Class 21

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Agenda:

Recap Chapter 10.1 and 10.2

Lecture Chapter 10.3

Recap Chapter 10.1 and 10.2

We form a paired difference from the data

Paired Difference

$$d = x_1 - x_2 (10.1)$$

This means that we are subtracting the sample value from

population 2 from the sample value from population 1.

So if the d_i 's are approximately normally distributed

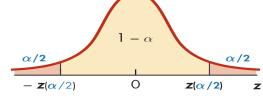
with a mean of μ_d and a standard deviation of σ_d , then

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$$
 is normally distributed (recall CLT)

with a mean $\mu_{\bar{d}} = \mu_d$, and standard deviation $\sigma_{\bar{d}} = \frac{\sigma_d}{\sqrt{n}}$.

This would allow us to form a z statistic for the mean of

differences \bar{d} , $z = \frac{\bar{d} - \mu_d}{\sigma_d / \sqrt{n}}$ with a standard normal distribution. We can then look up probabilities in the table,



find critical values $z(\alpha/2)$, construct confidence intervals

and test hypotheses using
$$z^* = \frac{\overline{d} - \mu_{0d}}{\sigma_d / \sqrt{n}}$$
 .

$$\bar{d} \pm z(\alpha/2) \frac{\sigma_d}{\sqrt{n}}$$

Figure from Johnson & Kuby, 2012.

However, as in Inferences for One Population, we never

know the true value of σ_d . So we estimate it with sample

standard deviation
$$s_d$$
. This changes $z = \frac{d - \mu_d}{\sigma_d / \sqrt{n}}$

standard deviation s_d . This changes $z = \frac{\bar{d} - \mu_d}{\sigma_d / \sqrt{n}}$ to $t = \frac{\bar{d} - \mu_d}{s_s / \sqrt{n}}$ and the distribution from standard normal

to Student t with
$$df=n-1$$
 where $s_d^2 = \frac{1}{n-1} \sum_{i=1}^n (d_i - \overline{d})^2$.

10: Inferences Involving Two Populations 10.2 Inference for Mean Difference Two Dependent Samples Confidence Interval Procedure

With σ_d unknown, a 1- α confidence interval for μ_d is:

Confidence Interval for Mean Difference (Dependent Samples)

$$\overline{d} - t(df, \alpha/2) \frac{s_d}{\sqrt{n}}$$
 to $\overline{d} + t(df, \alpha/2) \frac{s_d}{\sqrt{n}}$ where $df = n-1$ (10.2)

$$\overline{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$$
 (10.3) $s_d^2 = \frac{1}{n-1} \sum_{i=1}^{n} (d_i - \overline{d})^2$ (10.4)

Car	1	2	3	4	5	6
Brand A	125	64	94	38	90	106
Brand B	133	65	103	37	102	115

Example:

Construct a 95% CI for mean difference in Brand B – A tire wear.

$$d_i$$
's: 8, 1, 9, -1, 12, 9
 $n = 6$ $df = 5$ $t(df, \alpha/2) = 2.57$
 $\bar{d} = 6.3$ $\alpha = 0.05$
 $s_d = 5.1$ $\bar{d} \pm t(df, \alpha/2) \frac{s_d}{\sqrt{n}} \longrightarrow (0.090, 11.7)$

$$\overline{d} = \frac{1}{n} \sum_{i=1}^{n} d_{i}$$

$$s_{d}^{2} = \frac{1}{n-1} \sum_{i=1}^{n} (d_{i} - \overline{d})^{2}$$

Figure from Johnson & Kuby, 2012.

