

Class 14

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Agenda:

Recap Chapter 8.5

Lecture Chapter 9.1

Recap Chapter 8.5

8: Introduction to Statistical Inference

8.4 Hypothesis Test of Mean (σ Known): Probability Approach

Step 1 The Set-Up: Null (H_0) and alternative (H_a) hypotheses

$$H_0: \mu = 69'' \text{ vs. } H_a: \mu \neq 69''$$

Step 2 The Hypothesis Test Criteria: Test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \quad \sigma \text{ known, } n \text{ is "large" so by CLT } \bar{x} \text{ is normal} \\ z^* \text{ is normal}$$

Step 3 The Sample Evidence: Calculate test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{67.2 - 69}{4 / \sqrt{15}} = -1.74$$

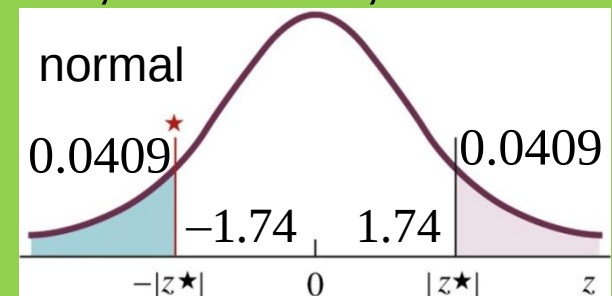
$$n=15, \bar{x}=67.2, \sigma=4$$

Step 4 The Probability Distribution:

$$P(z > |z^*|) = p\text{-value} \rightarrow 0.0819$$

Step 5 The Results:

$$p\text{-value} \leq \alpha, \text{ reject } H_0, \quad p\text{-value} > \alpha \text{ fail to reject } H_0 \quad \alpha = 0.05$$



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Step 1 The Set-Up: Null (H_0) and alternative (H_a) hypotheses

$$H_0: \mu = 69 \text{'' vs. } H_a: \mu \neq 69 \text{''}$$

Step 2 The Hypothesis Test Criteria: Test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \quad \sigma \text{ known, } n \text{ is "large" so by CLT } \bar{x} \text{ is normal} \quad z^* \text{ is normal}$$

Step 3 The Sample Evidence: Calculate test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{67.2 - 69}{4 / \sqrt{15}} = -1.74$$

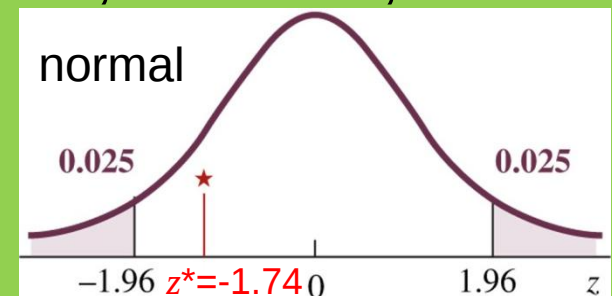
$$n=15, \bar{x}=67.2, \sigma=4$$

Step 4 The Probability Distribution:

$$\alpha = 0.05, z(\alpha/2) = 1.96$$

Step 5 The Results:

$$|z^*| > z(\alpha / 2), \text{ reject } H_0, |z^*| \leq z(\alpha / 2) \text{ fail to reject } H_0$$



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

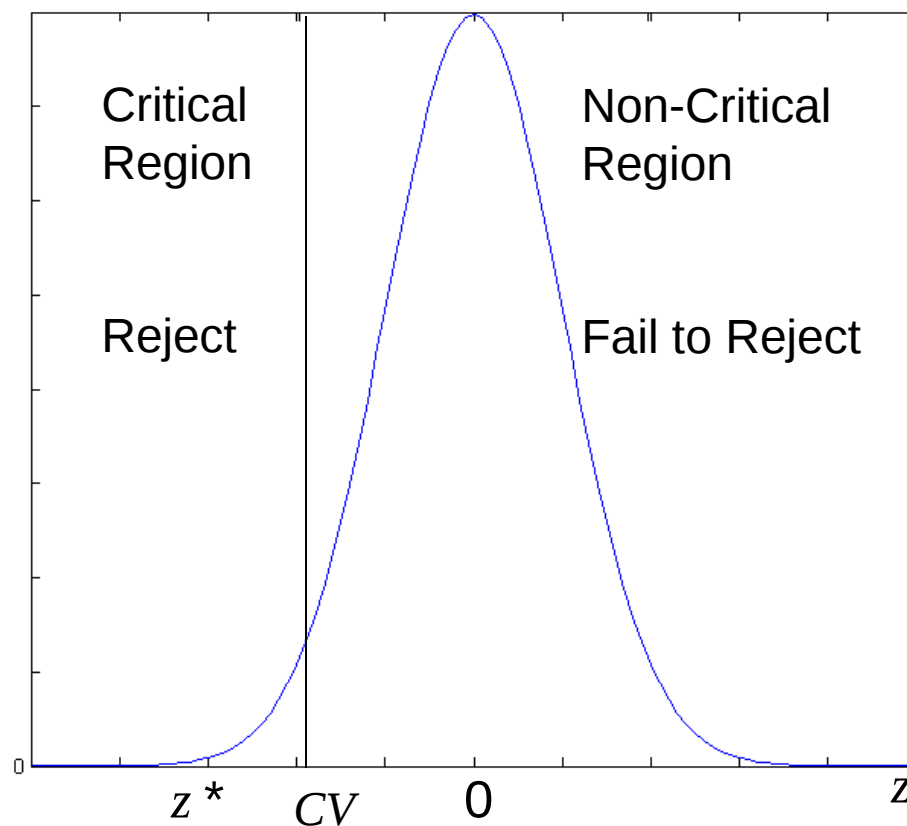
There are three possible hypothesis pairs for the mean.

$$H_0: \mu \geq \mu_0 \text{ vs. } H_a: \mu < \mu_0$$

Reject H_0 if less than

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \quad -z(\alpha)$$

data indicates $\mu < \mu_0$
because $\bar{x}$ is “a lot”
smaller than μ_0



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

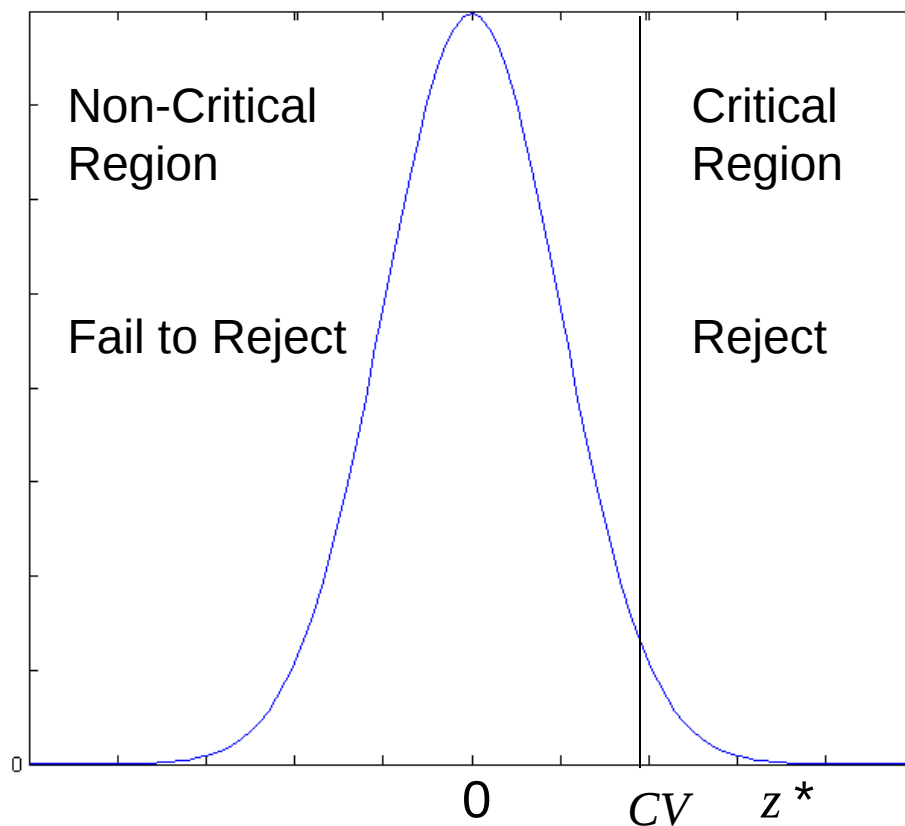
There are three possible hypothesis pairs for the mean.

$$H_0: \mu \leq \mu_0 \text{ vs. } H_a: \mu > \mu_0$$

Reject H_0 if z is greater than

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \quad z(\alpha)$$

data indicates $\mu > \mu_0$
because $\bar{x}$ is “a lot”
larger than μ_0



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

There are three possible hypothesis pairs for the mean.

