Class 14

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Agenda:

Recap Chapter 8.5

Lecture Chapter 9.1

Recap Chapter 8.5

8: Introduction to Statistical Inference

8.4 Hypothesis Test of Mean (σ Known): Probability Approach

Step 1 The Set-Up: Null (H_0) and alternative (H_a) hypotheses

H_0: μ = 69” vs. H_a: $\mu \neq$ 69”

Step 2 The Hypothesis Test Criteria: Test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

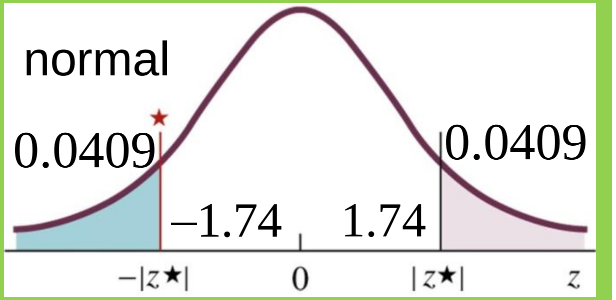
σ known, n is “large” so by CLT $\bar{x}$ is normal

z^* is normal

Step 3 The Sample Evidence: Calculate test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{67.2 - 69}{4 / \sqrt{15}} = -1.74$$

n=15, $\bar{x}$=67.2, $\sigma = 4$



Step 4 The Probability Distribution:

$P(z >| z^* |) = p - \text{value} \longrightarrow$ 0.0819

Step 5 The Results:

$p - \text{value} \leq \alpha$, **reject H_0**, $p - \text{value} > \alpha$ **fail to reject H_0**

α = 0.05

8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Step 1 The Set-Up: Null (H_0) and alternative (H_a) hypotheses

$$\boxed{H_0: \mu = 69\text{”} \quad \text{vs.} \quad H_a: \mu \neq 69\text{”}}$$

Step 2 The Hypothesis Test Criteria: Test statistic.

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

σ known, n is “large” so by CLT $\bar{x}$ is normal

z^* is normal

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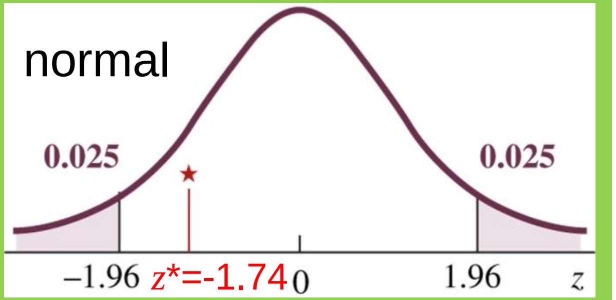
n=15, $\bar{x}$ =67.2, $\sigma = 4$



Step 4 The Probability Distribution:

α = 0.05, $z(\alpha/2)$=1.96

Step 5 The Results:

$|z^*| > z(\alpha/2)$, **reject H_0,** $\boxed{|z^*| \leq z(\alpha/2) \text{ fail to reject } H_0}$