We can test for differences in the population means:

$$\begin{array}{lll} H_0: \mu_1 \!\! \geq \!\! \mu_2 \ \text{VS.} \ H_a: \mu_1 \!\! < \!\! \mu_2 & \to & H_0: \mu_1 \!\! - \!\! \mu_2 \geq \!\! 0 \ \text{VS.} \ H_a: \mu_1 \!\! - \!\! \mu_2 < \!\! 0 \\ H_0: \mu_1 \!\! \leq \!\! \mu_2 \ \text{VS.} \ H_a: \mu_1 \!\! > \!\! \mu_2 & \to & H_0: \mu_1 \!\! - \!\! \mu_2 \leq \!\! 0 \ \text{VS.} \ H_a: \mu_1 \!\! - \!\! \mu_2 > \!\! 0 \\ H_0: \mu_1 \!\! = \!\! \mu_2 \ \text{VS.} \ H_a: \mu_1 \!\! \neq \!\! \mu_2 & \to & H_0: \mu_1 \!\! - \!\! \mu_2 = \!\! 0 \ \text{VS.} \ H_a: \mu_1 \!\! - \!\! \mu_2 \neq \!\! 0 \\ \mu_d \!\! = \!\! \mu_1 \!\! - \!\! \mu_2 & \to & H_0: \mu_d \!\! \geq \!\! 0 \ \text{VS.} \ H_a: \mu_d \!\! < \!\! 0 \\ (\mu_d \!\! = \!\! \mu_{before} \!\! - \mu_{after}) & H_0: \mu_d \!\! \leq \!\! 0 \ \text{VS.} \ H_a: \mu_d \!\! > \!\! 0 \\ H_0: \mu_d \!\! = \!\! 0 \ \text{VS.} \ H_a: \mu_d \!\! \neq \!\! 0 \end{array}$$

With σ_d unknown, the test statistic for μ_d is:

Test Statistic for Mean Difference (Dependent Samples)

$$t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}}$$
 where $df = n-1$

(10.5)

Go through the same five hypothesis testing steps.

There are three possible hypothesis pairs for the difference in

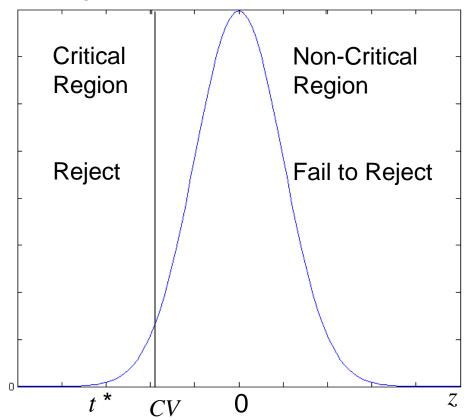
means.

$$H_0$$
: $\mu_d \ge \mu_{d0}$ vs. H_a : $\mu_d < \mu_{d0}$

Reject H_0 if less than

$$t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}} \qquad -t(df, \alpha)$$

data indicates $\mu_d < \mu_{d0}$ because \bar{d} is "a lot" smaller than μ_{d0}



There are three possible hypothesis pairs for the difference in

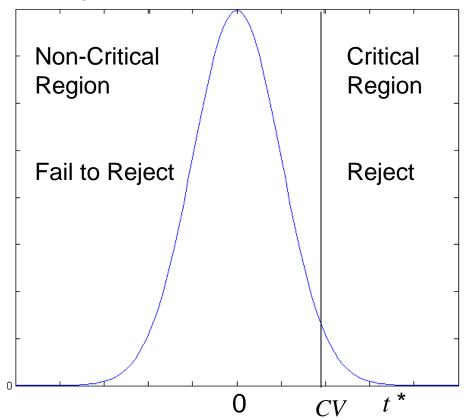
means.

$$H_0$$
: $\mu_d \le \mu_{d0}$ vs. H_a : $\mu_d > \mu_{d0}$

Reject H_0 if greater then

$$t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}} \qquad t(df, \alpha)$$

data indicates $\mu_d > \mu_{d0}$ because \bar{d} is "a lot" smaller than μ_{d0}



There are three possible hypothesis pairs for the difference in

means.

