

Class 7

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Agenda:

Recap Chapter 4.3 - 4.5

Lecture Chapter 5.1 - 5.3

Recap Chapter 4.3

4: Probability

4.3 Rules of Probability – “A or B”

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example: Pick Card, $A=\text{Heart}$, $B=\text{Ace}$

$P(\text{Heart or Ace})$

$$P(\text{Heart}) = 13 / 52$$

$$P(\text{Ace}) = 4 / 52$$

$$P(\text{Heart and Ace}) = 1 / 52$$

$$P(\text{Heart or Ace}) = P(\text{Heart}) + P(\text{Ace}) - P(\text{Heart and Ace})$$

$$P(\text{Heart or Ace}) = 13 / 52 + 4 / 52 - 1 / 52 = 16 / 52$$

double
counted

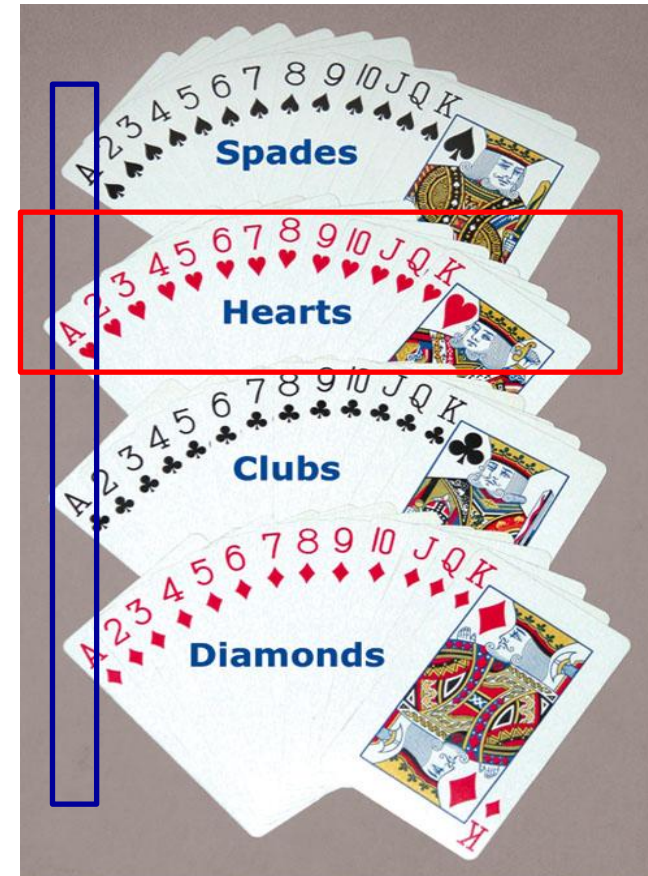


Figure from Johnson & Kuby, 2012.

4: Probability

4.3 Rules of Probability – “A and B”

$$P(A \text{ and } B) = P(B)P(A|B)$$

Example: Pick Card, $A=\text{Heart}$, $B=\text{Ace}$

$P(\text{Heart and Ace})$

$P(\text{Ace}) = 4 / 52$

$P(\text{Heart} | \text{Ace}) = 1 / 4$

$P(\text{Heart and Ace}) = P(\text{Ace})P(\text{Heart}|\text{Ace})$

$P(\text{Heart and Ace}) = (4 / 52)(1 / 4) = 1 / 52$

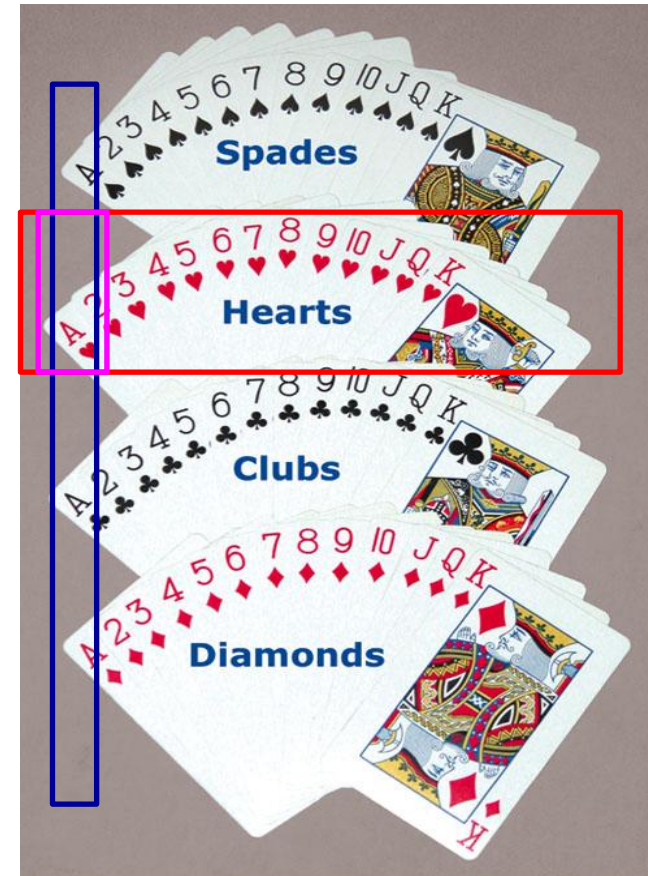


Figure from Johnson & Kuby, 2012.

