Class 26

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Agenda:

Recap Chapter 12.1

Statistical Inference

Recap Chapter 12.1

12: Analysis of Variance

12.1 Introduction to the Analysis of Variance

Previously we learned about hypothesis testing for:
One Population: μ, p, and σ^2.
Two Populations: $\mu_d=\mu_1-\mu_2$, $\mu_1-\mu_2$, p_1-p_2, and σ_1^2/σ_2^2.

(We also learned about hypothesis testing for contingency tables.)

Now we are going to study hypothesis testing for three or more populations.

Three Populations: at least two of $\mu_1,\mu_2,\mu_3,\ldots$ different.

12: Analysis of Variance

12.1 Introduction to the Analysis of Variance

If we are testing for differences in means,
…why are we analyzing variance?

As it turns out, we calculate two variances and take the ratio.

If all the means are truly the same, the two variances will be the same and the ratio will be 1.

12: Analysis of Variance

12.1 Introduction to the Analysis of Variance

Example: Hypothesis Test for Three Means, μ_1, μ_2, and μ_3.
It is believed that manufacturing plant temperature affects production rate.

	Temperature Levels		
	Sample from 68°F ($i = 1$)	Sample from 72°F ($i = 2$)	Sample from 76°F ($i = 3$)
	10	7	3
	12	6	3
	10	7	5
	9	8	4
		7	
Column totals	$C_1 = 41$ $\bar{x}_1 = 10.25$	$C_2 = 35$ $\bar{x}_2 = 7.0$	$C_3 = 15$ $\bar{x}_3 = 3.75$
	$k_1 = 4$	$k_2 = 5$	$k_3 = 4$
	μ_1	μ_2	μ_3

12: Analysis of Variance

12.1 Introduction to the Analysis of Variance

Hypothesis Testing Procedure

Step 1 $H_0: \mu_1 = \mu_2 = \mu_3$ vs.

H_a: *at least two μ's different*

Step 2

Temperature Levels		
Sample from 68°F ($i = 1$)	Sample from 72°F ($i = 2$)	Sample from 76°F ($i = 3$)
10	7	3
12	6	3
10	7	5
9	8	4
	7	
$C_1 = 41$	$C_2 = 35$	$C_3 = 15$
$\bar{x}_1 = 10.25$	$\bar{x}_2 = 7.0$	$\bar{x}_3 = 3.75$

Sum of Squares Due to Factor

$$SS(\text{factor}) = \left(\frac{C_1^2}{k_1} + \frac{C_2^2}{k_2} + \frac{C_3^2}{k_3} + \cdots\right) - \frac{(\sum x)^2}{n}$$

$$\text{df(factor)} = c - 1$$

$$\text{MS(factor)} = \frac{SS(\text{factor})}{\text{df(factor)}}$$

Sum of Squares Due to Error

$$SS(\text{error}) = \sum(x^2) - \left(\frac{C_1^2}{k_1} + \frac{C_2^2}{k_2} + \frac{C_3^2}{k_3} + \cdots\right)$$

$$\text{df(error)} = n - c$$

$$\text{MS(error)} = \frac{SS(\text{error})}{\text{df(error)}}$$

$$F\star = \frac{\text{MS(factor)}}{\text{MS(error)}}$$

$$\alpha = .05$$

Shortcut for Total Sum of Squares

$$SS(\text{total}) = \sum(x^2) - \frac{(\sum x)^2}{n}$$

$$\text{df(total)} = n - 1$$

12: Analysis of Variance

12.1 Introduction to the Analysis of Variance

Hypothesis Testing Procedure

$$SS(temperature) = \left(\frac{41^2}{4} + \frac{35^2}{5} + \frac{15^2}{4}\right) - \frac{(91)^2}{13} = 84.5$$

$$SS(error) = 731.0 - \left(\frac{41^2}{4} + \frac{35^2}{5} + \frac{15^2}{4}\right) = 9.5$$

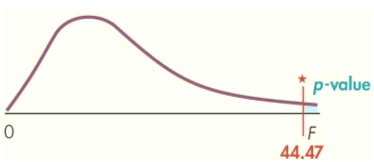
Source	df	SS	MS
Temperature	2	84.5	42.25
Error	10	9.5	0.95
Total	12	94.0	$F^* = 44.47$