8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

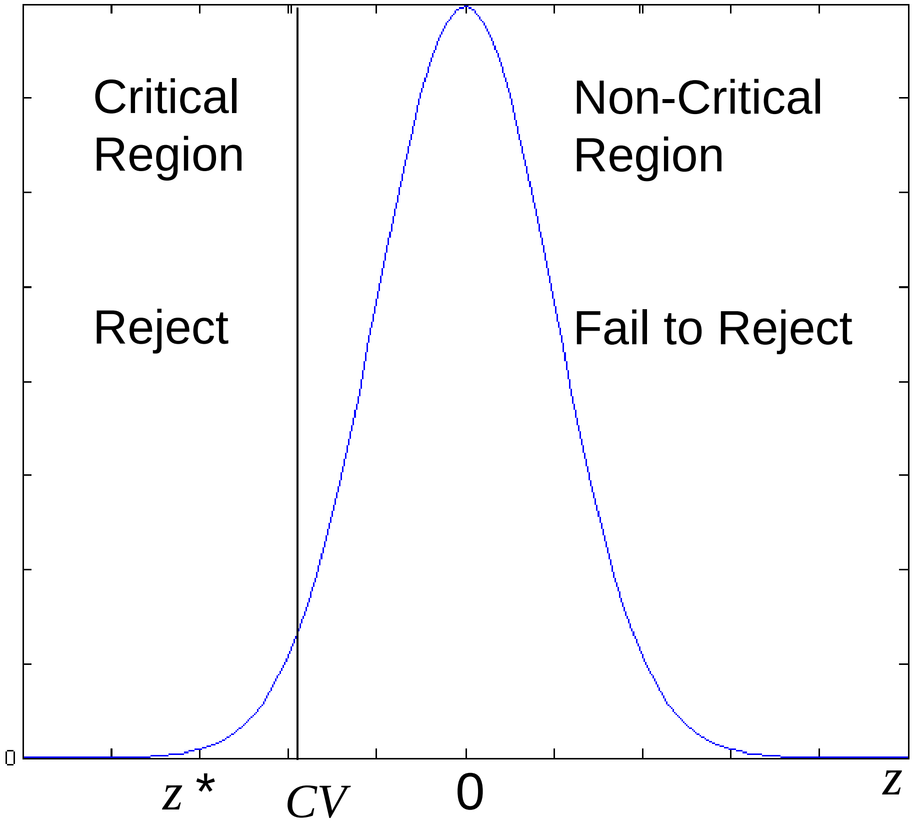
There are three possible hypothesis pairs for the mean.

H_0: $\mu \geq \mu_0$ vs. H_a: $\mu < \mu_0$

Reject H_0 if less than

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \qquad -z(\alpha)$$

data indicates $\mu < \mu_0$
because $\bar{x}$ is “a lot”
smaller than μ_0



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

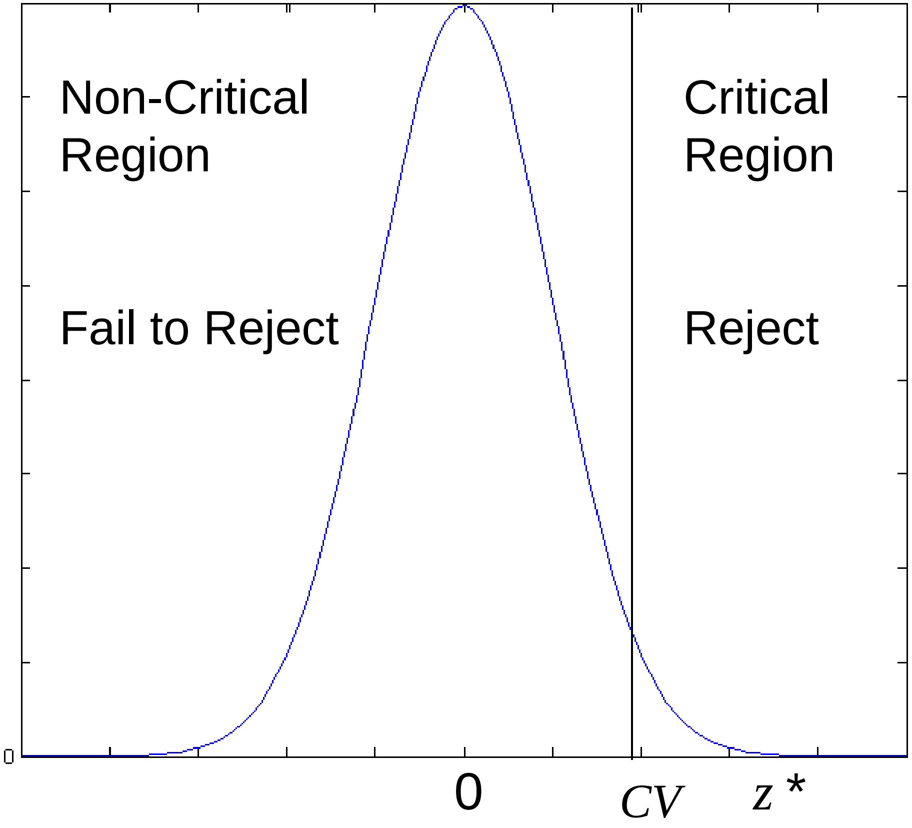
There are three possible hypothesis pairs for the mean.