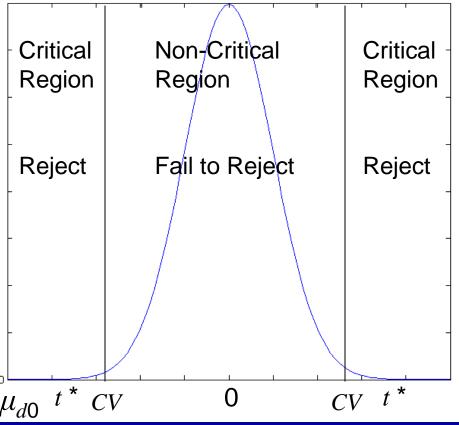
$$H_0$$
: $\mu_d = \mu_{d0}$ vs. H_a : $\mu_d \neq \mu_{d0}$

Reject H_0 if less than

$$t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}} - t(df, \alpha / 2)$$

or if __ is greater than

$$t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}} \quad t(df, \alpha / 2)$$
data indicates $\mu_d \neq \mu_{d0}, \overline{d}$ far from μ_{d0} t^* cv



n = 6 8, 1, 9, -1, 12, 9

Car	1	2	3	4	5	6
Brand A	125	64	94	38	90	106
Brand B	133	65	103	37	102	115

Example:

Test mean difference of Brand B minus Brand A is zero.

Step 1
$$H_0$$
: μ_d =0 vs. H_a : $\mu_d \neq 0$

Step 2

$$df = 5 \qquad t^* = \frac{\overline{d} - \mu_{d0}}{s_d / \sqrt{n}}$$

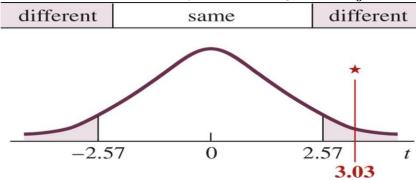
$$\alpha = .05$$

Step 3
$$\bar{d} = 6.3$$

 $s_d = 5.1$ $t^* = \frac{6.3 - 0}{5.1/\sqrt{6}} = 3.03$

Step 4
$$t(df, \alpha/2) = 2.57$$

Step 5 Since $t^* > t(df, \alpha/2)$, reject H_0



Conclusion: Significant difference in tread wear at .05 level.

Figures from Johnson & Kuby, 2012.

Chapter 10: Inferences Involving Two Populations

Questions?

Homework: Read Chapter 10.1-10.2

WebAssign

Chapter 10 #13, 15, 23, 25, 29, 31, 35

Chapter 10: Inference Involving Two Populations

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Background We said from SDSM that $mean(\bar{x}) = \mu$ and $variance(\bar{x}) = \frac{\sigma^2}{n}$.

We are often interested in comparisons between means $\overline{x}_1 - \overline{x}_2$.

There's a rule that says that if $\overline{x}_{\!\scriptscriptstyle 1}$ and $\overline{x}_{\!\scriptscriptstyle 2}$ have means $\mu_{\!\scriptscriptstyle 1}$ and $\mu_{\!\scriptscriptstyle 2}$,

and variances σ_1^2 and σ_2^2 ,

then mean
$$(\overline{x}_1 - \overline{x}_2) = \mu_1 - \mu_2$$

and variance
$$(\overline{x}_1 - \overline{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \longrightarrow \sigma_{\overline{x}_1 - \overline{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$
 if $x_1 \& x_2$ independent

Variances add not standard deviations.

10: Inferences Involving Two Populations

10.3 Inference for Mean Difference Two Independent Samples Means Using Two Independent Samples

If two populations are independent we can construct confidence intervals and test hypotheses for the difference in their means.

If independent samples of sizes n_1 and n_2 are drawn ... with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 , then the sampling distribution of $\overline{x}_1 - \overline{x}_2$... has

1. mean $\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$ and

2. standard error
$$\sigma_{\overline{x}_1 - \overline{x}_2} = \sqrt{\left(\frac{\sigma_1^2}{n_1}\right) + \left(\frac{\sigma_2^2}{n_2}\right)}$$

If both pops, are normal, then $\bar{x}_1 - \bar{x}_2$ is normal.