$$H_0: \mu = \mu_0 \text{ vs. } H_a: \mu \neq \mu_0$$

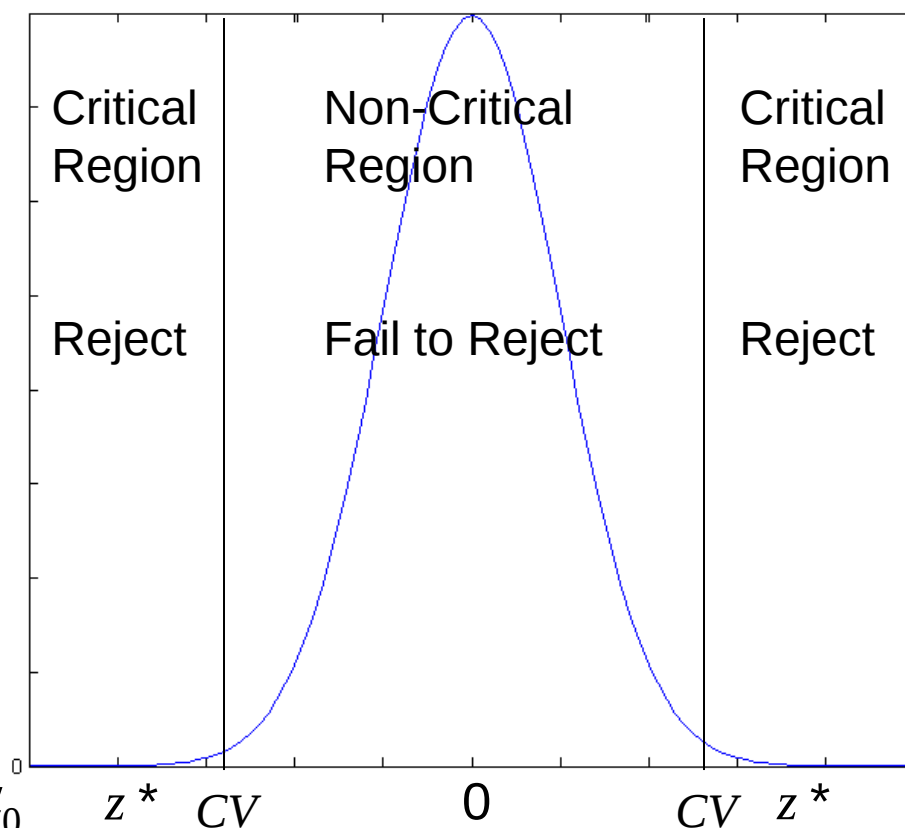
Reject H_0 if less than

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} < -z(\alpha / 2)$$

or if is greater than

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z(\alpha / 2)$$

data indicates $\mu \neq \mu_0$, $\bar{x}$ far from μ_0



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Let's examine the hypothesis test

$$H_0: \mu \leq 69'' \text{ vs. } H_a: \mu > 69''$$

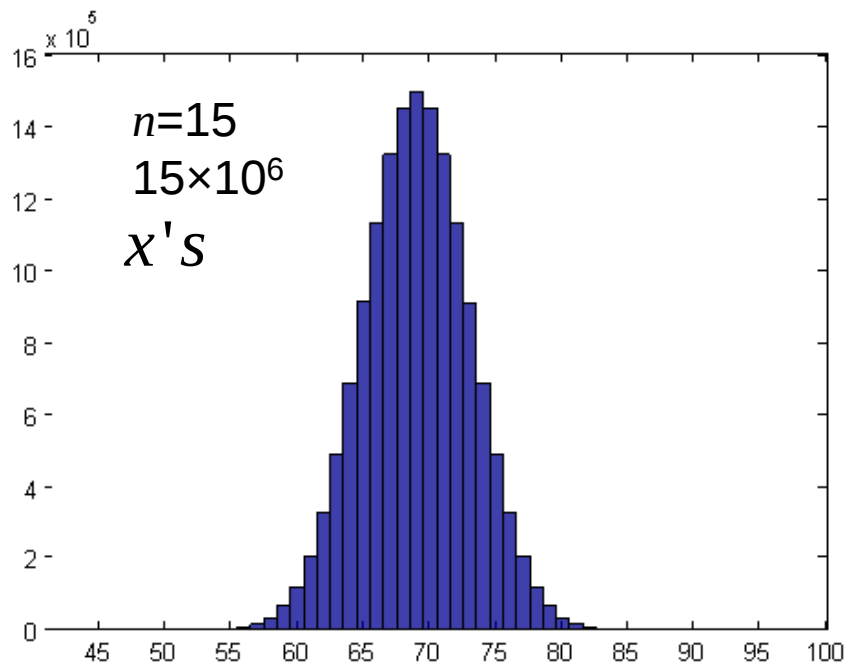
with $\alpha=.05$ for the heights of Math 1700 students.

Generate random data values.

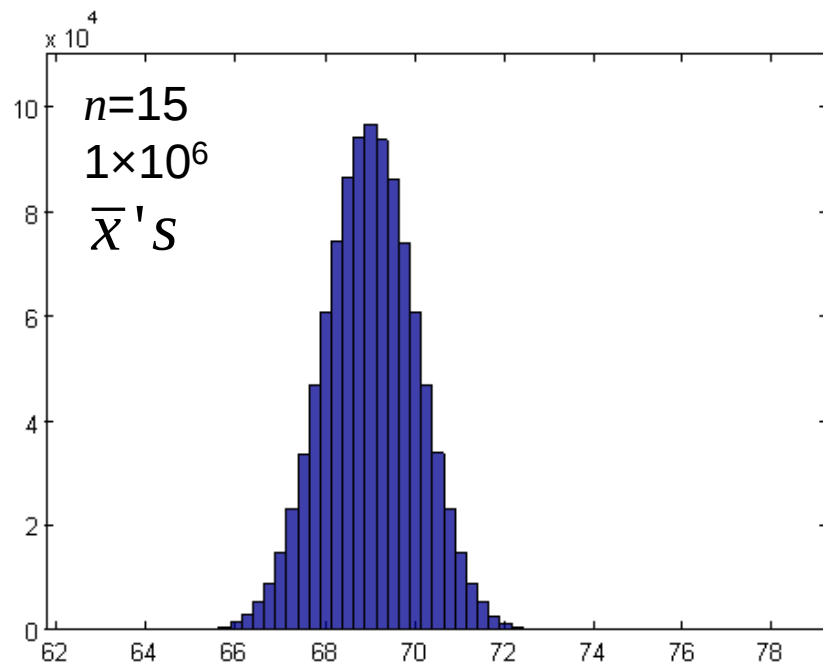
8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Generated 15×10^6
normal data values
from $\mu = 69$ " and $\sigma = 4$ ".

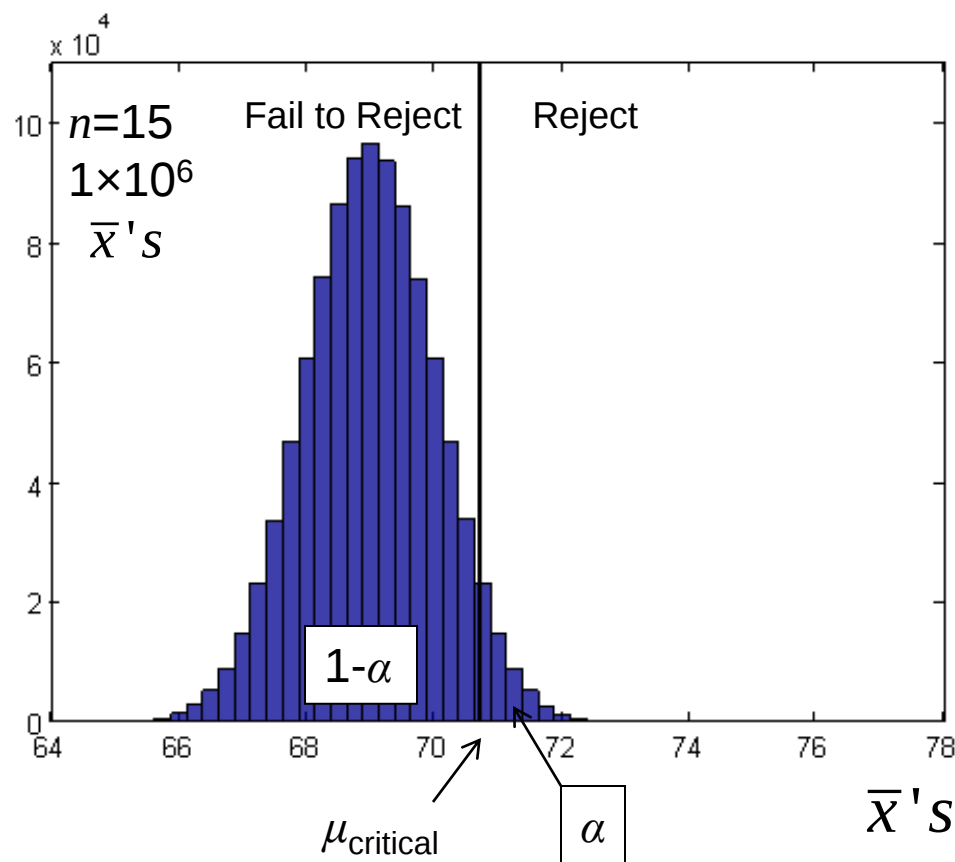


Calculated
 1×10^6 means with $n=15$.
(Will repeat for $\mu = 72$ ".)



