	5.17	2.94	4.34	8.55	2.62	3.00	1.86	2.15
2021	2.02	1.19	0.84	1.41	2.38	2.25	1.07	4.77	1.68	3.67	0.42	2.34
2022	0.46	0.89	2.95	5.15	2.96	3.16	4.79	3.02	6.23	1.59	1.26	2.28
2023	2.03	4.30	4.02	2.24	0.88	1.82	4.33	2.60	4.40	3.42	1.75	2.22
2024	3.32	0.79	5.50	3.80	6.52	4.71	2.62	4.38	1.83	1.41	2.89	0.98

Intro to Time Series Modeling & Forecasting

Questions?