(10.6)

However, the true population variances are never truly known

so we estimate σ_1^2 and σ_2^2 by s_1^2 and s_2^2 and the

standard error

$$\sigma_{\overline{x}_1 - \overline{x}_2} = \sqrt{\left(\frac{\sigma_1^2}{n_1}\right) + \left(\frac{\sigma_2^2}{n_2}\right)}$$
 (10.6)

by

$$S_{\bar{x}_1 - \bar{x}_2} = \sqrt{\left(\frac{S_1^2}{n_1}\right) + \left(\frac{S_2^2}{n_2}\right)} \quad . \tag{10.7}$$

10: Inferences Involving Two Populations 10.3 Inference for Mean Difference Two Independent Samples Confidence Interval Procedure

With σ_1 and σ_2 unknown, a 1- α confidence interval for $\mu_1 - \mu_2$ is:

Confidence Interval for Mean Difference (Independent

Samples)

$$(\overline{x}_1 - \overline{x}_2) - t(df, \alpha/2) \sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}$$
 to $(\overline{x}_1 - \overline{x}_2) + t(df, \alpha/2) \sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}$

where df is either calculated or smaller of df_1 , or df_2 (10.8)

Actually, this is for $\sigma_1 \neq \sigma_2$.

Next larger number than

$$df = \left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2 / \left(\frac{\left(s_1^2 / n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2 / n_2\right)^2}{n_2 - 1}\right)$$

If using a computer program.

If not using a computer program.

Need normal populations to use *t* critical values.

10: Inferences Involving Two Populations

10.3 Inference Mean Difference Confidence Interval

Sample	Number	Mean	Standard Deviation
Female (f)	$n_1 = 20$	$\overline{x}_f = 63.8$ $\overline{x}_m = 69.8$	$s_f = 2.18$
Male (m)	$n_2 = 30$		$s_m = 1.92$

Example:

Interested in difference in mean heights between men and women. The heights of 20 females and 30 males is measured. Construct a 95% confidence interval for $\mu_{m} - \mu_{f}$, σ_{m} & σ_{f} unknown

$$(\overline{x}_m - \overline{x}_f) \pm t(df, \alpha/2) \sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}$$

$$\alpha = 0.05$$
 $t(19,.025) =$

Figure from Johnson & Kuby, 2012.

10: Inferences Involving Two Populations

10.3 Inference Mean Difference Confidence Interval

Sample	Number	Mean	Standard Deviation
Female (f)	$n_1 = 20$	$\overline{x}_f = 63.8$	$s_f = 2.18$
Male (m)	$n_2 = 30$	$\frac{\overline{x}_f}{\overline{x}_m} = 63.8$	$s_m^{\prime} = 1.92$

Example:

Interested in difference in mean heights between men and women. The heights of 20 females and 30 males is measured. Construct a 95% confidence interval for $\mu_m - \mu_f$, $\sigma_m \& \sigma_f$ unknown

$$(\overline{x}_{m} - \overline{x}_{f}) \pm t(df, \alpha/2) \sqrt{\left(\frac{s_{m}^{2}}{n_{m}}\right) + \left(\frac{s_{f}^{2}}{n_{f}}\right)}$$

$$(69.8 - 63.8) \pm 2.09 \sqrt{\left(\frac{(1.92)^{2}}{30}\right) + \left(\frac{(2.18)^{2}}{20}\right)}$$
therefore 4.75 to 7.25

$$\alpha = 0.05$$

 $t(19,.025) = 2.09$

Figure from Johnson & Kuby, 2012.

We can test for differences in the population means:

$$H_0: \mu_1 \ge \mu_2 \text{ vs. } H_a: \mu_1 < \mu_2 \longrightarrow H_0: \mu_1 - \mu_2 \ge 0 \text{ vs. } H_a: \mu_1 - \mu_2 < 0$$
 $H_0: \mu_1 \le \mu_2 \text{ vs. } H_a: \mu_1 > \mu_2 \longrightarrow H_0: \mu_1 - \mu_2 \le 0 \text{ vs. } H_a: \mu_1 - \mu_2 > 0$
 $H_0: \mu_1 = \mu_2 \text{ vs. } H_a: \mu_1 \ne \mu_2 \longrightarrow H_0: \mu_1 - \mu_2 = 0 \text{ vs. } H_a: \mu_1 - \mu_2 \ne 0$