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



$$H_0: \mu \leq 69'' \text{ vs. } H_a: \mu > 69''$$

$$\alpha = .05$$

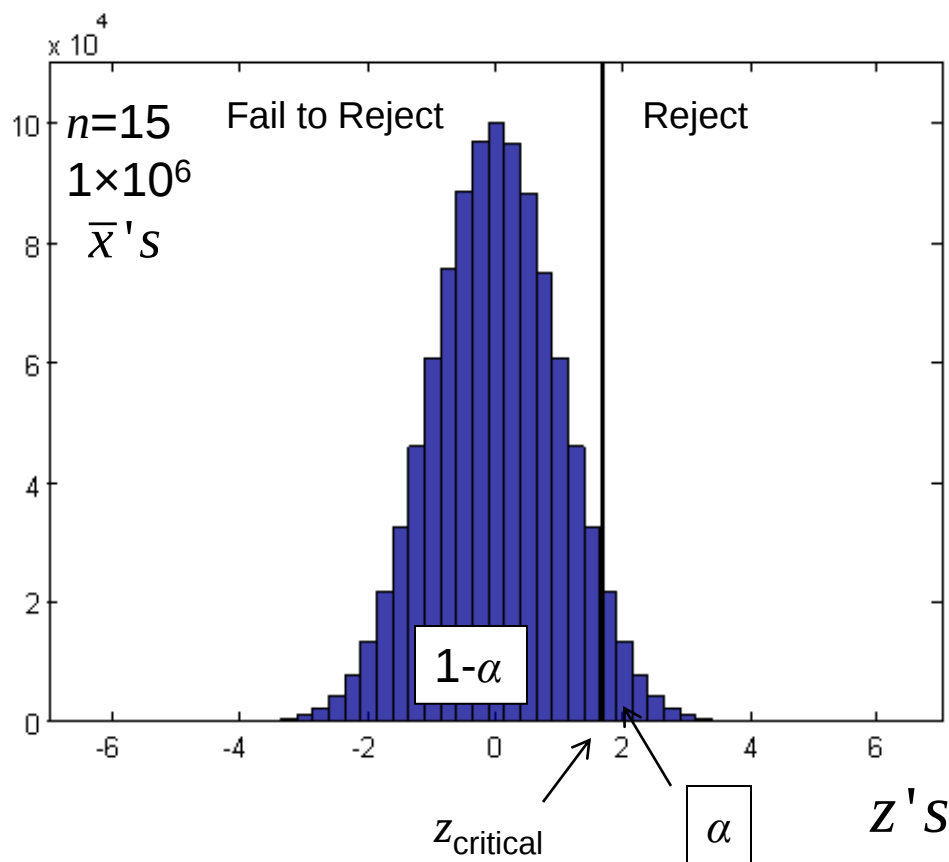
When the true mean $\mu = 69''$, we reject H_0 α fraction of the time.

Commit a Type I Error.

Given α , we want μ_{critical} .

8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



Instead of $\mu_{critical}$ we find critical z , $z_{critical} = z(\alpha)$.

Do this by assuming that $H_0: \mu = 69$ is true, then calculate

$$z = \frac{\bar{x} - 69}{4 / \sqrt{15}}$$

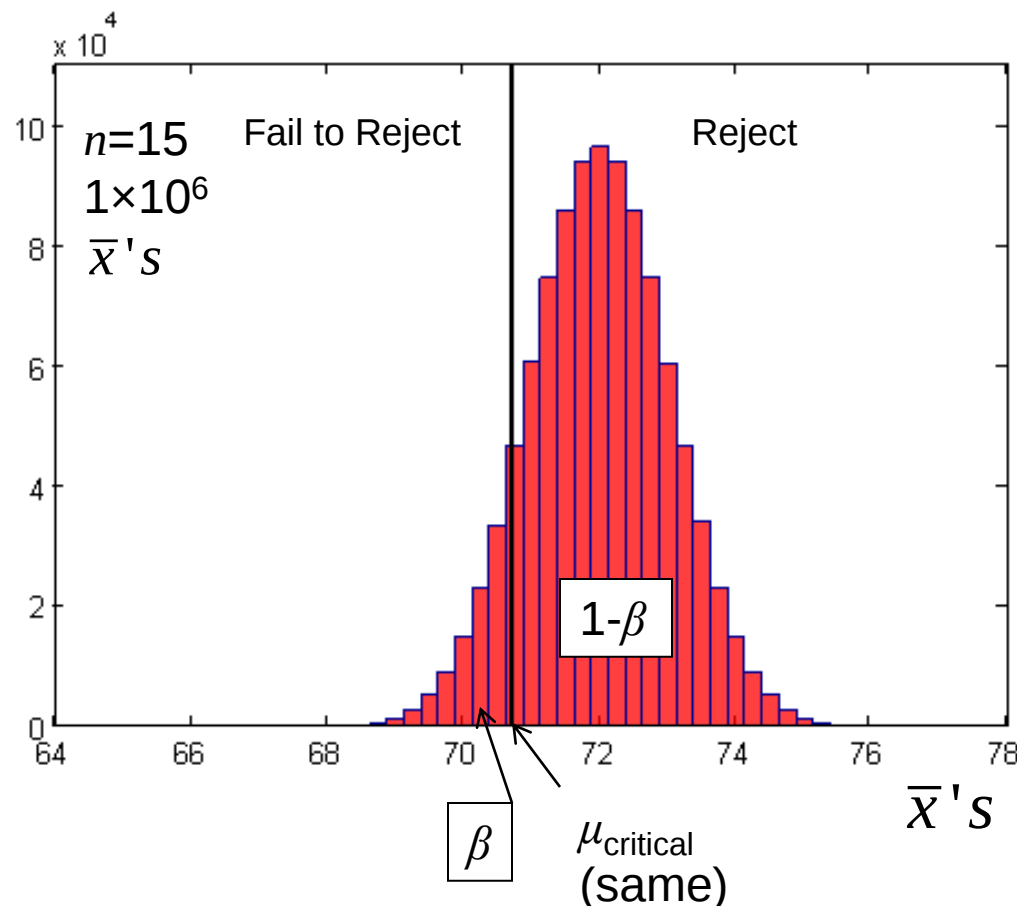
8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

$H_0: \mu \leq 69$ vs. $H_a: \mu > 69$
 $\alpha = .05$

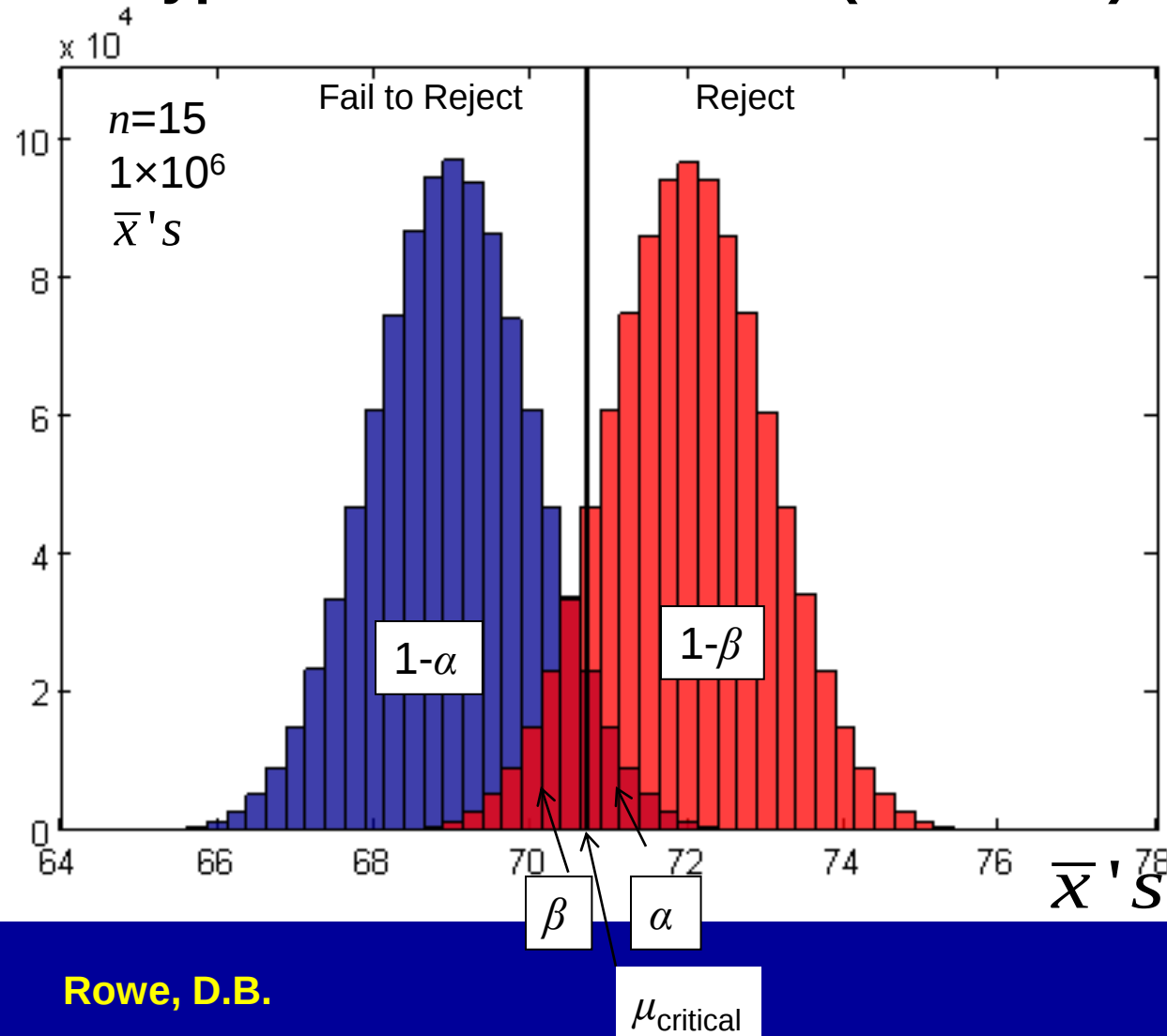
When the true mean $\mu = 72$ ",
 we do not reject H_0 β
 fraction of the time.