H_0: $\mu \le \mu_0$ vs. H_a: $\mu > \mu_0$

Reject H_0 if greater then

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \qquad z(\alpha)$$

data indicates $\mu > \mu_0$ because $\bar{x}$ is “a lot” larger than μ_0



8: Introduction to Statistical Inference

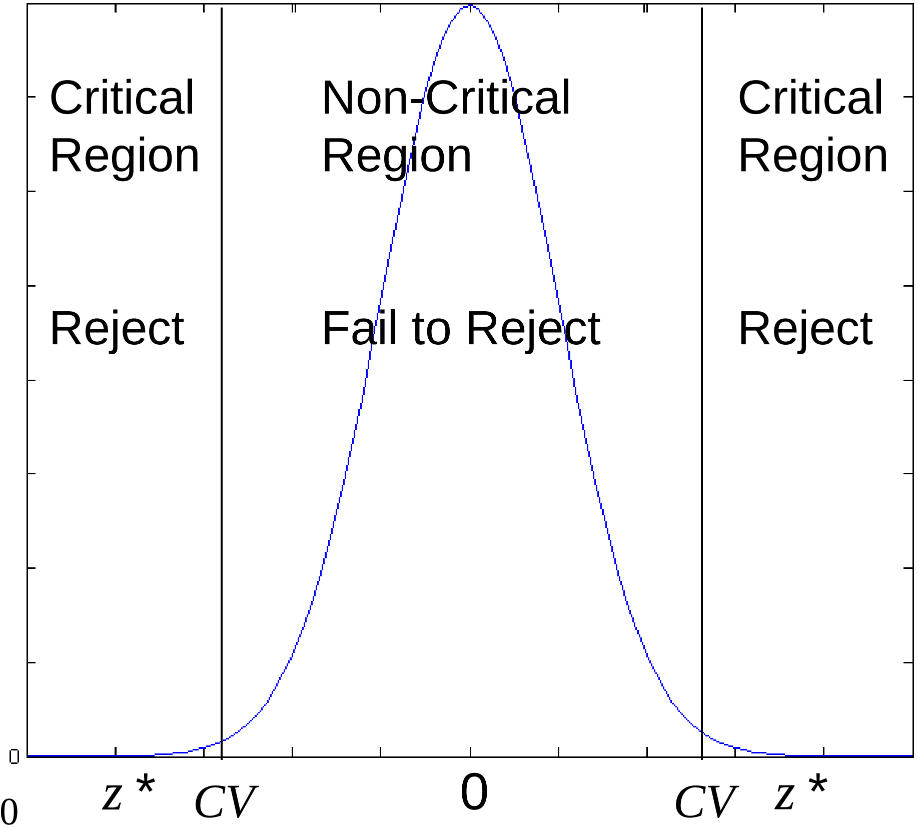
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

There are three possible hypothesis pairs for the mean.

H_0: $\mu = \mu_0$ vs. H_a: $\mu \neq \mu_0$

Reject H_0 if $z^* = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ less than $-z(\alpha / 2)$

or if $z^* = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ is greater than $z(\alpha / 2)$



data indicates $\mu \neq \mu_0$, $\bar{x}$ far from μ_0

8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Let's examine the hypothesis test