Step 3

$$F^* = \frac{42.25}{0.95} = 44.47$$

Step 4

p-value approach



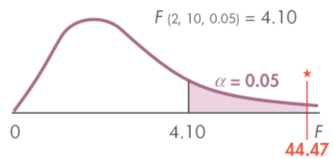
$\alpha = 0.01$

Degrees of Freedom for Numerator (columns); Degrees of Freedom for Denominator (rows)

	1	2	3	4	5	6	7	8	9	10
1	4052.	5000.	5403.	5625.	5764.	5859.	5928.	5981.	6022.	6056.
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30

$$.00 < p-value < .01$$

Classical approach



$\alpha = 0.05$

Degrees of Freedom for Numerator (columns); Degrees of Freedom for Denominator (rows)

	1	2	3	4	5	6	7	8	9	10
1	161.	200.	216.	225.	230.	234.	237.	239.	241.	242.
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75

$$F(2,10,0.05)=4.10$$

Step 5

Decision: Reject H_0

$$p-value < \alpha$$

$$.00 < p-value < .01$$

$$\alpha = 0.05$$

$$F^* > F_{crit}$$

$$F^* = 44.47$$

$$F_{crit} = 4.10$$

Statistical Inference

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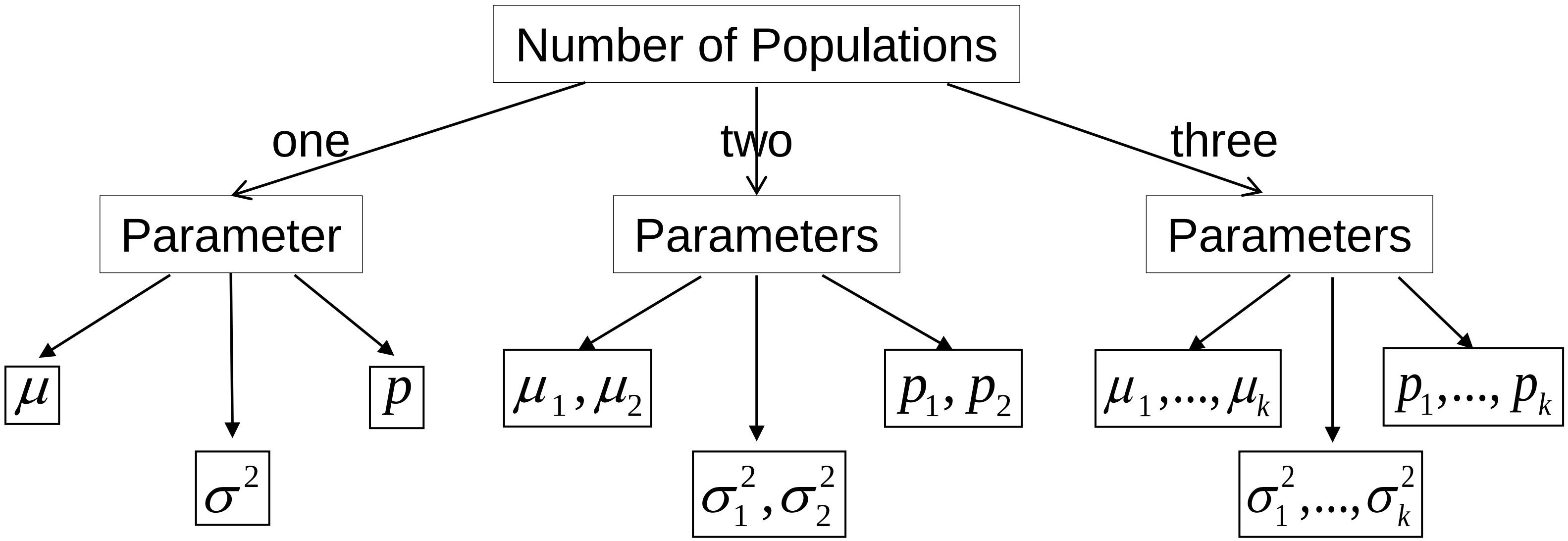