Commit a Type II Error



8: Introduction to Statistical Inference $H_0: \mu \leq \mu_0$ vs. $H_a: \mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



	H_0 True ($\mu=69''$)	H_0 False ($\mu=72''$)
Fail to Reject H_0	Correct Decision ($1-\alpha$)	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision ($1-\beta$)

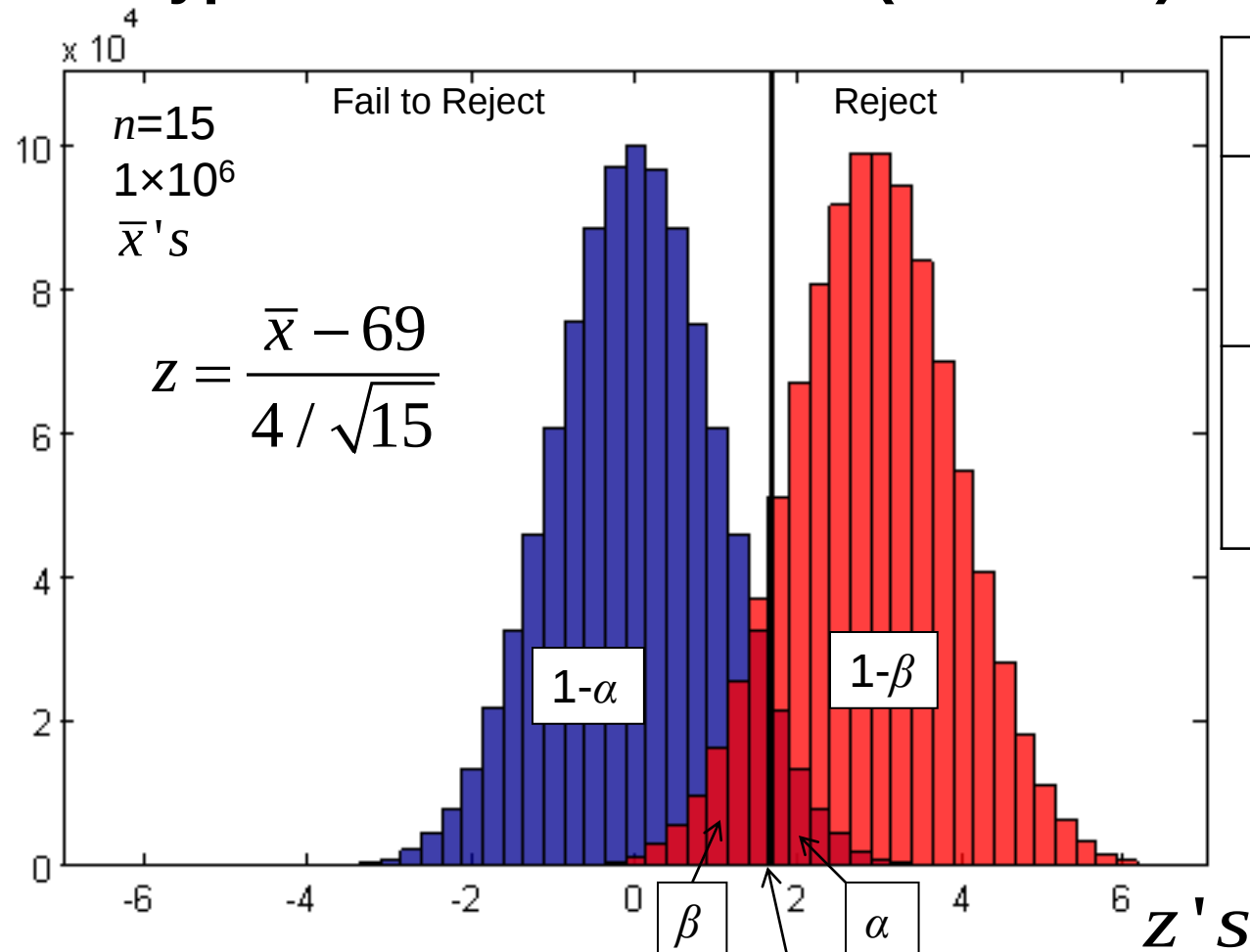
Power of the test:

$$1 - \beta = P(\text{Reject } H_0 \mid H_0 \text{ False})$$

Discrimination ability.
Ability to detect difference.

8: Introduction to Statistical Inference $H_0: \mu \leq \mu_0$ vs. $H_a: \mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



	H_0 True ($\mu=69''$)	H_0 False ($\mu=72''$)
Fail to Reject H_0	Correct Decision ($1-\alpha$)	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision ($1-\beta$)

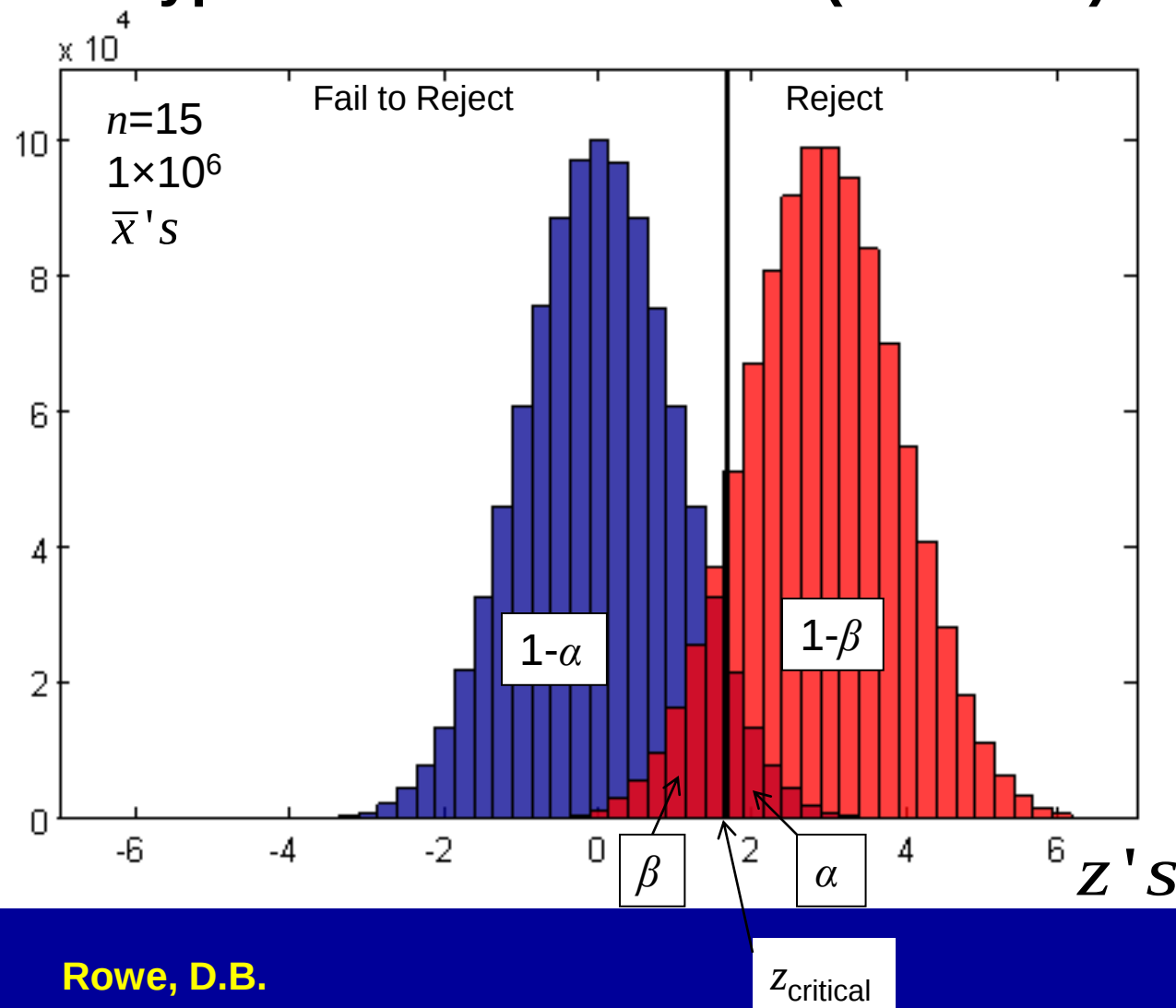
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8: Introduction to Statistical Inference $H_0: \mu \leq \mu_0$ vs. $H_a: \mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



We want our α ,
 Prob of Type I Error
 to be small.

So why not just
 decrease α ?

Decreasing α
 increases β .

And vice versa.

8: Introduction to Statistical Inference

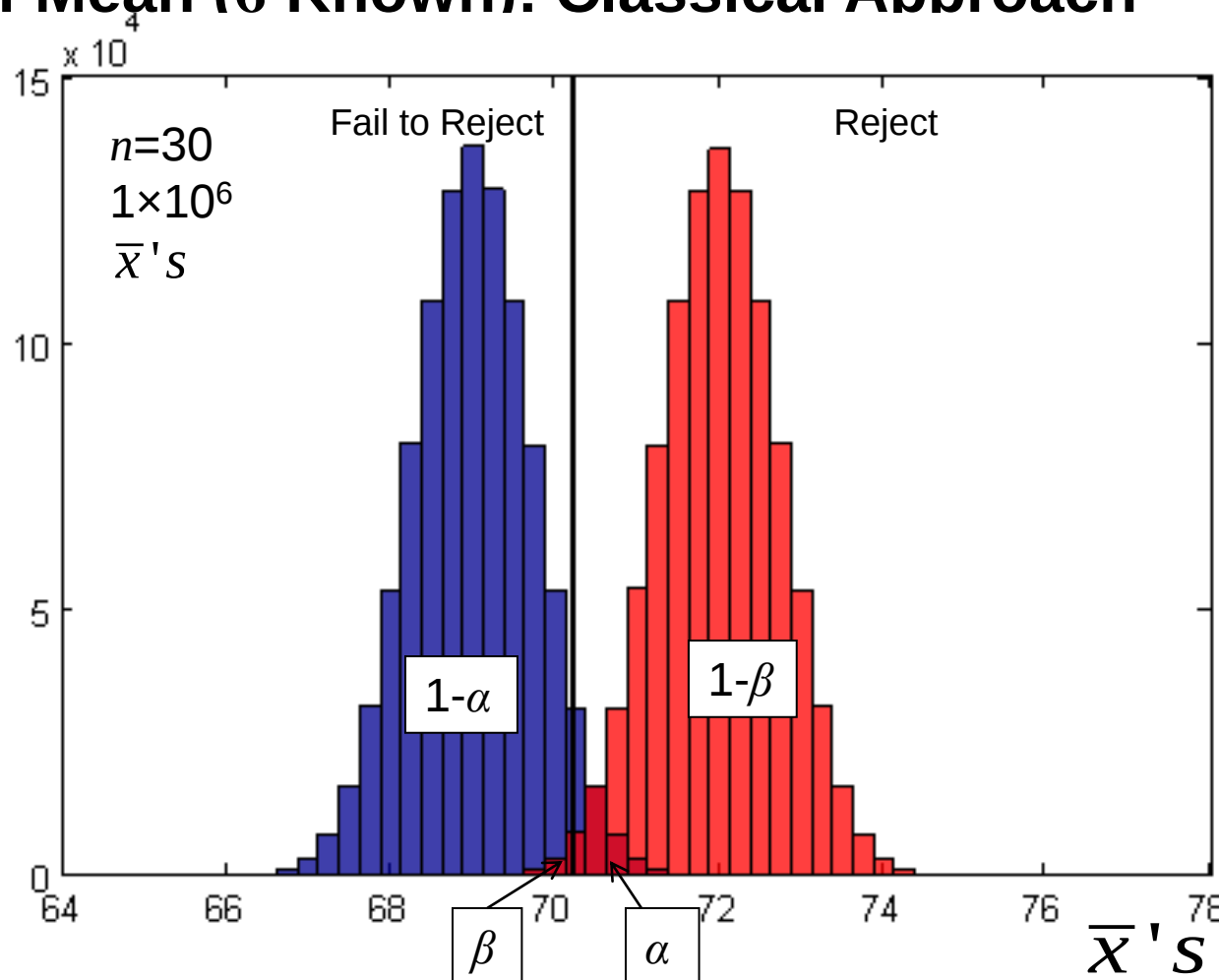
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