H_0: $\mu \leq 69”$ vs. H_a: $\mu > 69”$

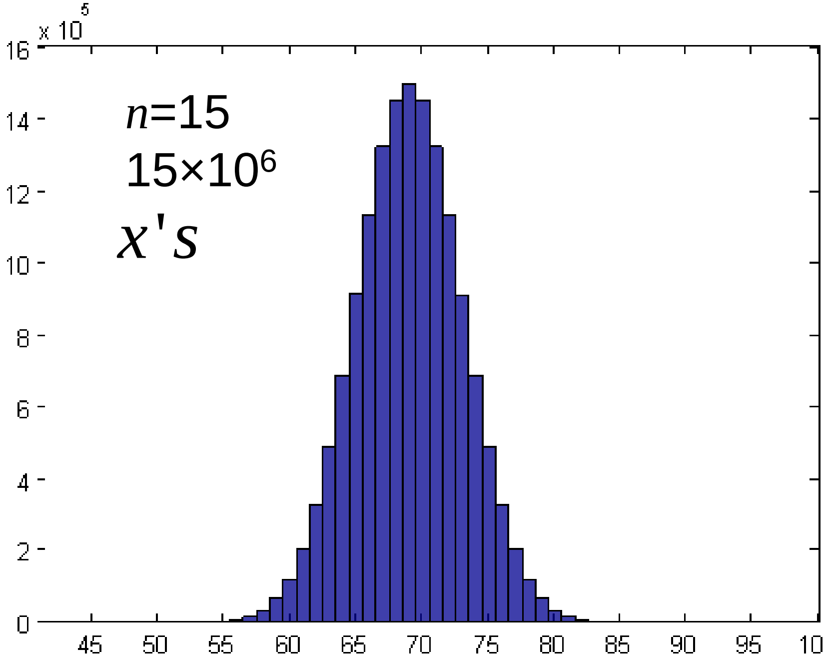
with α=.05 for the heights of Math 1700 students.

Generate random data values.

8: Introduction to Statistical Inference

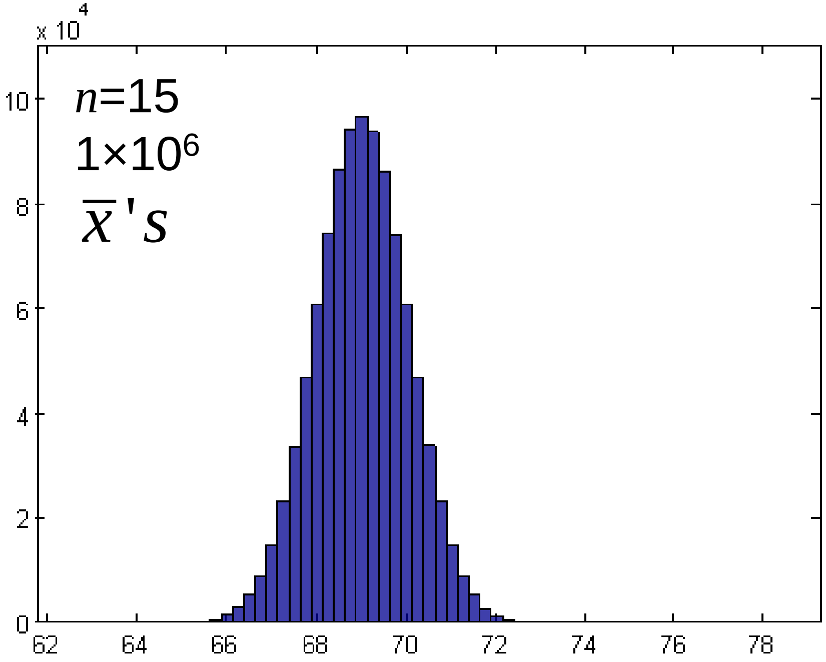
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

Generated 15×10^6 normal data values from μ = 69” and σ=4”.