4: Probability

Questions?

Homework: Read Chapter 4.3-4.5

Web Assign Chapter 4 # 59, 63, 65, 69,
85, 89, 91, 97, 105, 107, 113

Set one die to 4 (event B). Roll the other die 100 times.
Let A be that a 3 comes up (7 is the sum of the two die).
Calculate $P(A | B)$ using the empirical approach.

4: Probability

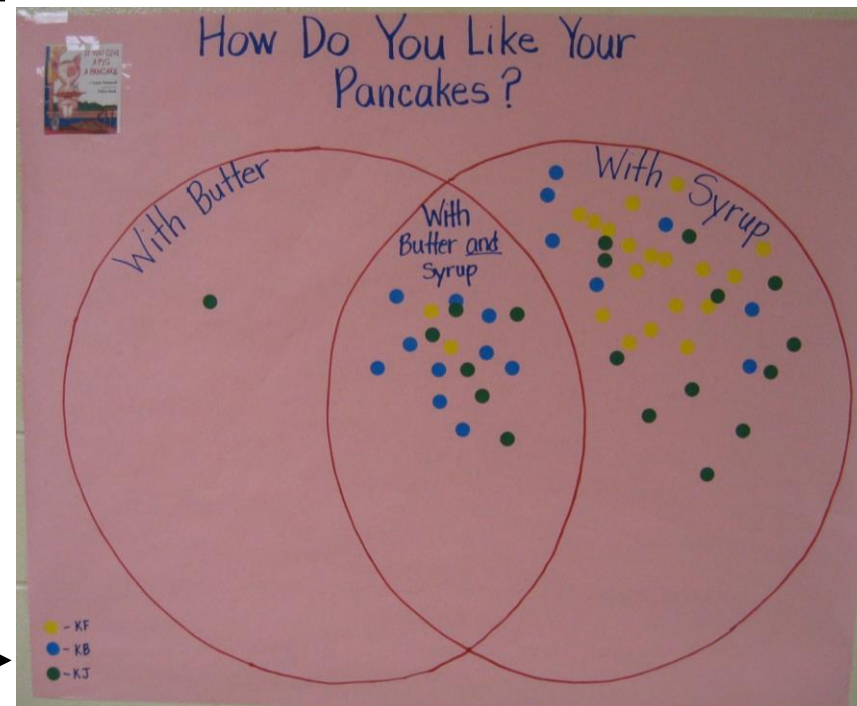
What is the probability that a random kindergartener likes syrup (B) on their pancakes given that they like butter (A) on their pancakes?

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$P(A) = \frac{n(A)}{n}$$

3 kindergarten
classes →



4: Probability



Homework: Watch Catch 21 Episode

<https://www.youtube.com/watch?v=N73GnjXfl48>

The bonus round begins at time 12:28.

After the $Q\spadesuit, J\clubsuit, A\spadesuit$ are drawn:

What is the probability of an A as the 4th card?

What is the probability of a “10” as the 4th card?

What is the probability of an A as the 5th card?

What is the probability of an A as the 6th card?

What is the probability of a 6 as the 7th card?

(10 Card is a 10 or a face card.)

Lecture Chapter 5.1 - 5.3

Chapter 5: Probability Distributions (Discrete Variables)

Daniel B. Rowe, Ph.D.

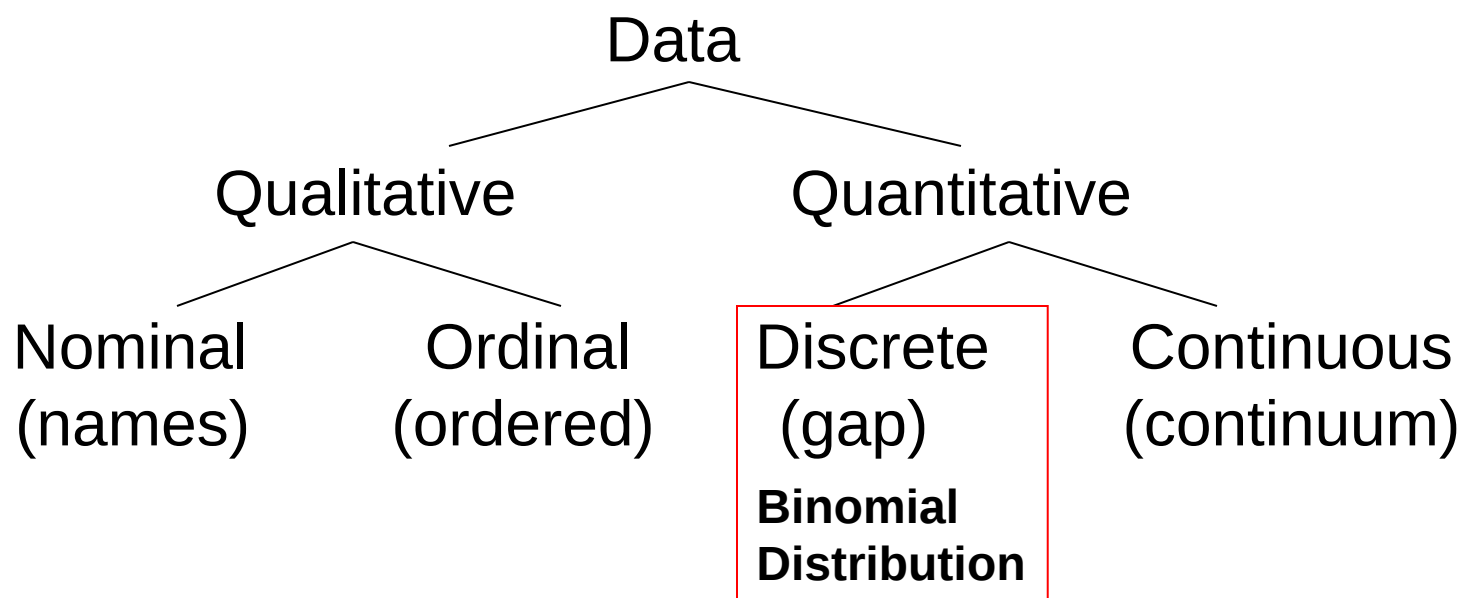
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5: Probability Distributions (Discrete Variables)