With σ_1 and σ_2 unknown, the test statistic for $\mu_1 - \mu_2$ is:

Test Statistic for Mean Difference (Independent Samples)

$$t^* = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_{0,1} - \mu_{0,2})}{\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}}$$

$$\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}$$

$$\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)^2}$$

$$\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)^2}$$
If not using a computer program.

where df is either calculated or smaller of df_1 , or df_2 (10.9) Actually, this is for $\sigma_1 \neq \sigma_2$.

Go through the same five hypothesis testing steps.

Need normal populations to use *t* critical values.

There are three possible hypothesis pairs for the difference in

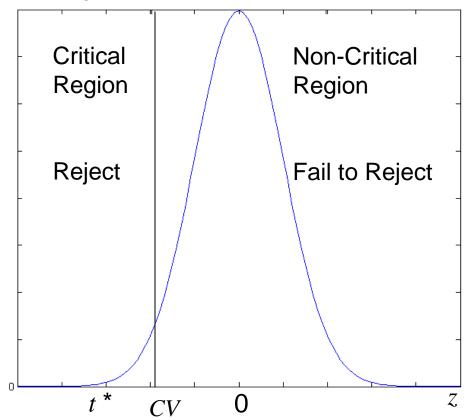
means.

$$H_0$$
: μ_1 - $\mu_2 \ge 0$ vs. H_a : μ_1 - $\mu_2 < 0$

Reject H_0 if less than

$$t^* = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}} - t(df, \alpha)$$

data indicates μ_1 - μ_2 < 0 because $\bar{x}_1 - \bar{x}_2$ is "a lot" smaller than 0.



There are three possible hypothesis pairs for the difference in

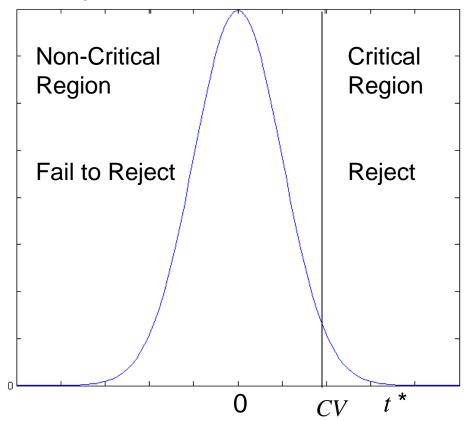
means.

$$H_0$$
: μ_1 - $\mu_2 \le 0$ vs. H_a : μ_1 - $\mu_2 > 0$

Reject H_0 if greater then

$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}} \qquad t(df, \alpha)$$

data indicates μ_1 - μ_2 > 0 because $\bar{x}_1 - \bar{x}_2$ is "a lot" smaller than 0.



There are three possible hypothesis pairs for the difference in

means.

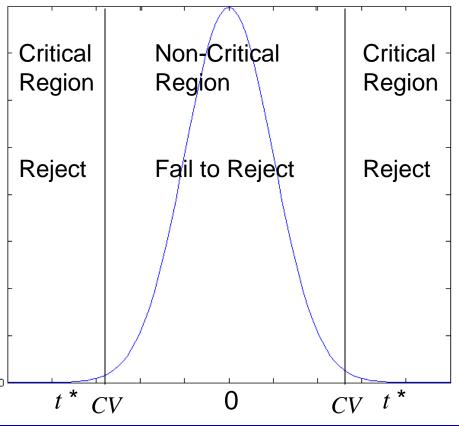
$$H_0$$
: μ_1 - μ_2 = 0 vs. H_a : μ_1 - $\mu_2 \neq 0$