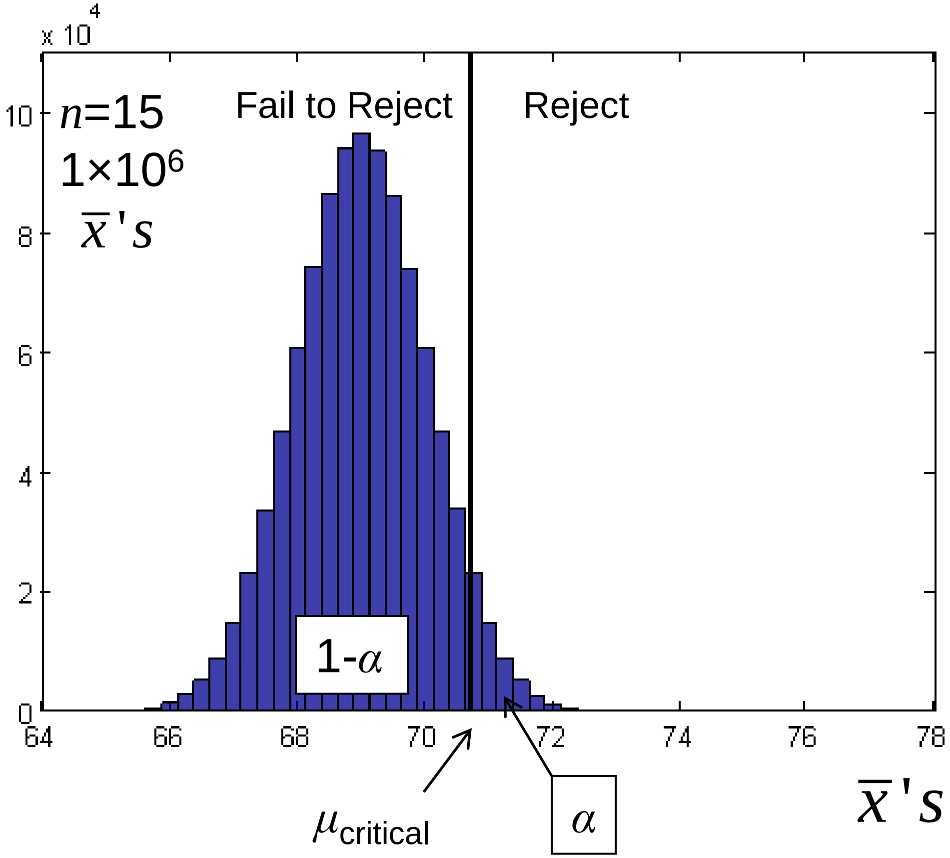
Calculated 1×10^6 means with n=15.

(Will repeat for μ = 72”.)



8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



H_0: $\mu \le 69$" vs. H_a: $\mu > 69$"
α=.05

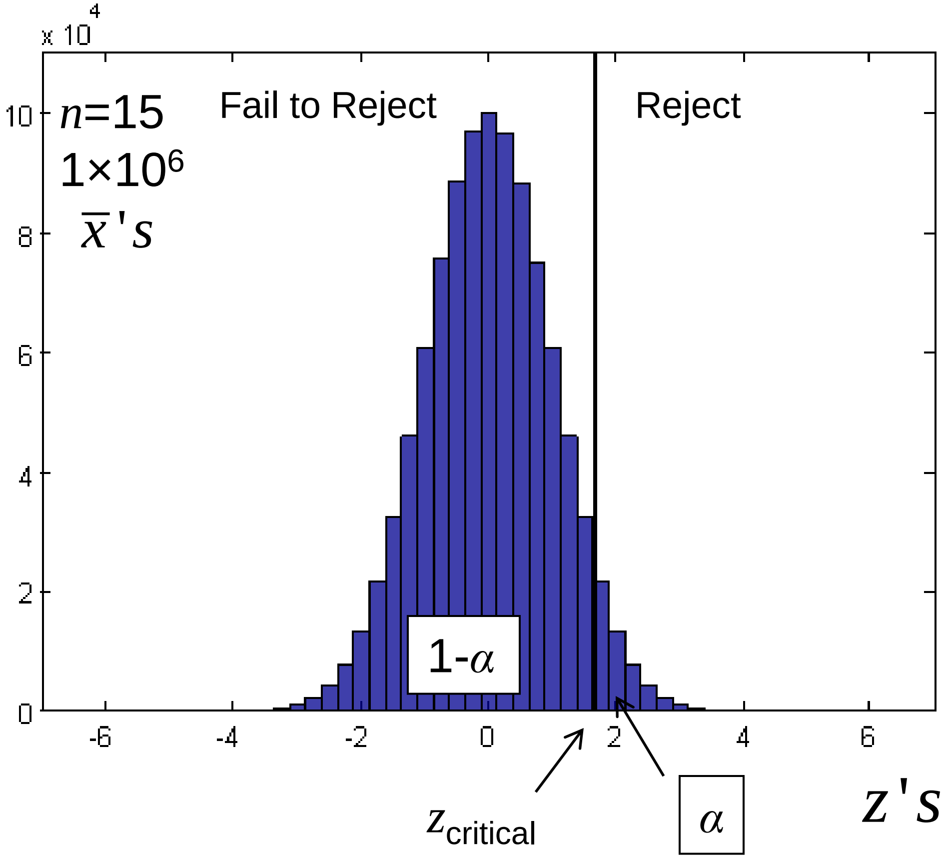
When the true mean μ=69", we reject H_0 α fraction of the time.

