MATH 2780 Chapter5 Worksheet

## Summary

### Steps in Model Building

Identify the response (dependent) variable $y$.
Classify each potential predictor (independent) variable as quantitative or qualitative.
Define qualitative dummy variables. Select one to be the base level, setup dummy variables for rest.
Consider higher-order terms (e.g., $x^2, x^3$) for quantitative variables.
Possibly code the quantitative variables in higher-order polynomials.
Consider interaction terms for both quantitative and qualitative independent variables.
Compare nested models using partial F-tests to arrive at a final model.

### General Form of the Multiple Regression Model:

$E(y)=\beta_0+\beta_1x_1+\beta_2x_1^2+\beta_3x_2+\beta_4x_3+\beta_5x_1x_2+\beta_6x_1x_3+\beta_7x_1^2x_2+\beta_8x_1^2x_3$ (1 Quantitative, 1 Qualitative at 3 levels )

### Coefficient and Residual Variance Estimation:

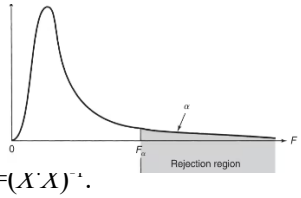
$$Y=X\beta+E$$
$$\hat{\beta}=(X'X)^{-1}X'y$$
$$s^2=\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{n-k-1}$$
$$MSE=s^2,\ s=\sqrt{s^2}$$
$$Y=\begin{bmatrix}y_1\\y_2\\y_3\\\vdots\\y_n\end{bmatrix},X=\begin{bmatrix}1&x_{11}&x_{21}&\cdots&x_{kn}\\1&x_{12}&x_{22}&\cdots&x_{k2}\\1&x_{13}&x_{23}&\cdots&x_{k3}\\\vdots&\vdots&\vdots&&\vdots\\1&x_{1n}&x_{2n}&\cdots&x_{kn}\end{bmatrix},\beta=\begin{bmatrix}\beta_0\\\beta_1\\\beta_2\\\vdots\\\beta_k\end{bmatrix},E=\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\vdots\\\varepsilon_n\end{bmatrix}$$

### Assumptions About the Random Error $\varepsilon$:

1. For any given $x_1,\ldots,x_k$, the error $\varepsilon$ has a normal distribution with, $E(\varepsilon)=0$ and $\text{var}(\varepsilon)=\sigma^2$.
2. The random errors are independent, $f(\varepsilon_i,\varepsilon_j)=f(\varepsilon_i)f(\varepsilon_j)$. Normal only needed for CIs and HTs.

**Model Test:** $H_0$: $\beta_1=\beta_2=\ldots=\beta_k=0$ *vs.* $H_a$: At least one $\beta_i\neq0$.

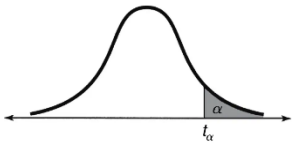
$$F=\frac{(SS_{yy}-SSE)/k}{SSE/(n-k-1)}=\frac{\text{Mean Square Model}}{MSE}=\frac{R^2/k}{(1-R^2)/(n-k-1)}$$




Reject if $F>F_{\alpha,k,n-k-1}$ or $\alpha>p\text{-value}=P(F>F_{\alpha,k,n-k-1})$.

**Individual Coefficient Test:** $t=\hat{\beta}_i/s_{\hat{\beta}_i}$, $s_{\hat{\beta}_i}=s\sqrt{W_{ii}}$, $W_{ii}$ is the $i^{\text{th}}$ diagonal of $W=(X'X)^{-1}$.

One Tailed: $H_0$: $\beta_i\geq0$ *vs.* $H_a$: $\beta_i<0$ w/ RR $t<-t_{\alpha,n-k-1}$ or $\alpha>p\text{-value}=P(t<-t_{\alpha,n-k-1})$,
One Tailed: $H_0$: $\beta_i\leq0$ *vs.* $H_a$: $\beta_i>0$ w/ RR $t>t_{\alpha,n-k-1}$ or $\alpha>p\text{-value}=P(t>t_{\alpha,n-k-1})$,
Two Tailed: $H_0$: $\beta_i=0$ *vs.* $H_a$: $\beta_i\neq0$ w/ RR $|t|>t_{\alpha/2,n-k-1}$ or $\alpha>p\text{-value}=2P(|t|>t_{\alpha,n-k-1})$.