What is the solution?

Increase n .

Figure shows n increased to $n=30$ from $n=15$.

Note α and β both smaller with larger n .



8: Introduction to Statistical Inference $H_0: \mu \leq \mu_0$ vs. $H_1: \mu > \mu_0$

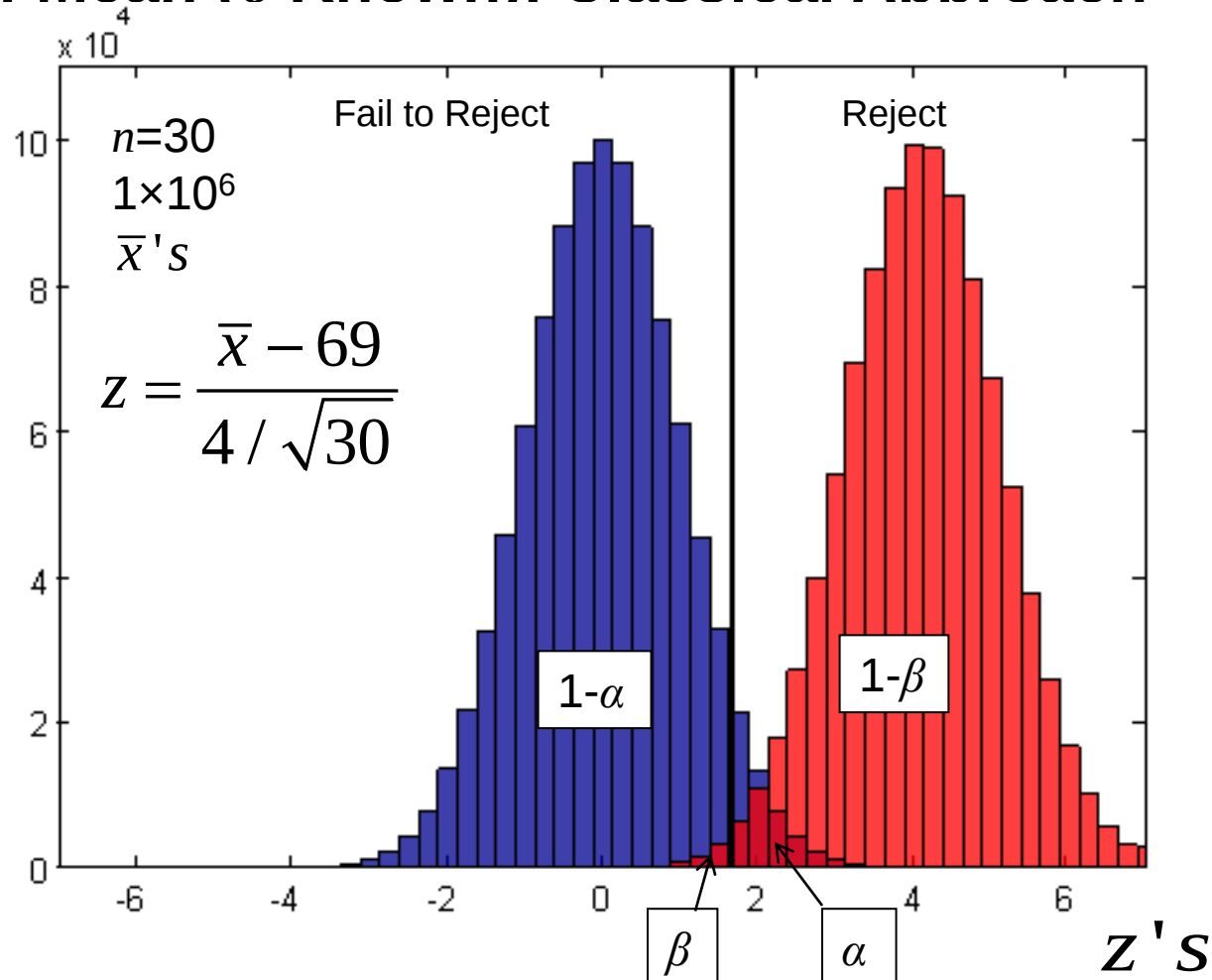
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

What is the solution?

Increase n .

Figure shows n increased to $n=30$ from $n=15$.

Note α and β both smaller with larger n .



Chapter 8: Introduction to Statistical Inference

Questions?

Homework: WebAssign

Lecture Chapter 9.1

Chapter 9: Inferences Involving One Population

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Department of Mathematical and Statistical Sciences



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

In Chapter 8, we performed hypothesis tests on the mean by

- 1) assuming that $\bar{X}$ was normally distributed (n “large”),
- 2) assuming the hypothesized mean μ_0 were true,
- 3) assuming that σ was known, so that we could form

$$z^* = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \quad \text{which with 1) – 3) has standard normal dist.}$$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

However, in real life, we never know σ for

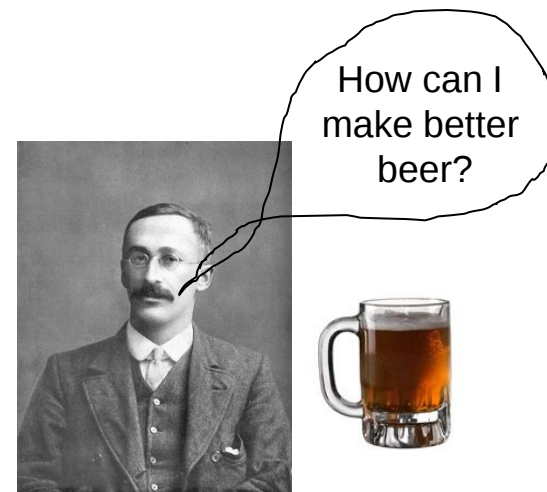
$$z^* = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

so we would like to estimate σ by s , then use

$$t^* = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} .$$

But t^* does not have a standard normal distribution.

It has what is called a Student t -distribution.



← Gosset
Guinness Brewery

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

What is the Student t -distribution and how do we get it?

Background Information

If the data comes from a normally distributed population, then

$$x \sim N(\mu, \sigma^2)$$

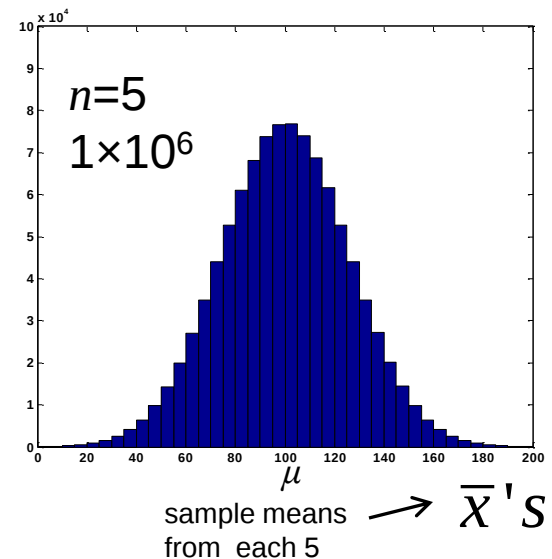
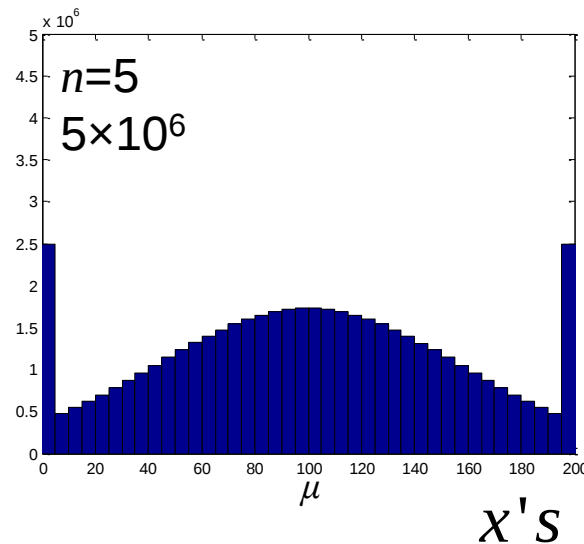
$\nwarrow$ mean $\swarrow$ variance

$$\bar{x} \sim N(\mu, \sigma^2 / n)$$

$\nwarrow$ mean $\swarrow$ variance

generate
 5×10^6
 random
 values

$\mu = 100$
 $\sigma = 57.7$
 $n = 5$



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$

$$\sigma = 57.7$$

$$n = 5$$

If we know the true population mean μ , then

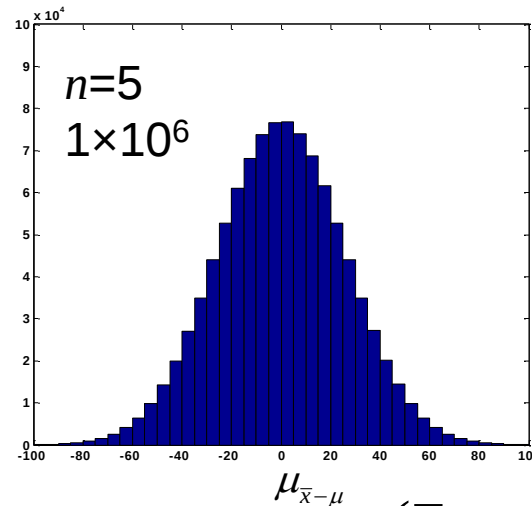
$$\bar{x} - \mu \sim N(0, \sigma^2 / n)$$