Commit a Type I Error.

Given α, we want μ_{critical}.

8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



Instead of μ_{critical} we find critical z, $z_{critical}$=$z(\alpha)$.

Do this by assuming that H_0: μ=69” is true, then calculate

$$z = \frac{\bar{x} - 69}{4 / \sqrt{15}}$$

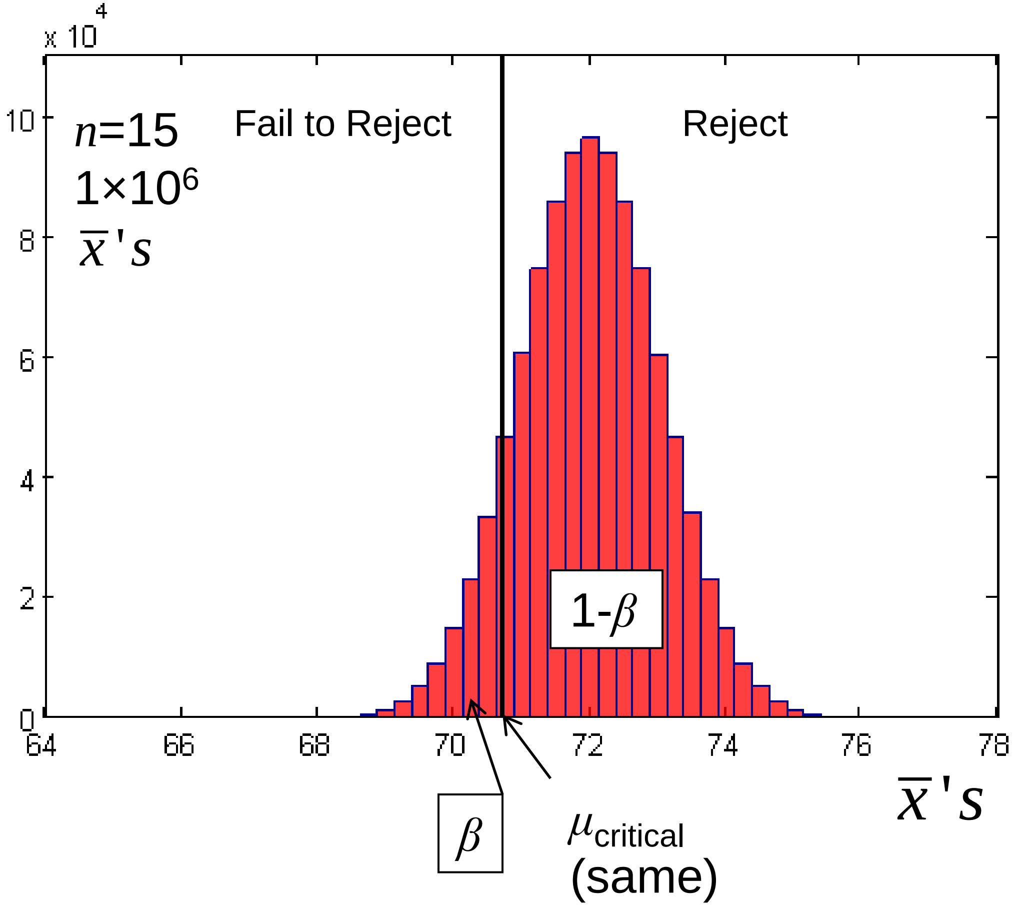
8: Introduction to Statistical Inference

8.5 Hypothesis Test of Mean (σ Known): Classical Approach

H_0: $\mu \le 69$ vs. H_a: $\mu > 69$
$\alpha=.05$

When the true mean μ=72”, we do not reject H_0 β fraction of the time.