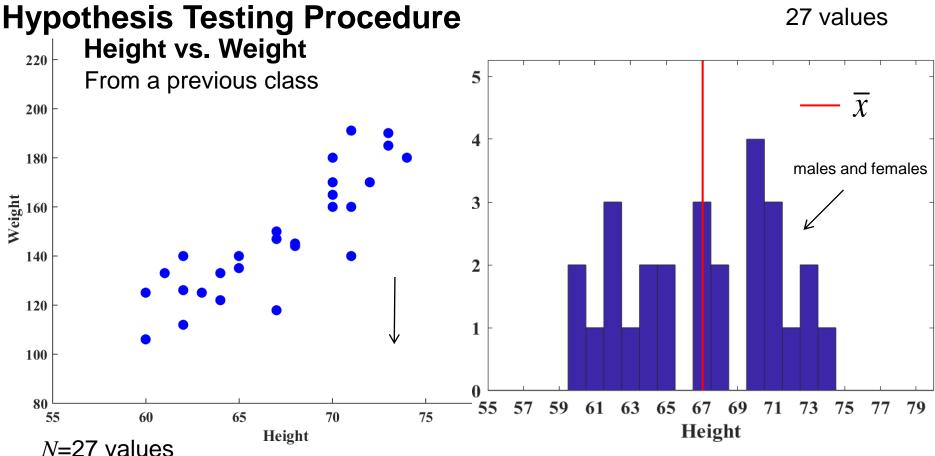
Reject H_0 if

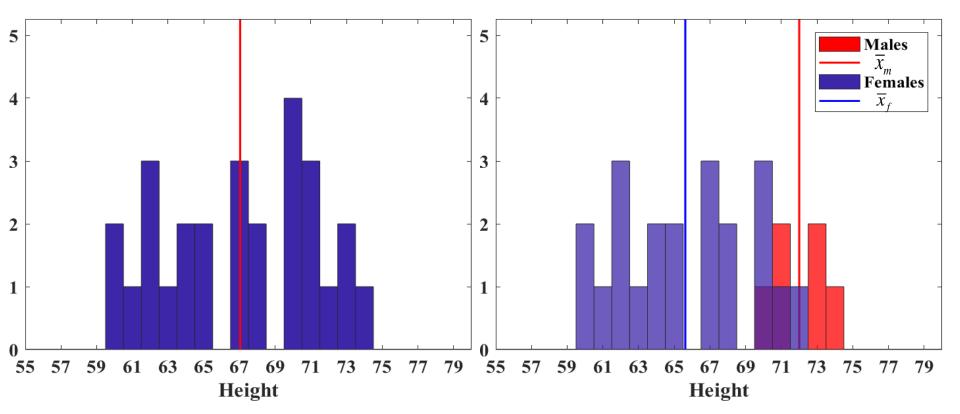
less than

$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}} - t(df, \alpha/2)$$
or if
$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}$$
 is greater than
$$t(df, \alpha/2)$$

data indicates μ_1 - $\mu_2 \neq 0$, $\overline{x}_1 - \overline{x}_2$







Is the height of males = height of females at α =.05?