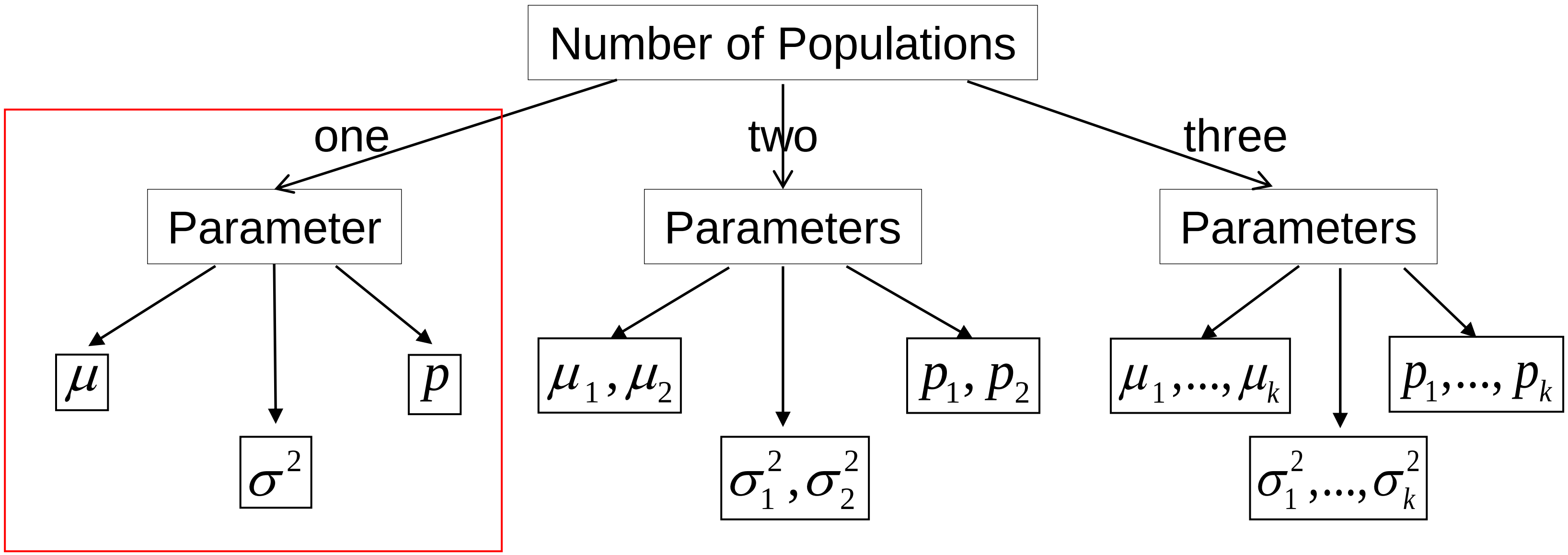
Department of Mathematical and Statistical

Statistical Inference

Statistical Inference:



Statistical Inference: 1 Population



Statistical Inference

H_0: $\mu \geq \mu_0$ vs. H_a: $\mu < \mu_0$
H_0: $\mu \leq \mu_0$ vs. H_a: $\mu > \mu_0$
H_0: $\mu = \mu_0$ vs. H_a: $\mu \neq \mu_0$

$$t^* = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

H_0: $\sigma^2 \geq \sigma_0^2$ vs. H_a: $\sigma^2 < \sigma_0^2$
H_0: $\sigma^2 \leq \sigma_0^2$ vs. H_a: $\sigma^2 > \sigma_0^2$
H_0: $\sigma^2 = \sigma_0^2$ vs. H_a: $\sigma^2 \neq \sigma_0^2$

$$\chi^{2*} = \frac{(n-1)s^2}{\sigma_0^2}$$

H_0: $p \geq p_0$ vs. H_a: $p < p_0$
H_0: $p \leq p_0$ vs. H_a: $p > p_0$
H_0: $p = p_0$ vs. H_a: $p \neq p_0$

$$z^* = \frac{p' - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Statistical Inference: Procedures for μ