5.1 Random Variables

At beginning of course we talked about types of data.



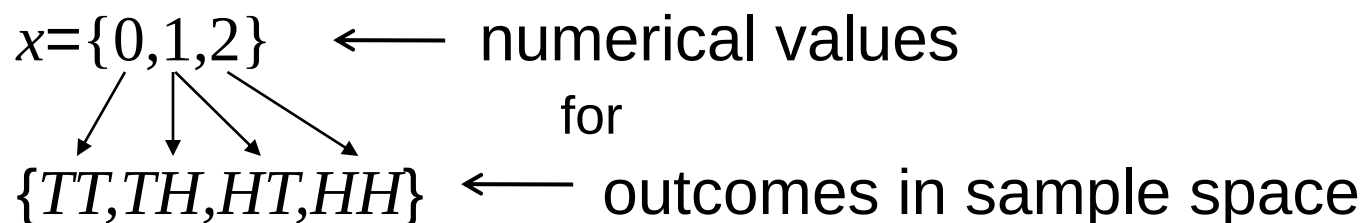
5: Probability Distributions (Discrete Variables)

5.1 Random Variables

Random Variables: A variable that assumes a unique numerical value for each of the outcomes in the sample space of a probability experiment.

Example:

Let x = the number of heads when we flip a coin twice.



5: Probability Distributions (Discrete Variables)

5.1 Random Variables

Discrete Random Variables: A quantitative random variable that can assume a countable number of values.

Continuous Random Variable: A quantitative random variable that can assume an uncountable number of values.

← Continuum of values.

Examples:

Discrete: Number of heads when we flip a coin ten times.

Continuous: Distance from earth center to sun center.

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

Probability Distribution: A distribution of the probabilities associated with each of the values of a random variable. The probability distribution is a theoretical distribution; it is used to represent populations.

Flipping a Coin

$x = \# H$	$P(x)$
0	0.25
1	0.50
2	0.25

Rolling a Die

$x = \text{face value}$	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Figures from Johnson & Kubly, 2012.

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

Probability Function: A rule $P(x)$ that assigns probabilities to the values of the random variables, x .

Example:

Let $x = \#$ of heads when we flip a coin twice.

$$x = \{0, 1, 2\}$$

$$P(x) = \frac{2!}{x!(2-x)!} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{2-x}$$

Note:

1. $0 \leq P(x) \leq 1$
2. $\sum P(x) = 1$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

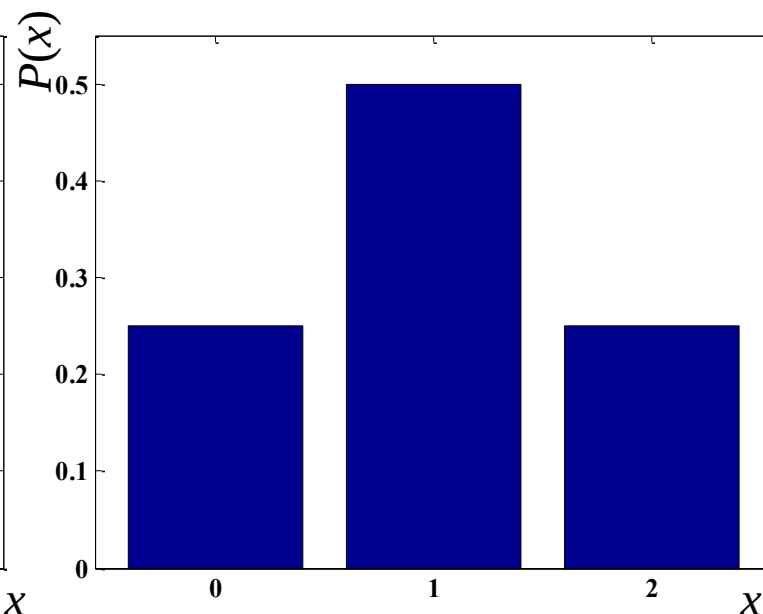
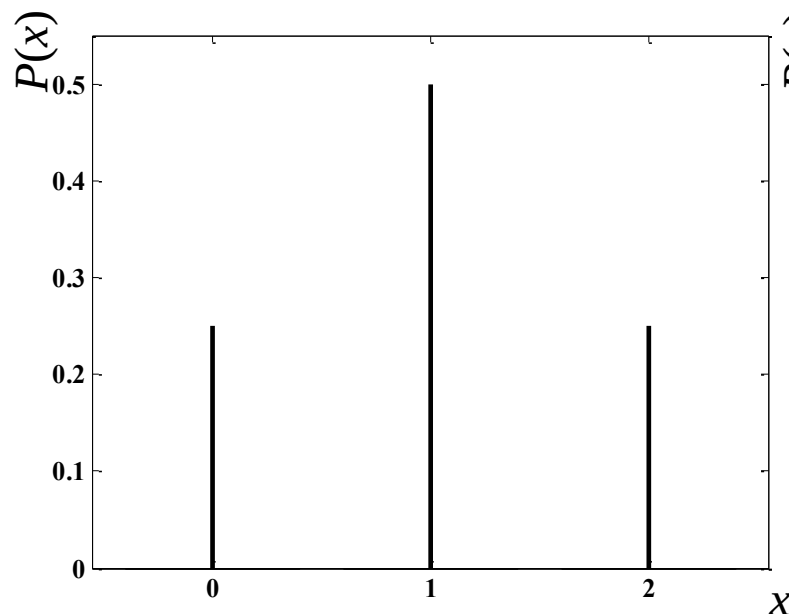
Example:

Let $x = \#$ of H when flip a coin twice.

$$P(x) = \frac{2!}{x!(2-x)!} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{2-x}$$

$x = 0, 1, 2$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$



1: Statistics

1.2 What is Statistics?

Data: The set of values collected from the variable from each of the elements that belong to the sample.

Experiment: A planned activity whose results yield a set of data.

Sample: Subset of the population.

Parameter: A numerical value summarizing all the data of an entire population.

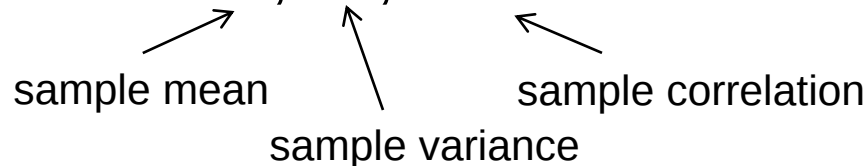
Statistic: A numerical value summarizing the sample data.

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

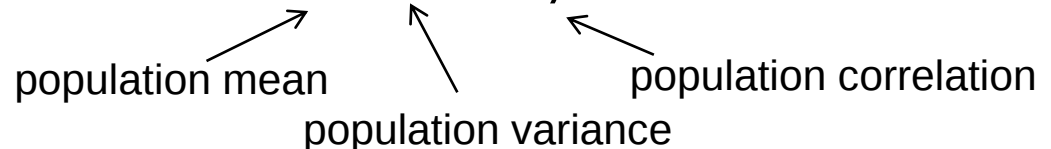
If we calculate a numerical summary from the sample of data, it is called a **statistic**. i.e. $\bar{x}$, s^2 , or r .

sample mean sample variance sample correlation



If we calculate a numerical summary from the population of data, it is called a **parameter**. i.e. μ , σ^2 , or ρ .

population mean population variance population correlation



μ is the Greek letter lower case mu.

σ is the Greek letter lower case sigma.

ρ is the Greek letter lower case rho.

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Mean of a discrete random variable (expected value):

The mean, μ , of a discrete random variable x is found by multiplying each possible value of x by its own probability, $P(x)$, and then adding all of the products together:

mean of x : μ = sum of (each x multiplied by its own probability)

$$\mu = \sum_{i=1}^n [x_i P(x_i)] \quad (5.1)$$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\mu = \sum_{i=1}^n [x_i P(x_i)] = x_1 P(x_1) + x_2 P(x_2) + \dots + x_n P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\mu = (x_1) \cdot P(x_1) + (x_2) \cdot P(x_2) + (x_3) \cdot P(x_3)$$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\mu = \sum_{i=1}^n [x_i P(x_i)] = x_1 P(x_1) + x_2 P(x_2) + \dots + x_n P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\mu = (x_1) \cdot P(x_1) + (x_2) \cdot P(x_2) + (x_3) \cdot P(x_3)$$

$$\mu = (0) \cdot P(0) + (1) \cdot P(1) + (2) \cdot P(2)$$

$$\mu = (0) \cdot (1/4) + (1) \cdot (1/2) + (2) \cdot (1/4)$$

$$\mu = 0 + 1/2 + 1/2$$

$$\mu = 1$$

	x	$P(x)$	
$\nearrow$	0	$\frac{1}{4}$	$\nwarrow$
x_1			$P(x_1)$
$\nearrow$	1	$\frac{1}{2}$	$\nwarrow$
x_2			$P(x_2)$
$\nearrow$	2	$\frac{1}{4}$	$\nwarrow$
x_3			$P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Variance of a discrete random variable: The variance, σ^2 , of a discrete random variable x is found by multiplying each possible value of the squared deviation, $(x - \mu)^2$, by its own probability, $P(x)$, and then adding all of the products together:

variance of x : sigma squared

= sum of (squared deviation times probability)

$$\sigma^2 = \sum_{i=1}^n [(x_i - \mu)^2 P(x_i)] \quad (5.2)$$

equivalent formula

$$\sigma^2 = \sum_{i=1}^n [x_i^2 P(x_i)] - \mu^2 \quad (5.3b)$$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\sigma^2 = \sum_{i=1}^n [(x_i - \mu)^2 P(x_i)] = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \dots + (x_n - \mu)^2 P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\sigma^2 = (x_1 - \mu)^2 \cdot P(x_1) + (x_2 - \mu)^2 \cdot P(x_2) + (x_3 - \mu)^2 \cdot P(x_3)$$