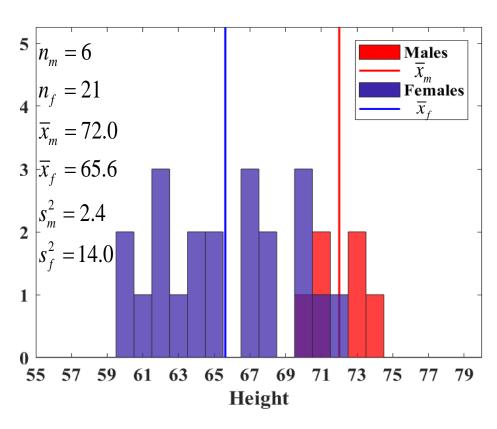
Step 1

Step 2

Step 3

Step 4





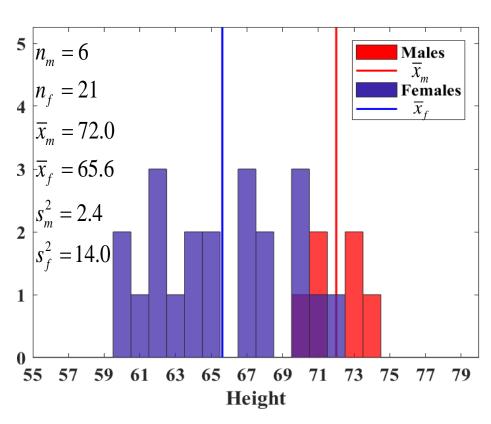
Step 1

$$H_0$$
: $\mu_f = \mu_m$ VS. H_a : $\mu_f \neq \mu_m$
Step 2

Step 3

Step 4

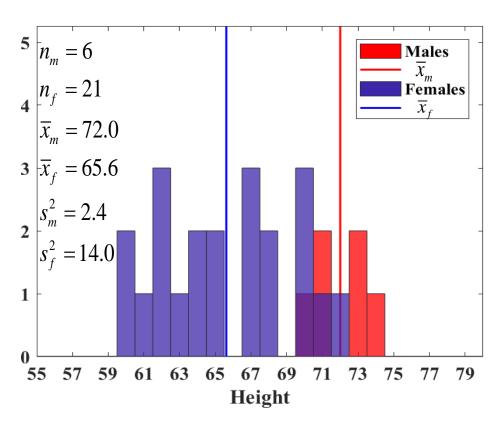




Step 1
$$H_0$$
: $\mu_f = \mu_m$ VS. H_a : $\mu_f \neq \mu_m$
Step 2
 $t^* = \frac{(\overline{x}_m - \overline{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$
Step 3

Step 4

Step 5



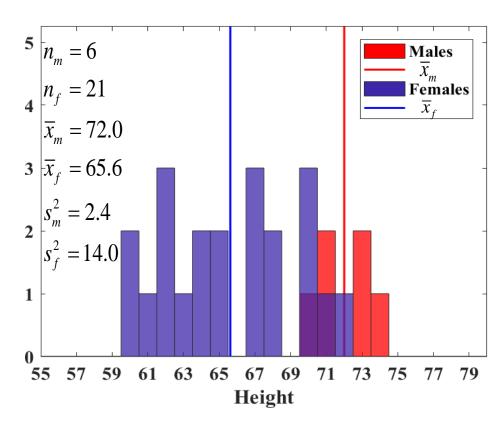
Step 1
$$H_0$$
: $\mu_f = \mu_m$ VS. H_a : $\mu_f \neq \mu_m$

Step 2
 $t^* = \frac{(\overline{x}_m - \overline{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$

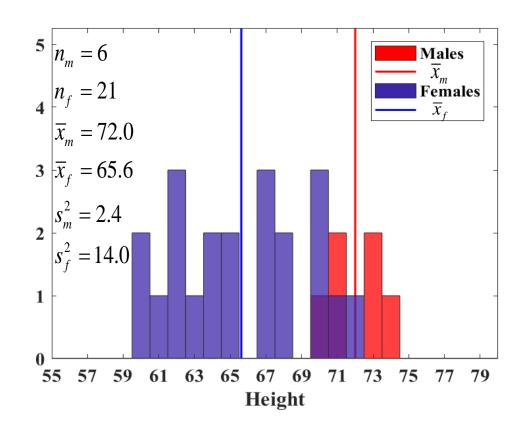
Step 3
 $t^* = \frac{(72.0 - 65.6) - (0)}{\sqrt{\left(\frac{2.4}{6}\right) + \left(\frac{14.0}{21}\right)}} = 6.17$

Step 4

Step 5



Step 1
$$H_0: \mu_f = \mu_m \text{ VS. } H_a: \mu_f \neq \mu_m$$
Step 2
 $t^* = \frac{(\bar{x}_m - \bar{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$
Step 3
 $t^* = \frac{(72.0 - 65.6) - (0)}{\sqrt{\left(\frac{2.4}{6}\right) + \left(\frac{14.0}{21}\right)}} = 6.17$
Step 4
 $t(df, \alpha/2) = 2.57$ Step 5