Assume or Test for Independent Observations

Normally Distributed Data
(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

- yes → σ^2 known?
  - yes → $z^* = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ — standard normal distribution
  - no → $t^* = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$ — Student-t distribution $df = n-1$
- no → n large?
  - yes → $z^* = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$ — standard normal distribution
  - no → sign-test $min(n(-), n(+))$ (nonparametric median) — custom distribution and table

Statistical Inference: Procedures for σ^2

Assume or Test for Independent Observations

Normally Distributed Data

(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

yes

$$\chi^2 * = \frac{(n-1)s^2}{\sigma_0^2}$$

Chi-Square distribution
$df=n$-1

no

Bootstrap Resampling

Select many samples of size n from $x_1,\ldots,x_n$ with replacement.
Calculate s^2 from each.
Calculate sample variance of s^2's.
Look at percentiles.

Statistical Inference: Procedures for p

n-Large

yes

$$z^* = \frac{p' - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Normal distribution

no

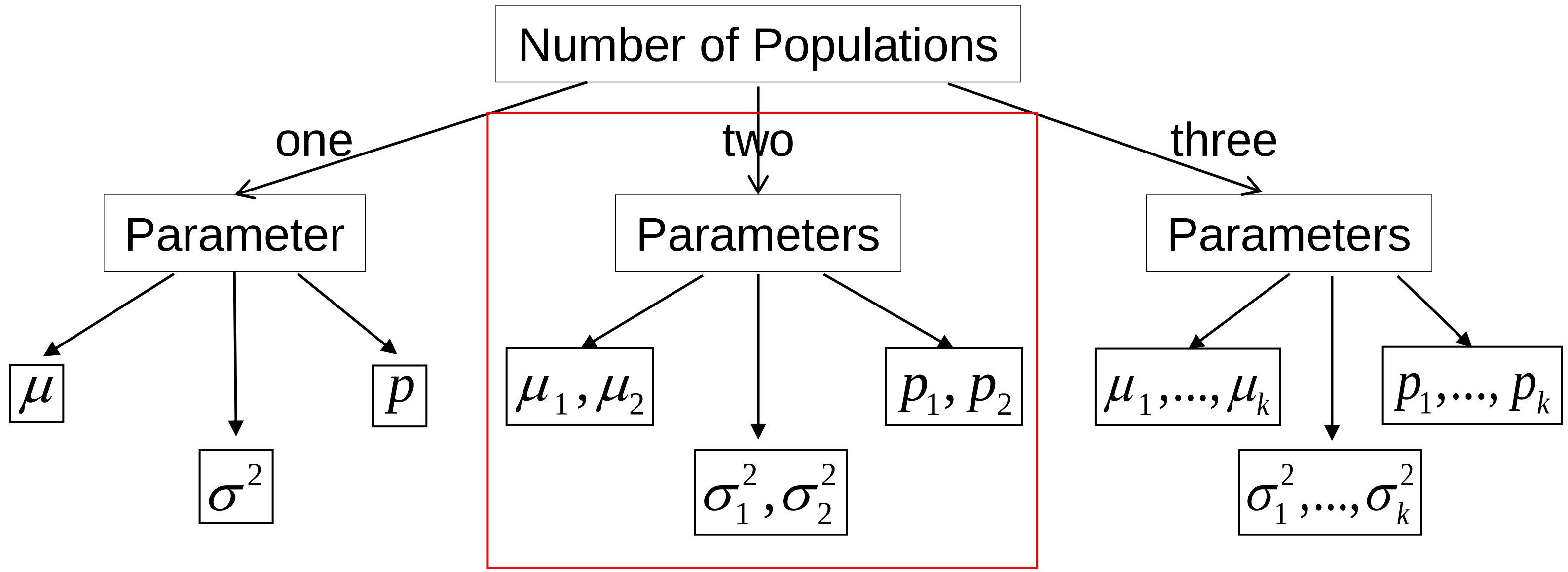
$$p' = \frac{x}{n}$$

Binomial Distribution

$$P(x) = \frac{n!}{x!(n-x)!} p_0^x (1-p_0)^{n-x}$$

$$p\text{-value} = \sum_{x>i} \frac{n!}{i!(n-i)!} p_0^i (1-p_0)^{n-i}$$

Statistical Inference: 2 Populations



Statistical Inference

H_0: $\mu_1 - \mu_2 \geq 0$ vs. H_a: $\mu_1 - \mu_2 < 0$
H_0: $\mu_1 - \mu_2 \leq 0$ vs. H_a: $\mu_1 - \mu_2 > 0$
H_0: $\mu_1 - \mu_2 = 0$ vs. H_a: $\mu_1 - \mu_2 \neq 0$