		$\mu = 1$		
		x	$P(x)$	
	$\nearrow$	0	$\frac{1}{4}$	$\nwarrow$
x_1				$P(x_1)$
	$\nearrow$	1	$\frac{1}{2}$	$\nwarrow$
x_2				$P(x_2)$
	$\nearrow$	2	$\frac{1}{4}$	$\nwarrow$
x_3				$P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\sigma^2 = \sum_{i=1}^n [(x_i - \mu)^2 P(x_i)] = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \dots + (x_n - \mu)^2 P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\begin{aligned} \sigma^2 &= (x_1 - \mu)^2 \cdot P(x_1) + (x_2 - \mu)^2 \cdot P(x_2) + (x_3 - \mu)^2 \cdot P(x_3) \\ \sigma^2 &= (0 - 1)^2 \cdot P(0) + (1 - 1)^2 \cdot P(1) + (2 - 1)^2 \cdot P(2) \\ \sigma^2 &= (-1)^2 \cdot (1/4) + (0)^2 \cdot (1/2) + (1)^2 \cdot (1/4) \\ \sigma^2 &= 1/4 + 0 + 1/4 \\ \sigma^2 &= 1/2 \end{aligned}$$

		$\mu = 1$	
	x	$P(x)$	
	0	$\frac{1}{4}$	
$\nearrow$			$\nwarrow$
x_1	1	$\frac{1}{2}$	$P(x_1)$
$\nearrow$			$\nwarrow$
x_2	2	$\frac{1}{4}$	$P(x_2)$
$\nearrow$			$\nwarrow$
x_3			$P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Standard deviation of a discrete variable: The positive square root of the variance.

$$\sigma = \sqrt{\sigma^2} \quad (5.4)$$

$$\sigma = \sqrt{\sum_{i=1}^n [(x_i - \mu)^2 P(x_i)]}$$

$$\sigma^2 =$$

$$\sigma =$$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Standard deviation of a discrete variable: The positive square root of the variance.

$$\sigma = \sqrt{\sigma^2} \quad (5.4)$$

$$\sigma = \sqrt{\sum_{i=1}^n [(x_i - \mu)^2 P(x_i)]}$$

$$\begin{aligned}\sigma^2 &= 1/2 \\ \sigma &= 1/\sqrt{2}\end{aligned}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Let's assume we have two independent events E_1 and E_2 .

We know that $P(E_1 \text{ and } E_2) = P(E_1)P(E_2)$. Page 211.

More generally, if we have n independent events $E_1, \dots, E_n$.

We know that $P(E_1 \text{ and } E_2 \cdots \text{ and } E_n) = P(E_1)P(E_2) \cdots P(E_n)$.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Let's assume we are flipping a coin twice.

E_1 =Head on first flip, E_2 =Tail on second flip.

The probability of heads on any given flip is $p = P(H)$.

The probability of tails (not heads) on any given flip is $q = (1-p)$.

$$\begin{array}{ll} \text{Then} & P(HT)=P(H)P(T) \\ & =p(1-p). \end{array} \quad \begin{array}{ll} \text{Similarly} & P(TH) = P(T)P(H) \\ & = (1-p)p. \end{array}$$

Let x = # of heads in two flips of a coin.

$$\begin{aligned} P(x=1) &= P(HT)+P(TH) \\ &= p(1-p)+(1-p)p = 2p(1-p). \end{aligned}$$


2 ways to get one H and one T
2 ways to get $x=1$ heads

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Bi means two like bicycle

An experiment with only two outcomes is called a Binomial exp.
 Call one outcome *Success* and the other *Failure*.
 Each performance of expt. is called a trial and are independent.

$$P(x) = \frac{n!}{\underbrace{x!(n-x)!}_{\text{num}(x \text{ successes})} \underbrace{p^x(1-p)^{n-x}}_{P(x \text{ successes and } n-x \text{ failures})}} \quad x = 0, \dots, n$$

Prob of exactly x successes

(5.5)

n = number of trials or times we repeat the experiment.

x = the number of successes out of n trials.

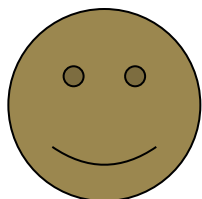
p = the probability of success on an individual trial.

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once.



O	$P(O)$
H	$1/2$
T	$1/2$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/2$
1	$1/2$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	1

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(0) =$$

num flips $\longrightarrow n=1$

num succ. $\longrightarrow x=1$

prob succ $\longrightarrow p=1/2$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once.