**Coefficient Confidence Interval:**

$\hat{\beta}_i\pm t_{\alpha/2,n-k-1}s\sqrt{W_{ii}}$, $s^2=MSE$ and $W_{ii}$ is the $i^{\text{th}}$ diagonal element of $W=(X'X)^{-1}$.

**Coefficient of determination $R^2$ and adjusted coefficient of determination $R^2_a$.** Fit quality.

$$R^2=1-SSE/SS_{yy},\quad 0\leq R^2\leq1,\ SSE=\sum(y_i-\hat{y}_i)^2,\ SS_{yy}=\sum(y_i-\bar{y})^2$$
$$R_a^2=1-[SSE/(n-k-1)]/[SS_{yy}/(n-1)]=1-[(n-1)/(n-k-1)](1-R^2),\quad R_a^2\leq R^2$$

**Estimated mean function at $x_0$:** $\hat{y}(x_0)=x_0\hat{\beta}$, $SE(\hat{y}_{x_0})=\sqrt{MSE(x_0(X'X)^{-1}x_0')}$

**Mean function confidence interval at $x_0$:** $CI=\hat{y}(x_0)\pm t_{\alpha/2,n-k-1}SE(\hat{y}_{x_0})$

**Mean function prediction interval at $x_0$:** $PI=\hat{y}(x_0)\pm t_{\alpha/2,n-k-1}\cdot\sqrt{MSE+(SE(\hat{y}_{x_0}))^2}$

### *F*-Test for Comparing Nested Models

Reduced Model: $E(y|x's)=\beta_0+\beta_1x_1+\ldots+\beta_gx_g$
Complete Model: $E(y|x's)=\beta_0+\beta_1x_1+\ldots+\beta_gx_g+\beta_{g+1}x_{g+1}+\ldots+\beta_kx_k$
$H_0$: $\beta_{g+1}=\ldots=\beta_k=0$ *vs.* $H_a$: At least one tested $\beta_i\neq0$.

$$F=\frac{(SSE_R-SSE_C)/(k-g)}{SSE_C/(n-k-1)}$$, Reject if $F>F_{\alpha,k-g,n-k-1}$ or $\alpha>p\text{-value}=P(F>F_{\alpha,k-g,n-k-1})$.

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**Example:** Solving a system of equations by inverting a matrix.
In algebra we learned that we can solve a system of equations to find $x_1$ and $x_2$.

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 4 \end{aligned}$$ add → $2x_1=6$ → $x_1=3$, $x_2=-1$.

Which can be written using arrays (matrices) as $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

We learned that $ax=b$ can solve for $x$ as $x=b/a=a^{-1}b$. We are going to do something similar.

Define $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Then, we can write the above system of equations as

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ or $Ax = b$.

We are going to multiply both sides by the inverse of $A$. (Uppercase letter for matrices and lower case for vectors and scalars. Matrix has more than one column while vector only has one column.)

$A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b$.

We can find that $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \rightarrow x = A^{-1}b = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

When $A$ is not square,

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, we need to multiply by transpose of $A$ first, $A'Ax = A'b$, then

multiply by the inverse of $A'A$. That is, $(A'A)^{-1}(A'A)x = (A'A)^{-1}A'b$, $x = (A'A)^{-1}A'b$ .

$$A'A = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}, \rightarrow (A'A)^{-1} = \begin{bmatrix} 0.3667 & -0.2000 \\ -0.2000 & 0.2000 \end{bmatrix} \rightarrow$$

$$(A'A)^{-1}A = \begin{bmatrix} 0.3667 & -0.2000 \\ -0.2000 & 0.2000 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0.1667 & 0.5667 & 0.1333 \\ 0.0000 & -0.4000 & 0.2000 \end{bmatrix}$$

$$b = (A'A)^{-1}Ax = \begin{bmatrix} 0.1667 & 0.5667 & 0.1333 \\ 0.0000 & -0.4000 & 0.2000 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3.4000 \\ -0.4000 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This is our most important equation in statistics $\hat{\beta} = (X'X)^{-1}X'y$ !