$$t^* = \frac{\bar{d} - \mu_{d0}}{s_d / \sqrt{n}}$$

or

$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_{0,1} - \mu_{0,2})}{\sqrt{\left(s_1^2/n_1\right) + \left(s_2^2/n_2\right)}}$$

H_0: $\sigma_1^2 \geq \sigma_2^2$ vs. H_a: $\sigma_1^2 < \sigma_2^2$
H_0: $\sigma_1^2 \leq \sigma_2^2$ vs. H_a: $\sigma_1^2 > \sigma_2^2$
H_0: $\sigma_1^2 = \sigma_2^2$ vs. H_a: $\sigma_1^2 \neq \sigma_2^2$

$$F^* = \frac{s_n^2}{s_d^2}$$

H_0: $p_1 \geq p_2$ vs. H_a: $p_1 < p_2$
H_0: $p_1 \leq p_2$ vs. H_a: $p_1 > p_2$
H_0: $p_1 = p_2$ vs. H_a: $p_1 \neq p_2$

$$z^* = \frac{(p_1' - p_2') - (p_{0,1} - p_{0,2})}{\sqrt{p_p' q_p' \left[1/n_1 + 1/n_2\right]}}$$

Statistical Inference: Procedures for μ_1-μ_2

Assume or Test for Independent Observations

Normally Distributed Data
(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

yes →

Independent Populations

- σ_1^2 & σ_2^2 known

  $$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

  normal

- σ_1^2 & σ_2^2 unknown → F test

  - $\sigma_1^2 = \sigma_2^2$
    - n large

      $$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

      normal
    - n small

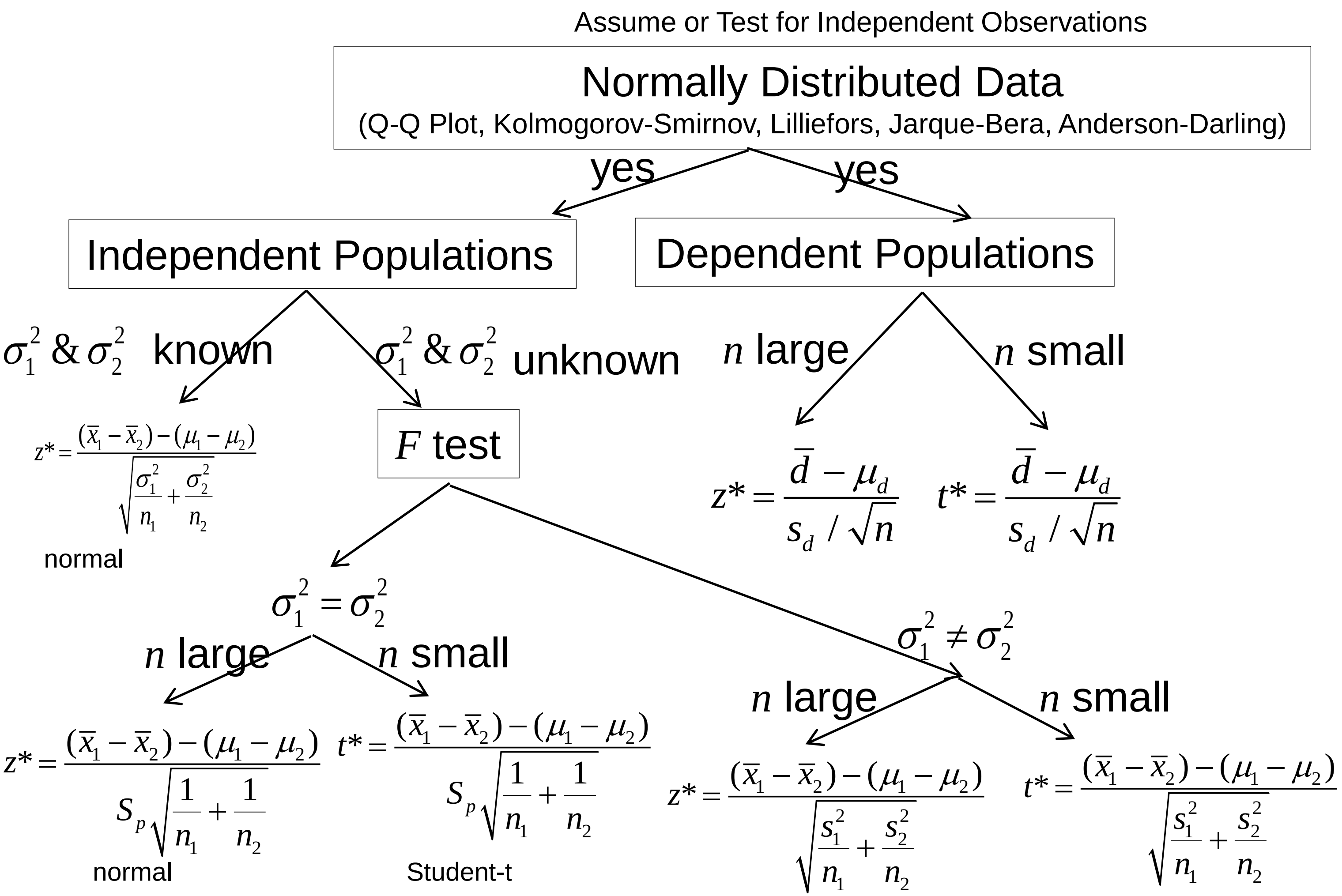
      $$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