O	$P(O)$
H	$1/2$
T	$1/2$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/2$
1	$1/2$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	1

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(0) = \frac{1!}{0!(1-0)!} \left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^{1-0} = \frac{1}{1 \times 1} \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$P(1) = \frac{1!}{1!(1-1)!} \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{1-1} = \frac{1}{1 \times 1} \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

num flips $\longrightarrow n=1$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=1/2$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice.



O	$P(O)$
HH	$1/4$
HT	$1/4$
TH	$1/4$
TT	$1/4$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/4$
1	$2/4$
2	$1/4$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	2
2	1

num flips $\longrightarrow n=2$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=1/2$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice.



O	$P(O)$
HH	$1/4$
HT	$1/4$
TH	$1/4$
TT	$1/4$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/4$
1	$2/4$
2	$1/4$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	2
2	1

num flips $\longrightarrow n=2$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=1/2$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{2!}{1!(2-1)!} \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{2-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times.

	<i>O</i>	<i>P(O)</i>
	<i>HHH</i>	1/8
	<i>HHT</i>	1/8
	<i>HTH</i>	1/8
	<i>HTT</i>	1/8
	<i>THH</i>	1/8
	<i>THT</i>	1/8
	<i>TTH</i>	1/8
	<i>TTT</i>	1/8

$x = \# \text{ of Heads}$

x	$P(x)$
0	1/8
1	3/8
2	3/8
3	1/8

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	3
2	3
3	1

$n=3$
 $x=1$
 $p=1/2$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times.

	<i>O</i>	<i>P(O)</i>
	<i>HHH</i>	1/8
	<i>HHT</i>	1/8
	<i>HTH</i>	1/8
	<i>HTT</i>	1/8
	<i>THH</i>	1/8
	<i>THT</i>	1/8
	<i>TTH</i>	1/8
	<i>TTT</i>	1/8

$x = \# \text{ of Heads}$

x	$P(x)$
0	1/8
1	3/8
2	3/8
3	1/8

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	3
2	3
3	1

$n=3$
 $x=1$
 $p=1/2$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{3!}{1!(3-1)!} \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{3-1}$$

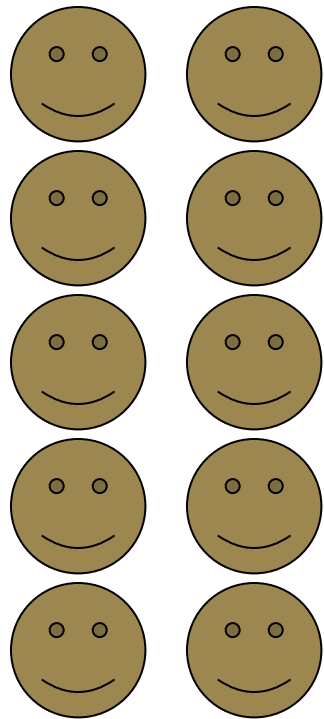
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin ten times.

$x = \#$ of Heads

$n(x) =$ ways to get x Heads



x	0	1	2	3	4	5	6	7	8	9	10
$n(x)$											
$P(x)$											

$$\begin{aligned} n &= 10 \\ x &= 0, \dots, 10 \\ p &= 1/2 \end{aligned}$$

$$n(x) = \frac{n!}{x!(n-x)!}$$

$$p^x (1-p)^{n-x} =$$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Note:

1. $0 \leq P(x) \leq 1$
2. $\sum P(x) = 1$

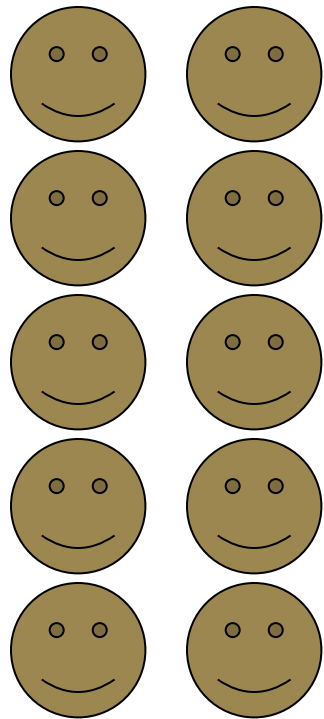
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin ten times.

$x = \#$ of Heads

$n(x) =$ ways to get x Heads



x	0	1	2	3	4	5	6	7	8	9	10
$n(x)$	1	10	45	120	210	252	210	120	45	10	1
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

$n=10$
 $x=0, \dots, 10$
 $p=1/2$

$$n(x) = \frac{n!}{x!(n-x)!}$$

$$p^x(1-p)^{n-x} = 1/1024$$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Note:
 1. $0 \leq P(x) \leq 1$
 2. $\sum P(x) = 1$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once. $p=2/3$



O	$P(O)$
H	$2/3$
T	$1/3$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/3$
1	$2/3$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	1

num flips $\longrightarrow n=1$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=2/3$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once. $p=2/3$



O	$P(O)$
H	$2/3$
T	$1/3$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/3$
1	$2/3$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	1

num flips $\longrightarrow n=1$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=2/3$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{1!}{1!(1-1)!} (2/3)^1 (1-2/3)^{1-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice. $p=2/3$



O	$P(O)$
HH	$4/9$
HT	$2/9$
TH	$2/9$
TT	$1/9$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/9$
1	$4/9$
2	$4/9$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	2
2	1

num flips $\longrightarrow n=2$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=2/3$