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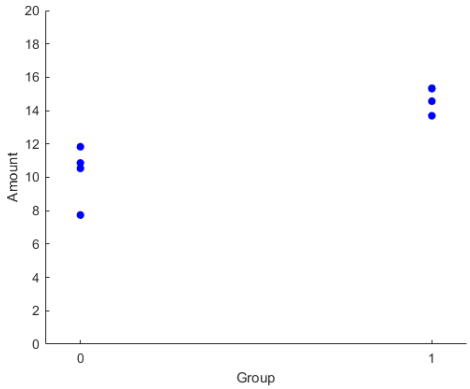
**Example 1:** One quantitative response variable $y$ depending on one qualitative variable $x$ with two levels.

a) Develop a coding scheme for the qualitative variable $x$ and an expectation model $E(y)$.

| Person | *Level* | $y$ |
|---|---|---|
| 1 | Low | 10.5 |
| 2 | Low | 11.8 |
| 3 | Low | 7.7 |
| 4 | Low | 10.9 |
| 5 | High | 15.3 |
| 6 | High | 13.7 |
| 7 | High | 14.6 |
| 8 | High | 15.3 |




| Person | $X$ | | $y$ |
|---|---|---|---|
| 1 | 1 | 0 | 10.5 |
| 2 | 1 | 0 | 11.8 |
| 3 | 1 | 0 | 7.7 |
| 4 | 1 | 0 | 10.9 |
| 5 | 1 | 1 | 15.3 |
| 6 | 1 | 1 | 13.7 |
| 7 | 1 | 1 | 14.6 |
| 8 | 1 | 1 | 15.3 |

$y = \beta_0 + \beta_1 x + \varepsilon$, $x$=0 if low and $x$=1 if high.

$E(y(0))=\beta_0+\beta_1(0)=\beta_0$, $E(y(1))=\beta_0+\beta_1(1)=\beta_0+\beta_1$, so $\beta_0=\mu_L$ and $\beta_1=\beta_0-\mu_H$.

b) Estimate the regression coefficients and MSE. $\hat{\beta} = (X'X)^{-1}X'y$

Step through code.

c) Compute the mean function at $x_0$=[1,0] and $x_0$=[1,1]. $\hat{y}(x_0) = x_0\hat{\beta}$, $SE(\hat{y}_{x_0}) = \sqrt{MSE(x_0(X'X)^{-1}x_0')}$

Step through code.

d) Compute the mean function CI at $x_0$. $CI = \hat{y}(x_0) \pm t_{\alpha/2,n-k-1} SE(\hat{y}_{x_0})$

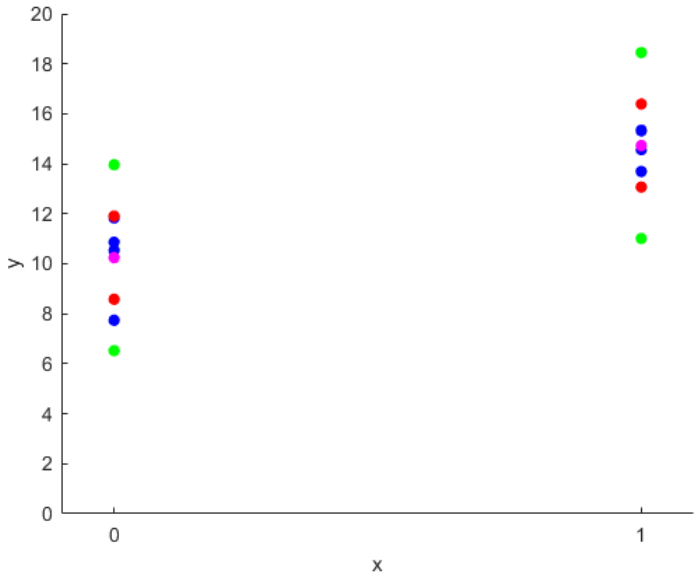
Step through code.

e) Compute the mean function PI at $x_0$. $PI = \hat{y}(x_0) \pm t_{\alpha/2,n-k-1} \cdot \sqrt{MSE + (SE(\hat{y}_{x_0}))^2}$

Step through code.

f) Test $H_0$: $\beta_2$=0 *vs.* $H_a$: $\beta_2 \neq 0$. $t = \hat{\beta}_i / s_{\hat{\beta}_i}$, $s_{\hat{\beta}_i} = s\sqrt{W_{ii}}$, $W_{ii}$ is the $i^{th}$ diagonal of $W$=$(X'X)^{-1}$.

Step through code.