$\swarrow$ mean $\nwarrow$ variance

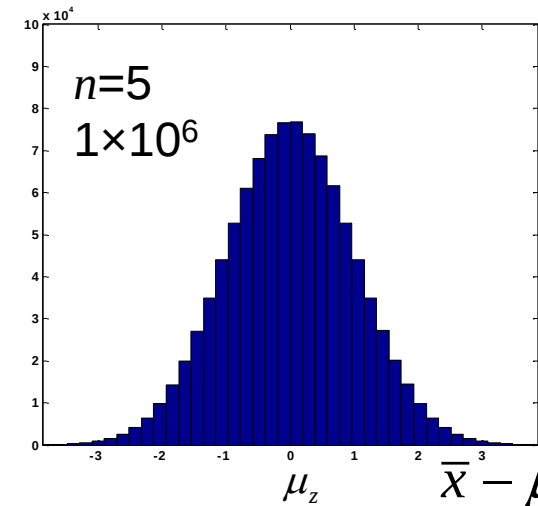
Given the variance of the mean σ^2/n , the distribution of

$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

$\swarrow$ mean $\nwarrow$ variance



$(\bar{x} - \mu)'s$



$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}'s$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$

$$\sigma = 57.7$$

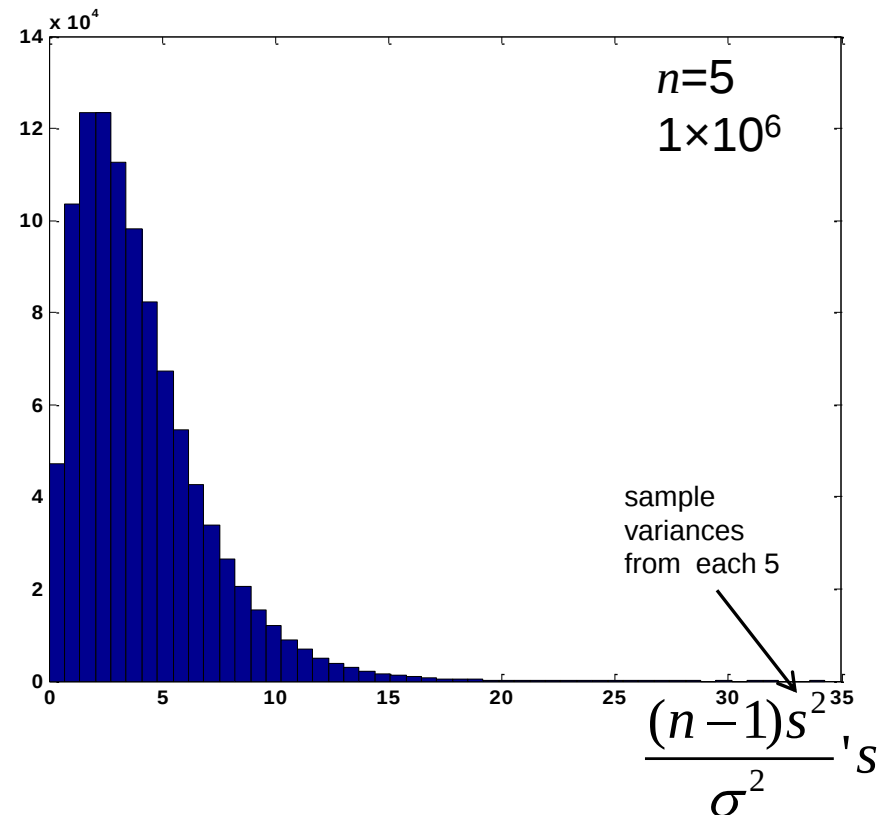
$$n = 5$$

It turns out that with the variance σ^2 known, the distribution of

$$\frac{(n-1)s^2}{\sigma^2} \quad \begin{array}{l} \leftarrow \text{sample variance} \\ \text{has a chi-square} \\ \leftarrow \text{population variance} \end{array}$$

distribution with $n-1$ degrees
of freedom.

(χ^2 distribution on Pages 453-454)



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$

$$\sigma = 57.7$$

$$n = 5$$

The ratio $\swarrow N(0,1)$

$$t = \left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right) / \sqrt{\frac{(n-1)s^2}{\sigma^2} / (n-1)}$$

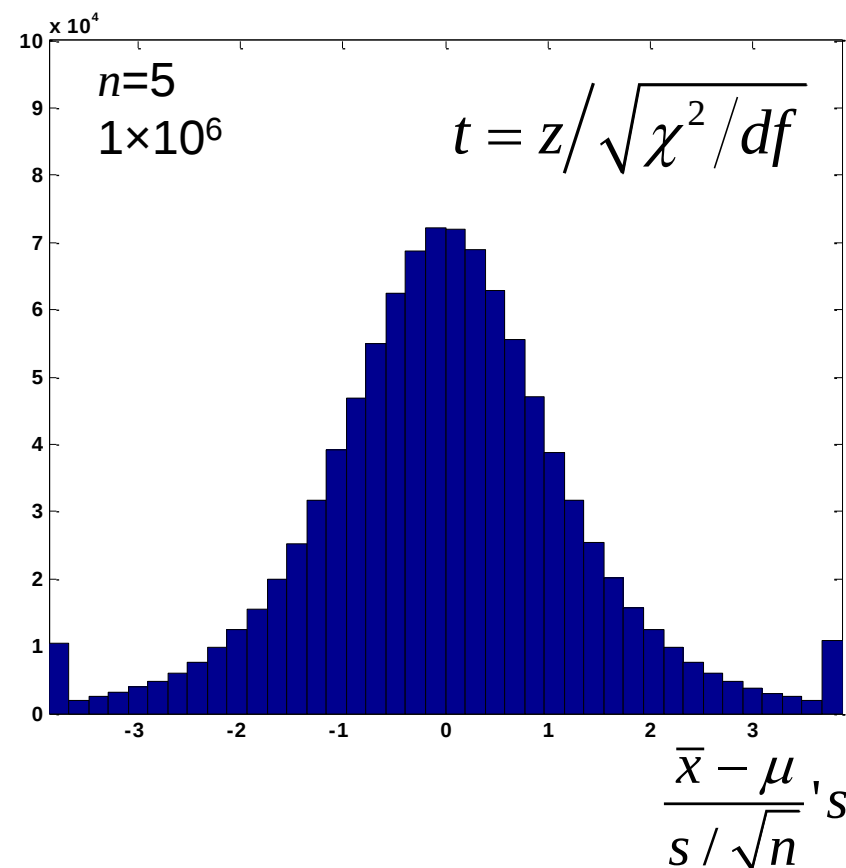
$\nwarrow \chi^2(n-1) \quad \nwarrow df$

with simplification

is
$$t = \frac{\bar{x} - \mu}{s / \sqrt{n}},$$

and has a Student

t -distribution with $n-1$ df.