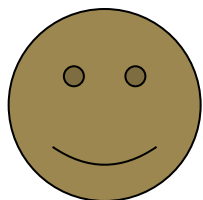
$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice. $p=2/3$



O	$P(O)$
HH	$4/9$
HT	$2/9$
TH	$2/9$
TT	$1/9$

$x = \# \text{ of Heads}$

x	$P(x)$
0	$1/9$
1	$4/9$
2	$4/9$

$n(x) = \text{ways to get } x \text{ Heads}$

x	$n(x)$
0	1
1	2
2	1

num flips $\longrightarrow n=2$
 num succ. $\longrightarrow x=1$
 prob succ $\longrightarrow p=2/3$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{2!}{1!(2-1)!} \left(\frac{2}{3}\right)^1 \left(1 - \frac{2}{3}\right)^{2-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times. $p=2/3$ $x = \# \text{ of Heads}$ $n(x) = \text{ways to get } x \text{ Heads}$

	O	$P(O)$	x	$P(x)$	x	$n(x)$
	HHH	$8 / 27$	0	$1 / 27$	0	1
	HHT	$4 / 27$	1	$6 / 27$	1	3
	HTH	$4 / 27$	2	$12 / 27$	2	3
	HTT	$2 / 27$	3	$8 / 27$	3	1
	THH	$4 / 27$				
	THT	$2 / 27$				
	TTH	$2 / 27$				
	TTT	$1 / 27$				

$n=3$
 $x=1$
 $p=2/3$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times. $p=2/3$ $x = \# \text{ of Heads}$ $n(x) = \text{ways to get } x \text{ Heads}$

	O	$P(O)$	x	$P(x)$	x	$n(x)$
	HHH	$8 / 27$	0	$1 / 27$	0	1
	HHT	$4 / 27$	1	$6 / 27$	1	3
	HTH	$4 / 27$	2	$12 / 27$	2	3
	HTT	$2 / 27$	3	$8 / 27$	3	1
	THH	$4 / 27$				
	THT	$2 / 27$				
	TTH	$2 / 27$				
	TTT	$1 / 27$				

$n=3$
 $x=1$
 $p=2/3$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{3!}{1!(3-1)!} \left(\frac{2}{3}\right)^1 \left(1 - \frac{2}{3}\right)^{3-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Example: $n=10, p=0.5$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

These Binomial probabilities can also be found in the back of the book. Table 2 in Appendix B

Please turn to page 713

TABLE 2 page 713

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x} \right]$

prob of success on a trial

number
of success
in n trialsnumber
of trials

n	x	P													x
		0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2
3	0	.970	.857	.729	.512	.343	.216	.125	.064	.027	.008	.001	0+	0+	0
	1	.029	.135	.243	.384	.441	.432	.375	.288	.189	.096	.027	.007	0+	1
	2	0+	.007	.027	.096	.189	.288	.375	.432	.441	.384	.243	.135	.029	2
	3	0+	0+	.001	.008	.027	.064	.125	.216	.343	.512	.729	.857	.970	3
4	0	.961	.815	.656	.410	.240	.130	.062	.026	.008	.002	0+	0+	0+	0
	1	.039	.171	.292	.410	.412	.346	.250	.154	.076	.026	.004	0+	0+	1
	2	.001	.014	.049	.154	.265	.346	.375	.346	.265	.154	.049	.014	.001	2
	3	0+	0+	.004	.026	.076	.154	.250	.346	.412	.410	.292	.171	.039	3
	4	0+	0+	0+	.002	.008	.026	.062	.130	.240	.410	.656	.815	.961	4
...															
8	0	.923	.663	.430	.168	.058	.017	.004	.001	0+	0+	0+	0+	0+	0
	1	.075	.279	.383	.336	.198	.090	.031	.008	.001	0+	0+	0+	0+	1
	2	.003	.051	.149	.294	.296	.209	.109	.041	.010	.001	0+	0+	0+	2
	3	0+	.005	.033	.147	.254	.279	.219	.124	.047	.009	0+	0+	0+	3
	4	0+	0+	.005	.046	.136	.232	.273	.232	.136	.046	.005	0+	0+	4
	5	0+	0+	0+	.009	.047	.124	.219	.279	.254	.147	.033	.005	0+	5
	6	0+	0+	0+	.001	.010	.041	.109	.209	.296	.294	.149	.051	.003	6
	7	0+	0+	0+	0+	.001	.008	.031	.090	.198	.336	.383	.279	.075	7
	8	0+	0+	0+	0+	0+	.001	.004	.017	.058	.168	.430	.663	.923	8

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Figure from Johnson & Kuby, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2 page 713

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x} \right]$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

n	x	P													x
		0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2

$n=2, p=1/2$

x	P(x)
0	1 / 4
1	2 / 4
2	1 / 4



$$P(0) = \frac{2!}{0!(2-0)!} (1/2)^0 (1-1/2)^{2-0}$$

$$P(1) = \frac{2!}{1!(2-1)!} (1/2)^1 (1-1/2)^{2-1}$$

$$P(2) = \frac{2!}{2!(2-2)!} (1/2)^2 (1-1/2)^{2-2}$$

Figure from Johnson & Kuby, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2 page 713