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```
#R Code
# set seed to same default value
set.seed(NULL)

# yi=b0+b1*xi+ei, y=X*b+e
nA<-4
nB<-4
n <-nA+nB
x <-c(rep(0,nA),rep(1,nB))
X <-cbind(rep(1,n),x)

k    <-ncol(X)-1
beta <-c(10,5)
sigma<-1

y<-X%*%beta+rnorm(n)
cbind(X,y)

plot(x,y,xlab="Level",ylab="Amount",pch=19)

# estimate coefficients
W  <-solve(t(X)%*%X)
b  <- W%*%t(X)%*%y
SSE<-t(y-X%*%b)%*%(y-X%*%b)
MSE<-SSE/(n-k-1)

# plot
x0<-rbind(c(1,0),c(1,1))
m <-nrow(x0)

# calculate fitted, CIs, and PIs
yfit   <- x0%*%b;
yfitCIL<-c(rep(0,m))
yfitCIU<-c(rep(0,m))
yfitPIL<-c(rep(0,m))
yfitPIU<-c(rep(0,m))
tcrit<-qt(1-alph,n-k-1)
for(i in 1:m)
{
  yfitCIL[i]<-x0[i,]%*%b
-tcrit*sqrt(MSE*   t(x0[i,])%*%W%*%x0[i,])
  yfitCIU[i]<-x0[i,]%*%b
+tcrit*sqrt(MSE*   t(x0[i,])%*%W%*%x0[i,])
  yfitPIL[i]<-x0[i,]%*%b
-tcrit*sqrt(MSE*(1+t(x0[i,])%*%W%*%x0[i,]))
  yfitPIU[i]<-x0[i,]%*%b
+tcrit*sqrt(MSE*(1+t(x0[i,])%*%W%*%x0[i,]))
}
```

```
# plot fit, CIs, and PIs
plot  (X[,2] ,y   ,xlab="Level",ylab="Amount",
pch=19,
cex = .5, col = "blue",xlim = c(0,1),ylim = c(0,16))
points(x0[,2],yfit,xlab="Level",ylab="Amount",
pch=19,cex = .5,col = "magenta",xlim = c(0,1),
ylim = c(0,16))
points(x0[,2],yfitCIL,xlab="Level",ylab="Amount",
pch=19,cex = .5,col = "red",xlim = c(0,1),
ylim = c(0,16))
points(x0[,2],yfitCIU,xlab="Level",ylab="Amount",
pch=19, cex = .5,col = "red",xlim = c(0,1),
ylim = c(0,16))
points(x0[,2],yfitPIL,xlab="Level",ylab="Amount"
,pch=19, cex = .5,col = "green",xlim = c(0,1),
ylim = c(0,16))
points(x0[,2],yfitPIU,xlab="Level",ylab="Amount",
pch=19, cex = .5,col = "green",xlim = c(0,1),
ylim = c(0,16))

# t-test if beta1=0
alph<-0.05;
tb0  <-b[1]/sqrt(MSE*W[1,1])
tb1  <-b[2]/sqrt(MSE*W[2,2])
```

```
% Matlab Code
% set seed to same default value
rng('default')

% yi=b0+b1*xi+ei, y=X*b+e
nA=4; nB=4; n=nA+nB;
x=[zeros(nA,1);ones(nA,1)];
X=[ones(n,1),x];
k=size(X,2)-1;
beta=[10;5];
sigma=1;

y=X*beta+randn(n,1);
[X,y]

figure;
scatter(X(:,2),y,'filled','blue')
xlabel('Level'), ylabel('Amount')
set(gca,'xtick',[0:1])
xlim([-0.1,1.1]),ylim([0,20])

% estimate coefficients
W    =inv(X'*X)
b=W*X'*y
SSE=(y-X*b)'*(y-X*b);
MSE=SSE/(n-k-1)

% plot
x0=[1,0;1,1]
m=size(x0,1);
yfit = x0*b;
yfitCIL=zeros(m,1);
yfitCIU=zeros(m,1);
yfitPIL=zeros(m,1);
yfitPIU=zeros(m,1);
```