9: Inferences Involving One Population

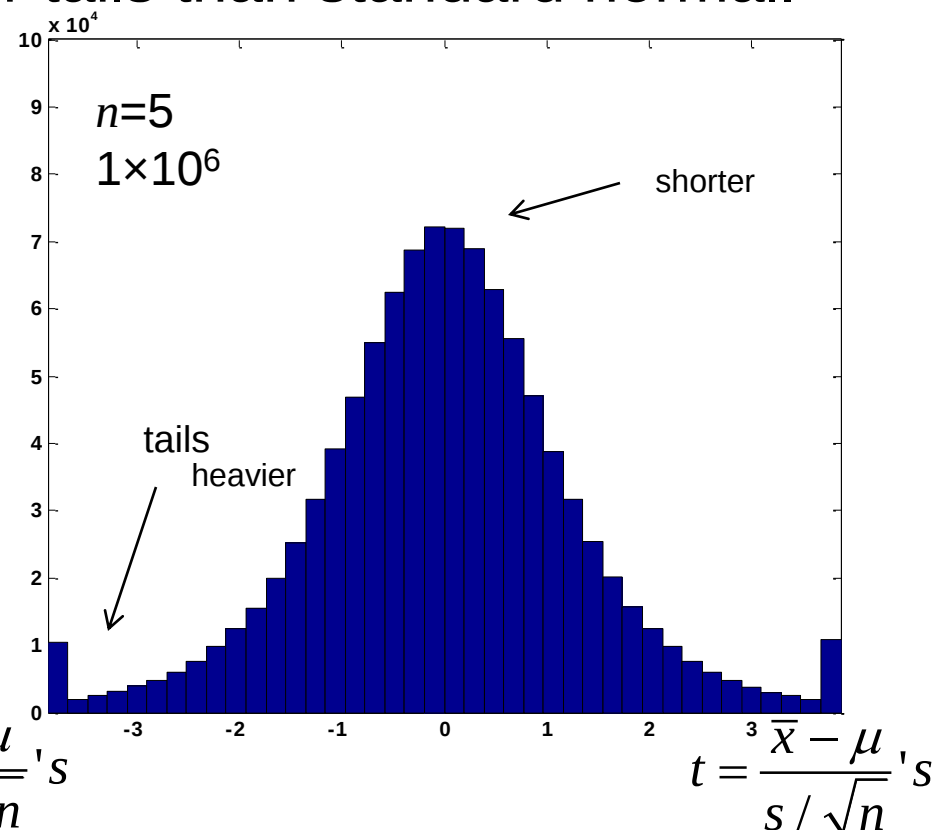
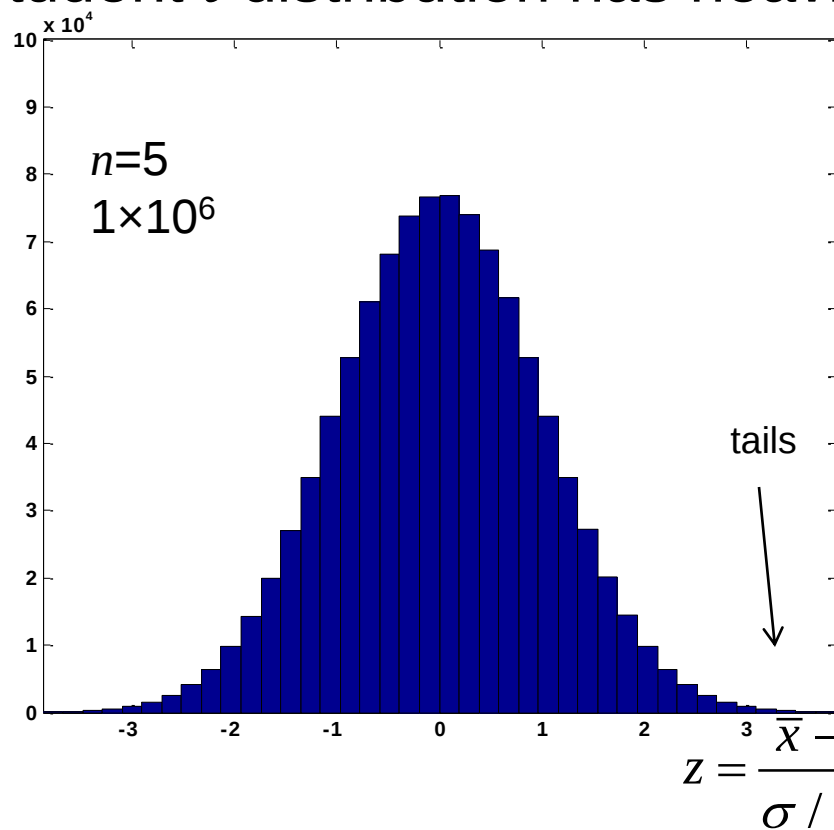
9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$

$$\sigma = 57.7$$

$$n = 5$$

Student t -distribution has heavier tails than standard normal.



9: Inferences Involving One Population

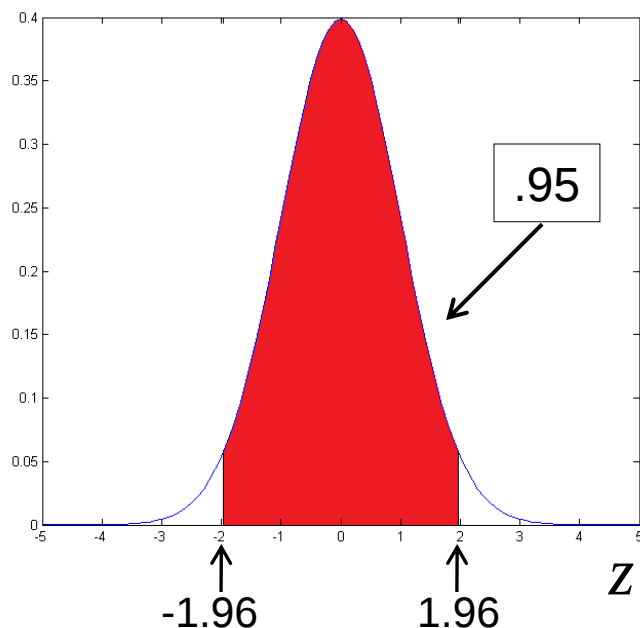
9.1 Inference about the Mean μ (σ Unknown)

$$\nu = df = n - 1$$

The standard normal dist is:

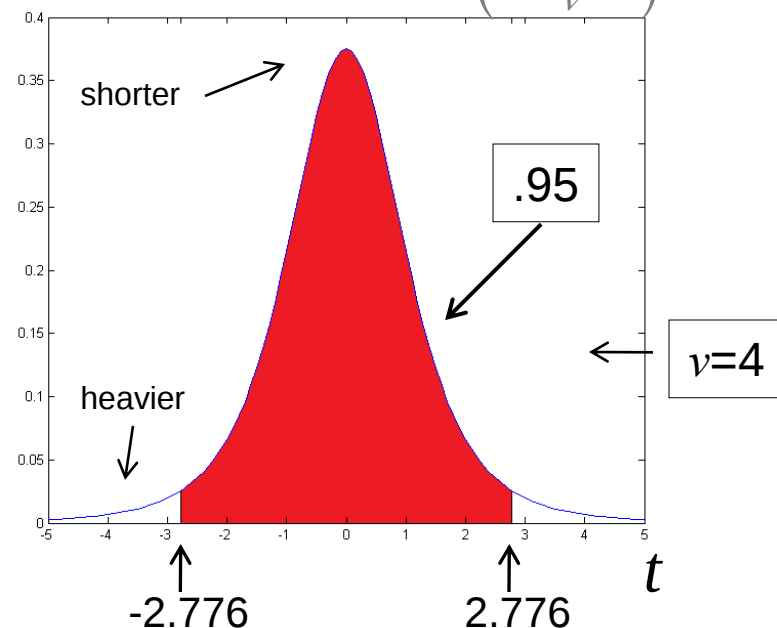
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

← ignore →



The Student-t distribution is:

$$f(t | \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1 + \frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

The standard normal dist. is:

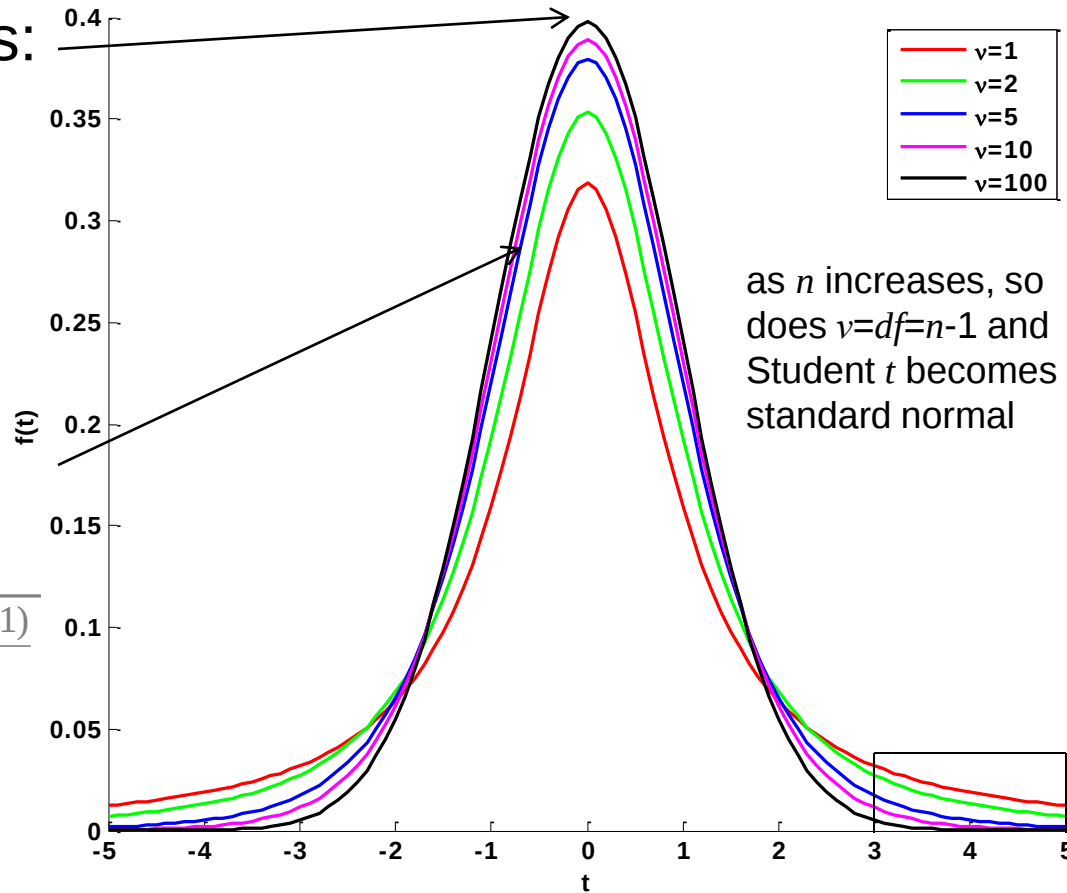
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← ignore