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x} \right]$ (continued)

$n=10, x=7, p=1/2$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Figure from Johnson & Kubly, 2012.

n	x	p													x
		0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	
10	0	.904	.599	.349	.107	.028	.006	.001	0+	0+	0+	0+	0+	0+	0
	1	.091	.315	.387	.268	.121	.040	.010	.002	0+	0+	0+	0+	0+	1
	2	.004	.075	.194	.302	.233	.121	.044	.011	.001	0+	0+	0+	0+	2
	3	0+	.010	.057	.201	.267	.215	.117	.042	.009	.001	0+	0+	0+	3
	4	0+	.001	.011	.088	.200	.251	.205	.111	.037	.006	0+	0+	0+	4
	5	0+	0+	.001	.026	.103	.201	.246	.201	.103	.026	.001	0+	0+	5
	6	0+	0+	0+	.006	.037	.111	.205	.251	.200	.088	.011	.001	0+	6
	7	0+	0+	0+	.001	.009	.042	.117	.215	.267	.201	.057	.010	0+	7
	8	0+	0+	0+	0+	.001	.011	.044	.121	.233	.302	.194	.075	.004	8
	9	0+	0+	0+	0+	0+	.002	.010	.040	.121	.268	.387	.315	.091	9
	10	0+	0+	0+	0+	0+	0+	.001	.006	.028	.107	.349	.599	.904	10
x		0	1	2	3	4	5	6	7	8	9	10			
P(x)		$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$			

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x} \right]$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

n	x	p													x
		0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2

$n=2, p=2/3$

x	P(x)
0	1/9
1	4/9
2	4/9

p=2/3
.111
.444
.444

Not every p is on the table.

Figure from Johnson & Kubly, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 4, 5, or 6 heads?

$$P(4 \leq x \leq 6) = P(4) + P(5) + P(6)$$

$$P(4 \leq x \leq 6) =$$

$$P(4 \leq x \leq 6) =$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 4, 5, or 6 heads?

$$P(4 \leq x \leq 6) = P(4) + P(5) + P(6)$$

$$P(4 \leq x \leq 6) = 210/1024 + 252/1024 + 210/1024$$

$$P(4 \leq x \leq 6) = 672/1024 \approx 0.6123$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 7 or fewer heads?

$$P(x \leq 7) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7)$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 7 or fewer heads?

$$P(x \leq 7) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7)$$

$$P(x \leq 7) = (1 + 10 + 45 + 120 + 210 + 252 + 210 + 120) / 1024$$

$$P(x \leq 7) = 968 / 1024$$

$$P(x \leq 7) \approx 0.9453$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 7 or fewer heads?

$$P(x \leq 7) = 1 - P(x \geq 8)$$

$$P(x \leq 7) =$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: $n=10$, $p=1/2$

What is the probability of getting 7 or fewer heads?

$$P(x \leq 7) = 1 - P(x \geq 8)$$

$$P(x \leq 7) = 1 - (45 + 10 + 1)/1024$$

$$P(x \leq 7) = 968/1024$$

$$P(x \leq 7) \approx 0.9453$$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

An experiment with two outcomes does not have to be H and T .

More general than H and T , call one *Success* and other *Failure*.

Generally call the one we're interested in the *Success*.

Now that we've established that the formula works.

We can determine the theoretical mean number of heads, μ , and the theoretical variance for the number of heads, σ^2 .

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical mean μ of a discrete distribution is:

$$\mu = \sum_{i=1}^n [x_i P(x_i)]$$

$$x_1=0, x_2=1, x_3=2, \dots$$

And upon insertion of the Binomial distribution

$$\mu = \sum_{x=0}^n x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np$$

(5.7)

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical variance σ^2 of a discrete distribution is:

$$\sigma^2 = \sum_x (x - \mu)^2 P(x)$$

$$x_1=0, x_2=1, x_3=2, \dots$$

And upon insertion of the Binomial distribution

$$\sigma^2 = \sum_{x=0}^n (x - \mu)^2 \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np(1-p) \longrightarrow \sigma = \sqrt{np(1-p)} \quad (5.8)$$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example:

Before, using $\mu = \sum_{x=0}^n [xP(x)]$, we found $\mu = 1$.

$$\begin{aligned} n &= 2 \\ x &= 1 \\ p &= 1/2 \end{aligned}$$

Now using $\mu = np$, we get $\mu = (2) \cdot (1/2) = 1$.

Before, using $\sigma^2 = \sum_{x=0}^n [(x - \mu)^2 P(x)]$, we found $\sigma^2 = 1/2$.

Now using $\sigma^2 = np(1 - p)$,

we get $\sigma^2 = (2) \cdot (1/2) \cdot (1/2) = 1/2$.

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example: $n=10$, $p=1/2$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

What are μ and σ^2 ?

$$\mu = np =$$

$$\sigma^2 = np(1-p) =$$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example: $n=10, p=1/2$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

What are μ and σ^2 ?

$$\mu = np = 5$$

$$\sigma^2 = np(1-p) = 2.5$$

5: Probability Distributions (Discrete Variables)

Questions?

Homework: Read Chapter 5.1-5.3

WebAssign

Chapter 5 # 15, 17, 19, 29, 31, 43,
55a,b, 63, 77, 85, 89