```
for i=1:m
   yfitCIL(i,1)=x0(i,:)*b...
    -tinv(1-0.05/2,n-k-1)...
    *sqrt(MSE*   x0(i,:)*W*x0(i,:)' );
   yfitCIU(i,1)=x0(i,:)*b...
    +tinv(1-0.05/2,n-k-1)...
    *sqrt(MSE*   x0(i,:)*W*x0(i,:)' );
   yfitPIL(i,1)=x0(i,:)*b...
    -tinv(1-0.05/2,n-k-1)...
    *sqrt(MSE*(1+x0(i,:)*W*x0(i,:)'));
   yfitPIU(i,1)=x0(i,:)*b...
    +tinv(1-0.05/2,n-k-1)...
    *sqrt(MSE*(1+x0(i,:)*W*x0(i,:)'));
end

figure;
scatter(X(:,2),y,       'filled','blue')
hold on
scatter(x0(:,2),yfit, 'filled','magenta')
scatter(x0(:,2),yfitCIL,'filled','red')
scatter(x0(:,2),yfitCIU,'filled','red')
scatter(x0(:,2),yfitPIL,'filled','green')
scatter(x0(:,2),yfitPIU,'filled','green')
xlabel('x'), ylabel('y')
set(gca,'xtick',[0:1])
xlim([-0.1,1.1]),ylim([0,20])