      Student-t
  - $\sigma_1^2 \neq \sigma_2^2$
    - n large

      $$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$
    - n small

      $$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Statistical Inference: Procedures for μ_1-μ_2



Statistical Inference: Procedures for μ_1-μ_2

Assume or Test for Independent Observations

Normally Distributed Data
(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

yes → Independent Populations

σ_1^2 & σ_2^2 known

$$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

normal

σ_1^2 & σ_2^2 unknown → *F* test

$\sigma_1^2 = \sigma_2^2$

- *n* large:

$$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

normal

- *n* small:

$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Student-t

$\sigma_1^2 \neq \sigma_2^2$

- *n* large:

$$z^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- *n* small:

$$t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

yes → Dependent Populations

- *n* large:

$$z^* = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

- *n* small:

$$t^* = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

no → Independent Populations

Mann–Whitney *U* test

$$U^* = \sum_{i=1}^{n}\sum_{j=1}^{m} S(x_i, y_j)$$

$$S(x_i, y_j) = \begin{cases} 1, & x_i > y_j \\ 1/2, & x_i = y_j \\ 0, & x_i < y_j \end{cases}$$

custom distribution and table or large *n* normal

no → Dependent Populations

sign-test

$$min(n(-), n(+))$$

(nonparametric median)

Statistical Inference: Procedures for σ_1^2, σ_2^2

Assume or Test for Independent Observations

Normally Distributed Data
(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

yes

$$F^* = \frac{s_1^2}{s_2^2}$$

F-distribution
$df_1 = n_1 - 1$
$df_2 = n_2 - 1$

no

Levene test

$$W^* = \frac{(n-k)\sum_{i=1}^{k} n_i(\bar{A}_{i.} - \bar{A}_{..})^2}{(k-1)\sum_{i=1}^{k}\sum_{j=}^{n_i}(A_{ij} - \bar{A}_{i.})^2}$$

F-distribution
$df_1 = n_1 - 1$
$df_2 = n_2 - 1$

$$A_{ij} = | y_{ij} - \bar{y}_{i.} |$$

Pop i Obs j
"." = sum over

Bartlet test

$$T^* = \frac{(n-k)\ln s_p^2 - \sum_{i=1}^{k}(n_i - 1)\ln s_i^2}{1 + (1/(3(k-1)))((\sum_{i=1}^{k} 1/(n_i - 1)) - 1/(n-k)}$$