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

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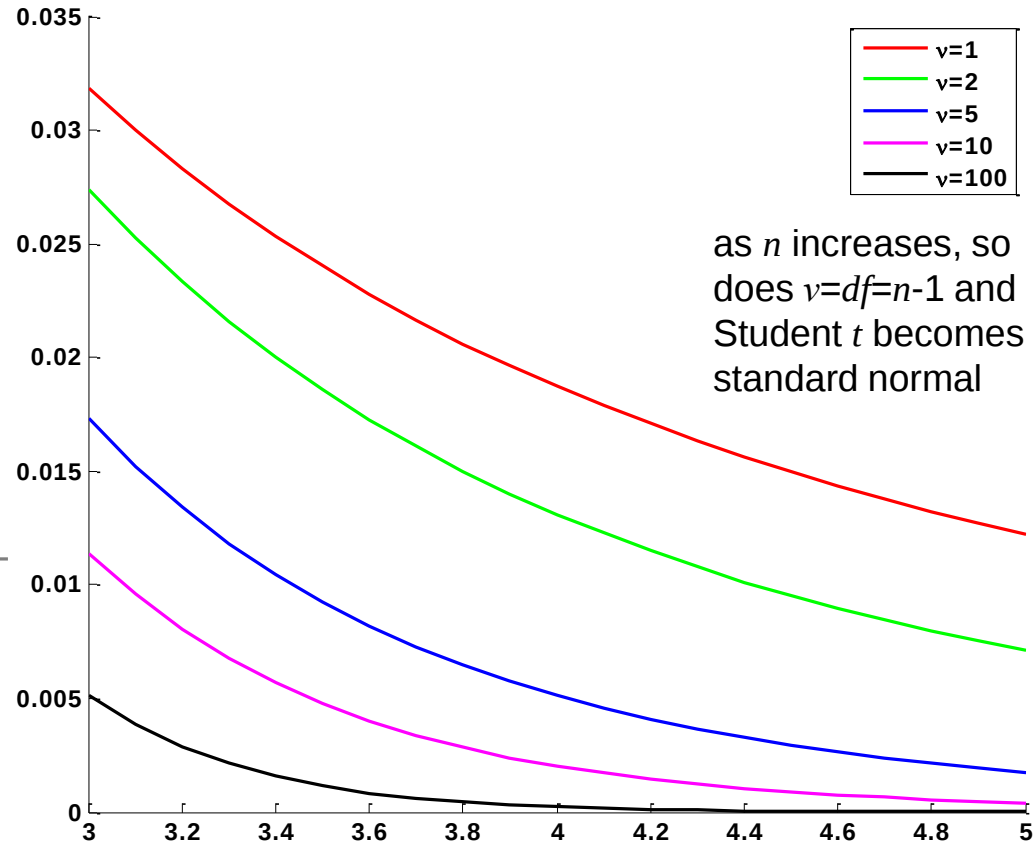
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← ignore



9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Using the t -Distribution Table

Finding critical value from a Student t -distribution, $df=n-1$

$t(df, \alpha)$, t value with α area larger than it

with df degrees
of freedom

Table 6
Appendix B
Page 719.

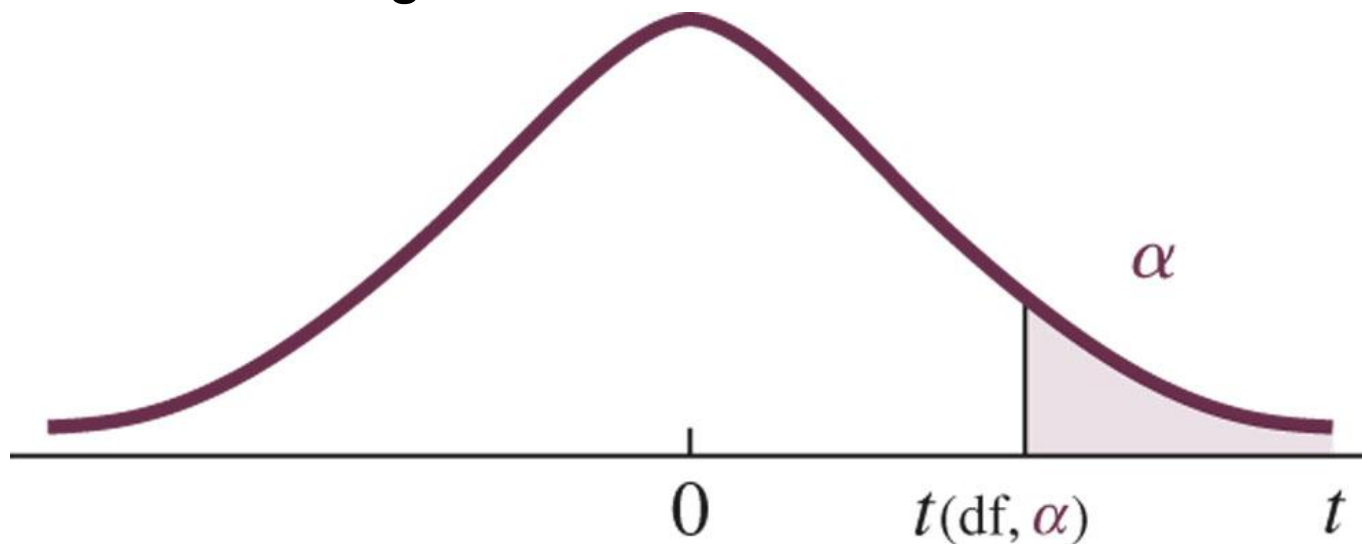


Figure from Johnson & Kubly, 2012.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Example: Find the value of $t(10, 0.05)$,
 $df=10$, $\alpha=0.05$.

Area in One Tail

	0.25	0.10	0.05	0.025	0.01	0.005
--	------	------	------	-------	------	-------

Area in Two Tails

df	0.50	0.20	0.10	0.05	0.02	0.01
3	0.765	1.64	2.35	3.18	4.54	5.84
4	0.741	1.53	2.13	2.78	3.75	4.60
5	0.727	1.48	2.02	2.57	3.36	4.03
6	0.718	1.44	1.94	2.45	3.14	3.71
7	0.711	1.41	1.89	2.36	3.00	3.50
8	0.706	1.40	1.86	2.31	2.90	3.36
9	0.703	1.38	1.83	2.26	2.82	3.25
10	0.700	1.37	1.81	2.23	2.76	3.17

⋮

35	0.682	1.31	1.69	2.03	2.44	2.72
40	0.681	1.30	1.68	2.02	2.42	2.70
50	0.679	1.30	1.68	2.01	2.40	2.68
70	0.678	1.29	1.67	1.99	2.38	2.65
100	0.677	1.29	1.66	1.98	2.36	2.63
df > 100	0.675	1.28	1.65	1.96	2.33	2.58

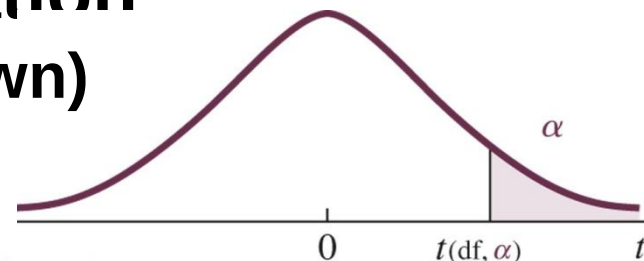


Table 6
 Appendix B
 Page 719.

Go to 0.05
 One Tail
 column and
 down to 10
 df row.

Figures from
 Johnson & Kubly, 2012.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

When making a confidence interval for μ when σ unknown, we assume that the population is normal, not just mean, but when n is “large,” can often use for nonnormal distributions.

The assumption for inferences about the mean μ when σ is unknown: The sampled population is normally distributed.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Discussed a confidence interval for the μ when σ was known,

Confidence Interval for Mean:

$$\bar{x} - z(\alpha / 2) \frac{\sigma}{\sqrt{n}} \quad \text{to} \quad \bar{x} + z(\alpha / 2) \frac{\sigma}{\sqrt{n}} \quad (8.1)$$

now, with σ unknown, the CI for the mean is

Confidence Interval for Mean:

$$\bar{x} - t(df, \alpha / 2) \frac{s}{\sqrt{n}} \quad \text{to} \quad \bar{x} + t(df, \alpha / 2) \frac{s}{\sqrt{n}} \quad (9.1)$$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$. Assume $\sigma=4.0$. Construct a 95% CI for μ .

$$\bar{x} \pm z(\alpha / 2) \frac{\sigma}{\sqrt{n}} \quad \text{Fill In}$$

Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$ and $s=3.5$. Construct a 95% CI for μ

$$\bar{x} \pm t(df, \frac{\alpha}{2}) \frac{s}{\sqrt{n}} \quad \text{Fill In}$$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$. Assume $\sigma=4.0$. Construct a 95% CI for μ .

$$\bar{x} \pm z(\alpha / 2) \frac{\sigma}{\sqrt{n}} \longrightarrow 67.2 - 1.96 \frac{4.0}{\sqrt{15}} \quad \text{to} \quad 67.2 + 1.96 \frac{4.0}{\sqrt{15}}$$

65.2 to 69.2

9: Inferences Involving One Population

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vs.

$$65.2 \quad \text{to} \quad 69.2$$

Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$ and $s=3.5$. Construct a 95% CI for μ .

$$\bar{x} \pm t(df, \frac{\alpha}{2}) \frac{s}{\sqrt{n}}$$

9: Inferences Involving One Population

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Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$. Assume $\sigma=4.0$. Construct a 95% CI for μ .

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vs.

65.2 to 69.2

Example: A random sample of $n=15$ math1700 student heights yielded $\bar{x} = 67.2$ and $s=3.5$. Construct a 95% CI for μ .

$$\bar{x} - t(df, \frac{\alpha}{2}) \frac{s}{\sqrt{n}} \longrightarrow 67.2 - 2.14 \frac{3.5}{\sqrt{15}} \text{ to } 67.2 + 2.14 \frac{3.5}{\sqrt{15}}$$

65.3 to 69.1

Chapter 9: Inferences Involving One Population

Questions?

Homework: Read Chapter 9.1

WebAssign

Chapter 9 # 9, 21, 23, 45, 55