% t-test if beta1=0
alph=0.05;
tb1  =b(2,1)/sqrt(MSE*W(2,2))
tcrit=tinv(1-alph,n-k-1)
```

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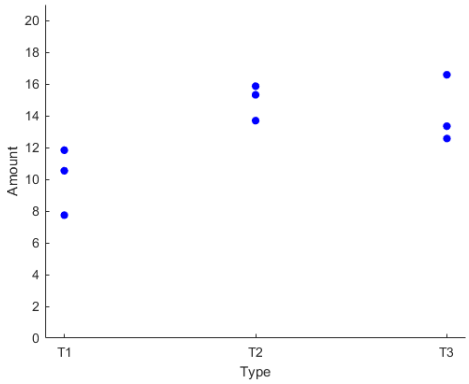
**Example 2:** One quantitative response variable $y$ depending on one qualitative variable with three types.

a) Develop a coding scheme for the qualitative variable and an expectation model $E(y)$.

| Person | *Type* | $y$ |
|---|---|---|
| 1 | T1 | 10.5 |
| 2 | T1 | 11.8 |
| 3 | T1 | 7.7 |
| 4 | T2 | 15.9 |
| 5 | T2 | 15.3 |
| 6 | T2 | 13.7 |
| 7 | T3 | 12.6 |
| 8 | T3 | 13.3 |
| 9 | T3 | 16.6 |




| Person | $X$ | | | $y$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 10.5 |
| 2 | 1 | 0 | 0 | 11.8 |
| 3 | 1 | 0 | 0 | 7.7 |
| 4 | 1 | 1 | 0 | 15.9 |
| 5 | 1 | 1 | 0 | 15.3 |
| 6 | 1 | 1 | 0 | 13.7 |
| 7 | 1 | 0 | 1 | 12.6 |
| 8 | 1 | 0 | 1 | 13.3 |
| 9 | 1 | 0 | 1 | 16.6 |

Arbitrarily select one level to be the base level, then setup dummy variables for the remaining levels.

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$ , $x_1$=1 if $T_2$ and $x_2$=1 if $T_3$, zero otherwise.

$E(y(0,0))=\beta_0+\beta_1(0)+\beta_2(0)=\beta_0$, $E(y(0,1))=\beta_0+\beta_1(1)+\beta_2(0)=\beta_0+\beta_1$, $E(y(1,1))=\beta_0+\beta_1(1)+\beta_2(1)=\beta_0+\beta_1+\beta_2$, so $\beta_0=\mu_{T1}$, $\beta_1=\mu_{T2}-\mu_{T1}$, and $\beta_2=\mu_{T3}-\mu_{T1}$.

b) Estimate the regression coefficients and MSE. $\hat{\beta} = (X'X)^{-1}X'y$

Step through code.

c) Compute the mean function at $x_0$=[1,0,1], $x_0$=[1,1,0], and $x_0$=[1,1,1].

$$\hat{y}(x_0) = x_0\hat{\beta},\ SE(\hat{y}_{x_0}) = \sqrt{MSE(x_0(X'X)^{-1}x_0')}$$

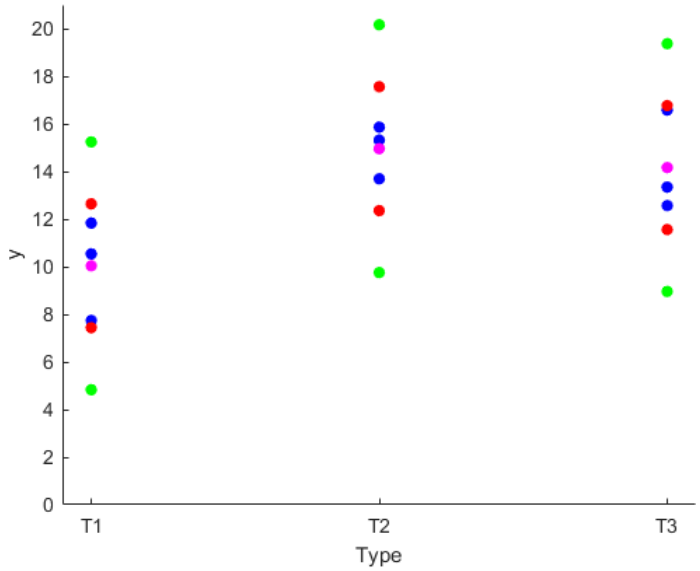
Step through code.

d) Compute the mean function CI at $x_0$. $CI = \hat{y}(x_0) \pm t_{\alpha/2,n-k-1} SE(\hat{y}_{x_0})$

Step through code.

e) Compute the mean function PI at $x_0$. $PI = \hat{y}(x_0) \pm t_{\alpha/2,n-k-1} \cdot \sqrt{MSE + (SE(\hat{y}_{x_0}))^2}$

Step through code.



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```
% set seed to same default value
rng('default')

% yi=b0+b1*x1i+b2i+ei, y=X*b+e
n1=3; n2=3; n3=3; n=n1+n2+n3;
x1=[zeros(n1,1); ones(n2,1);zeros(n3,1)];
x2=[zeros(n1,1);zeros(n2,1); ones(n3,1)];
X=[ones(n,1),x1,x2];
k=size(X,2)-1;
beta=[10;5;3];
sigma=1;

y=X*beta+randn(n,1);
[X,y]

X*beta

xx=[1*ones(n1,1);2*ones(n2,1);3*ones(n3,1)];
figure;
scatter(xx,y,'filled','blue')
xlabel('Type'), ylabel('Amount')
set(gca,'xtick',[1,2,3])
xticklabels({'T1','T2','T3'})
xlim([0.9,3.1]),ylim([0,21])

% estimate coefficients
W    =inv(X'*X)
b=W*X'*y %10.0376, 4.9202, 4.1249
SSE=(y-X*b)'*(y-X*b);
MSE=SSE/(n-k-1)

mean(y(1:3)) %10.0376
mean(y(4:6)) %14.9578
mean(y(7:9)) %14.1625

mean(y(4:6))-mean(y(1:3))  % 4.9202

mean(y(7:9))-mean(y(1:3))  % 4.1249
```

```
% plot
x0=[1,0,0;1,1,0;1,0,1]
m=size(x0,1);
yfit = x0*b;
yfitCIL=zeros(m,1); yfitCIU=zeros(m,1);
yfitPIL=zeros(m,1); yfitPIU=zeros(m,1);
for i=1:m
    yfitCIL(i,1)=x0(i,:)*b...
        -tinv(1-0.05/2,n-k-1)...
        *sqrt(MSE*   x0(i,:)*W*x0(i,:)' );
    yfitCIU(i,1)=x0(i,:)*b...
        +tinv(1-0.05/2,n-k-1)...
        *sqrt(MSE*   x0(i,:)*W*x0(i,:)' );
    yfitPIL(i,1)=x0(i,:)*b...
        -tinv(1-0.05/2,n-k-1)...
        *sqrt(MSE*(1+x0(i,:)*W*x0(i,:)'));
    yfitPIU(i,1)=x0(i,:)*b...
        +tinv(1-0.05/2,n-k-1)...
        *sqrt(MSE*(1+x0(i,:)*W*x0(i,:)'));
end
figure;
scatter(xx,y,       'filled','blue')
hold on
scatter([1;2;3],yfit,   'filled','ma-
genta')
scatter([1;2;3],yfitCIL,'filled','red')
scatter([1;2;3],yfitCIU,'filled','red')
scatter([1;2;3],yfitPIL,'filled','green')
scatter([1;2;3],yfitPIU,'filled','green')
xlabel('Type'), ylabel('y')
set(gca,'xtick',[1;2;3])
xticklabels({'T1','T2','T3'})
xlim([0.9,3.1]),ylim([0,21])
```