Chi-Square distribution
$df = k - 1$

Statistical Inference: Procedures for p_1-p_2

n-Large

yes | yes | no

$$z^* = \frac{(p_1' - p_2') - (p_{0,1} - p_{0,2})}{\sqrt{p_p'(1-p_p')\left[\frac{1}{n_1}+\frac{1}{n_2}\right]}} \quad \overset{=}{\longleftrightarrow} \quad \chi^{2*} = \sum_{j=1}^{k} \frac{(O_j - E_j)^2}{E_j}$$

Normal distribution

$$p_1' = \frac{x_1}{n_1} \qquad p_2' = \frac{x_2}{n_2}$$

$$p_p' = \frac{x_1 + x_2}{n_1 + n_2}$$

Chi-Square distribution

$df=k$-1

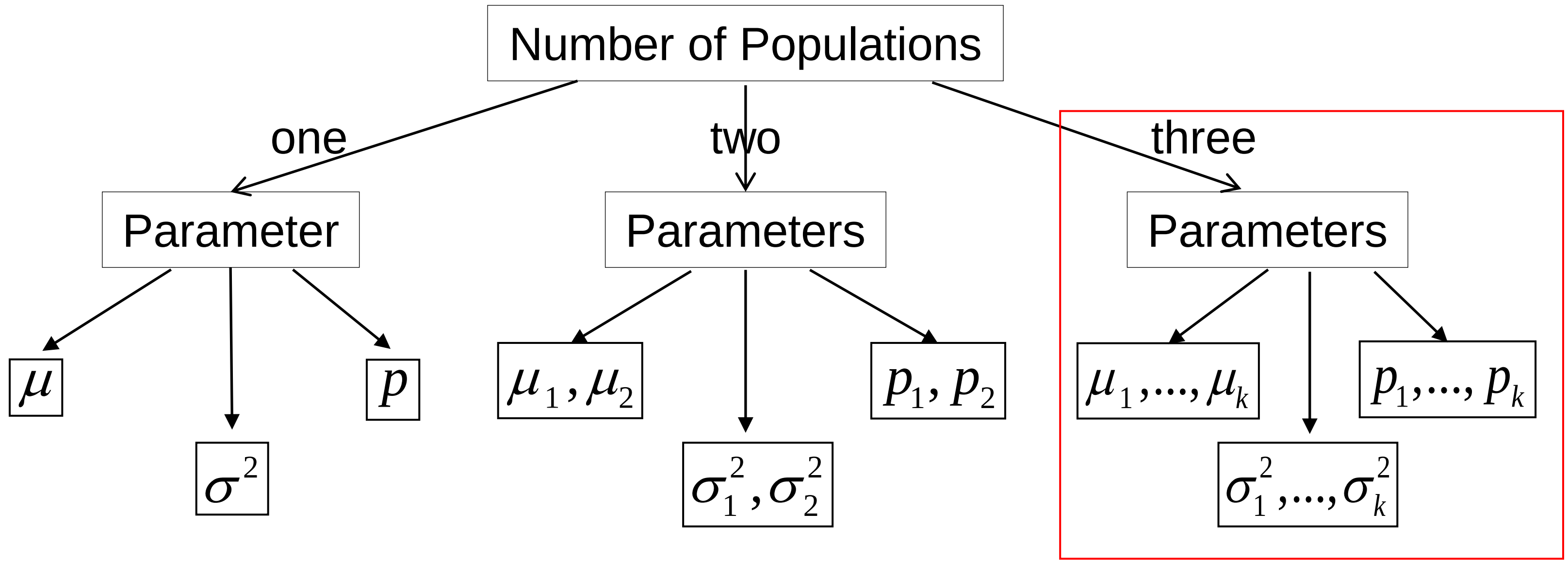
Fisher's exact test

$$p_1' - p_2'$$

$$P(x_1, x_2) = \frac{\binom{x_1+x_2}{x_1}\binom{n_1-x_1+n_2-x_2}{n_1-x_1}}{\binom{n_1+n_2}{n_1}}$$