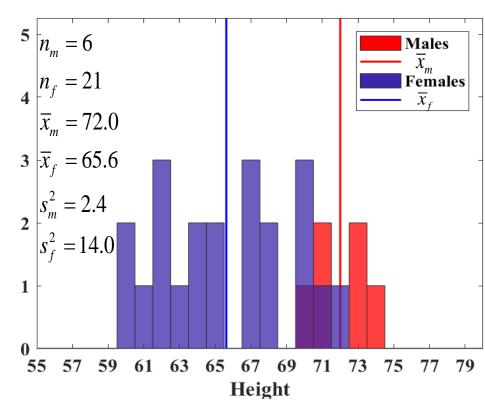


Step 1
$$H_0$$
: $\mu_f = \mu_m$ vs. H_a : $\mu_f \neq \mu_m$

Step 2
 $t^* = \frac{(\overline{x}_m - \overline{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$

Step 3
 $t^* = \frac{(72.0 - 65.6) - (0)}{\sqrt{\left(\frac{2.4}{6}\right) + \left(\frac{14.0}{21}\right)}} = 6.17$

Classical Step 4



 $t(df, \alpha/2) = 2.57$ Step 5 Reject $H_0 6.17 > 2.57$, height males \neq height females

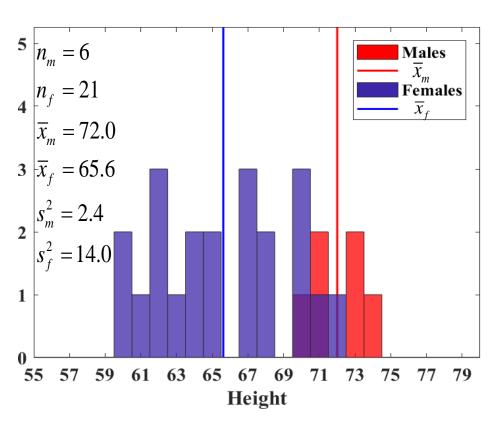
Step 1
$$H_0$$
: $\mu_f = \mu_m$ vs. H_a : $\mu_f \neq \mu_m$

Step 2
 $t^* = \frac{(\overline{x}_m - \overline{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$

Step 3
 $t^* = \frac{(72.0 - 65.6) - (0)}{\sqrt{\left(\frac{2.4}{6}\right) + \left(\frac{14.0}{21}\right)}} = 6.17$

Step 4

$$P(|t^*| > 6.17) = .0016$$
 Step 5



Step 1

$$H_0$$
: $\mu_f = \mu_m$ VS. H_a : $\mu_f \neq \mu_m$

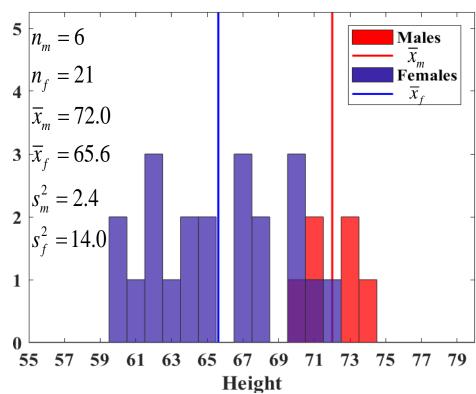
Step 2

 $t^* = \frac{(\overline{x}_m - \overline{x}_f) - (\mu_m - \mu_f)}{\sqrt{\left(\frac{s_m^2}{n_m}\right) + \left(\frac{s_f^2}{n_f}\right)}}$

Step 3

 $t^* = \frac{(72.0 - 65.6) - (0)}{\sqrt{\left(\frac{2.4}{6}\right) + \left(\frac{14.0}{21}\right)}} = 6.17$

Step 4



 $P(|t^*| > 6.17) = .0016$ Step 5 Reject H_0 .002 < .05 , height males \neq height females

Chapter 10: Inferences Involving Two Populations

Questions?

Homework: Read Chapter 10.3

WebAssign

Chapter 10 # 41, 45, 53, 57, 58, 59, 63