Sum probabilities of more extreme values to get p-value

Statistical Inference: k Populations



Statistical Inference

$H_0: \mu_1 = \ldots = \mu_k$
vs.
H_a: *at least two μ's are different*

$$F^* = \frac{MS(factor)}{MS(error)}$$

$H_0: \sigma_1^2 = \ldots = \sigma_k^2$
vs.
H_a: *at least two σ^2's are different*

We Didn't Learn

$H_0: p_{11} = \ldots = p_{rc}$
vs.
H_a: *at least two p's are different*

$$\chi^{2*} = \sum_{all\ cells} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

Statistical Inference: Procedures for $\mu_1,\ldots,\mu_k$

Assume or Test for Independent Observations

Normally Distributed Data
(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

- yes → **Equal variances (Levene or Bartlet Test)**
  - yes → **ANOVA Test**
  - no → **Welch's Test**
- no → **Kruskal-Wallis Test**

ANOVA Test

$$F^* = \frac{s_1^2}{s_2^2}$$

F-distribution
$df_1=n_1$-1
$df_2=n_2$-1

Welch's Test

$$F^* = \frac{s_1^2}{s_2^2}$$

F-distribution
$df_1=n_1$-1
$df_2=1/\Lambda$

$$\Lambda = \frac{3\sum_{j=1}^{k} \frac{\left(1-\frac{w_j}{\sum_{j=1}^{k} w_j}\right)^2}{n_j - 1}}{k^2 - 1}$$

Kruskal-Wallis Test

$$H^* = (n-1)\frac{\sum_{i=1}^{k}(\bar{r}_{i.} - \bar{r})^2}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(r_{ij} - \bar{r})^2}$$

r_{ij}=rank of obs j in pop i

$$\bar{r}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} r_{ij}$$

$$\bar{r} = \frac{1}{2}(n+1)$$

Special distribution and table.

Statistical Inference: Procedures for $\sigma_1^2,\ldots,\sigma_k^2$

Assume or Test for Independent Observations

Normally Distributed Data

(Q-Q Plot, Kolmogorov-Smirnov, Lilliefors, Jarque-Bera, Anderson-Darling)

yes

$$F^* = \frac{s_1^2}{s_2^2}$$

F-distribution

$df_1=n_1$-1

$df_2=n_2$-1

no

Levene test

$$W^* = \frac{(n-k)\sum_{i=1}^{k} n_i(\bar{A}_{i.} - \bar{A}_{..})^2}{(k-1)\sum_{i=1}^{k}\sum_{j=}^{n_i}(A_{ij} - \bar{A}_{i.})^2}$$

F-distribution

$df_1=n_1$-1

$df_2=n_2$-1

$$A_{ij} = | y_{ij} - \bar{y}_{i.} |$$

Pop i Obs j

"."=sum over

Bartlet test

$$T^* = \frac{(n-k)\ln s_p^2 - \sum_{i=1}^{k}(n_i-1)\ln s_i^2}{1+(1/(3(k-1)))((\sum_{i=1}^{k} 1/(n_i-1)) - 1/(n-k)}$$

Chi-Square distribution

$df=k$-1

Statistical Inference: Procedures for $p_1,\ldots,p_k$

n-Large

yes

$$\chi^2{*} = \sum_{j=1}^{k} \frac{(O_j - E_j)^2}{E_j}$$

Chi-Square distribution
$df=k-1$

no

Freeman-Halton exact test
or
p-value via simulation

Statistical Inference

Hypothesis tests also exist for:

Regression Coefficients,	$H_0\!:\beta_1=0$ vs. $H_a\!:\beta_1\neq 0$
Correlation Coefficient,	$H_0\!:\rho=0$ vs. $H_a\!: \rho\neq 0$
Temporal Autocorrelation,	$H_0\!:\rho=0$ vs. $H_a\!: \rho\neq 0$
Correlation Matrices,	$H_0\!:R$ is diagonal vs. $H_a\!: R$ not Diagonal

Two-way ANOVA

…

Statistical Inference

Questions?