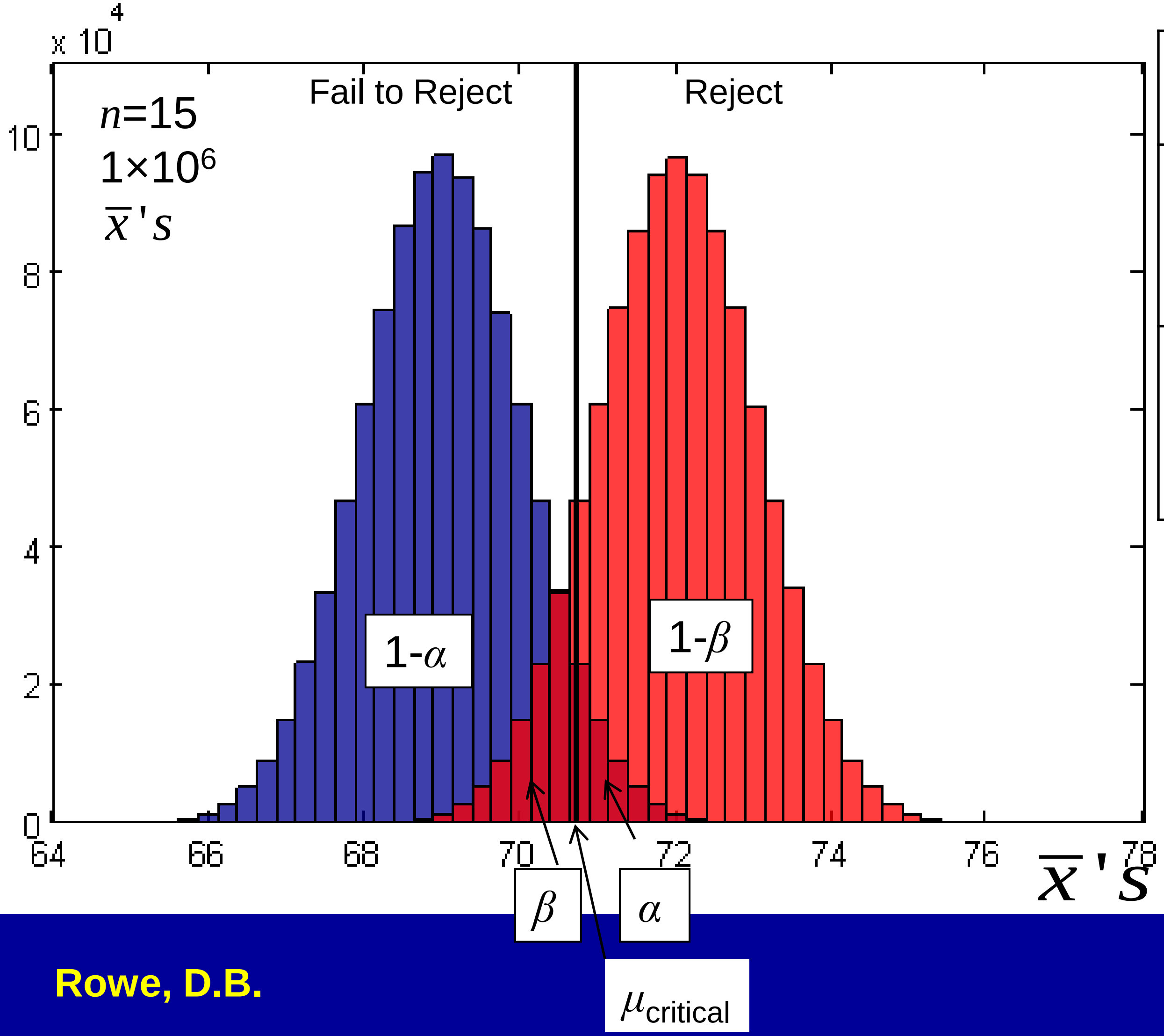
Commit a Type II Error



8: Introduction to Statistical Inference

H_0: $\mu \le \mu_0$ vs. H_a: $\mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



	H_0 True (μ=69”)	H_0 False (μ=72”)
Fail to Reject H_0	Correct Decision (1-α)	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision (1-β)

Power of the test:

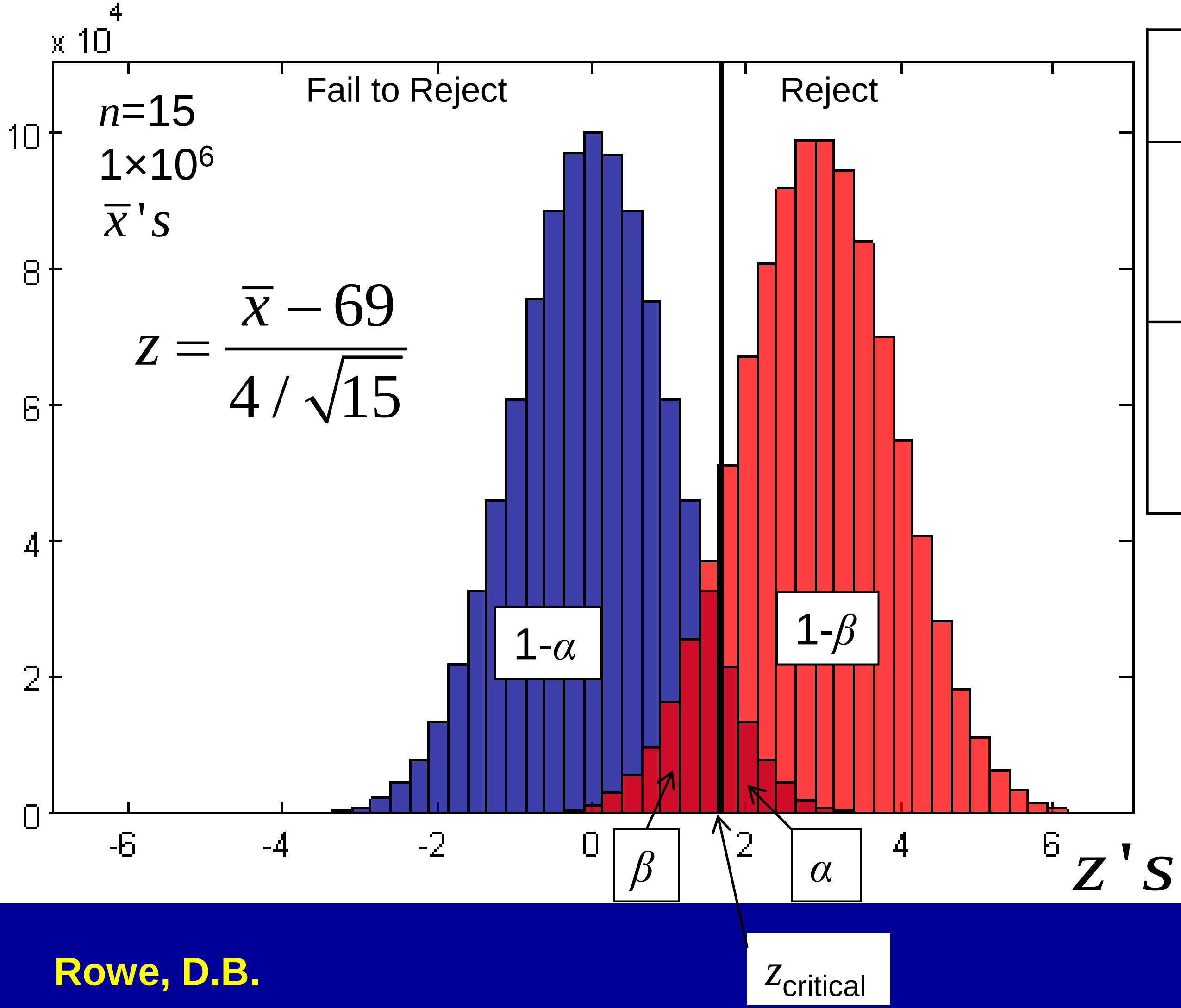
$$1-\beta = P(\text{Reject } H_0 \mid H_0 \text{ False})$$

Discrimination ability.
Ability to detect difference.

8: Introduction to Statistical Inference

H_0: $\mu \le \mu_0$ vs. H_a: $\mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



	H_0 True (μ=69”)	H_0 False (μ=72”)
Fail to Reject H_0	Correct Decision (1-α)	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision (1-β)

Power of the test:

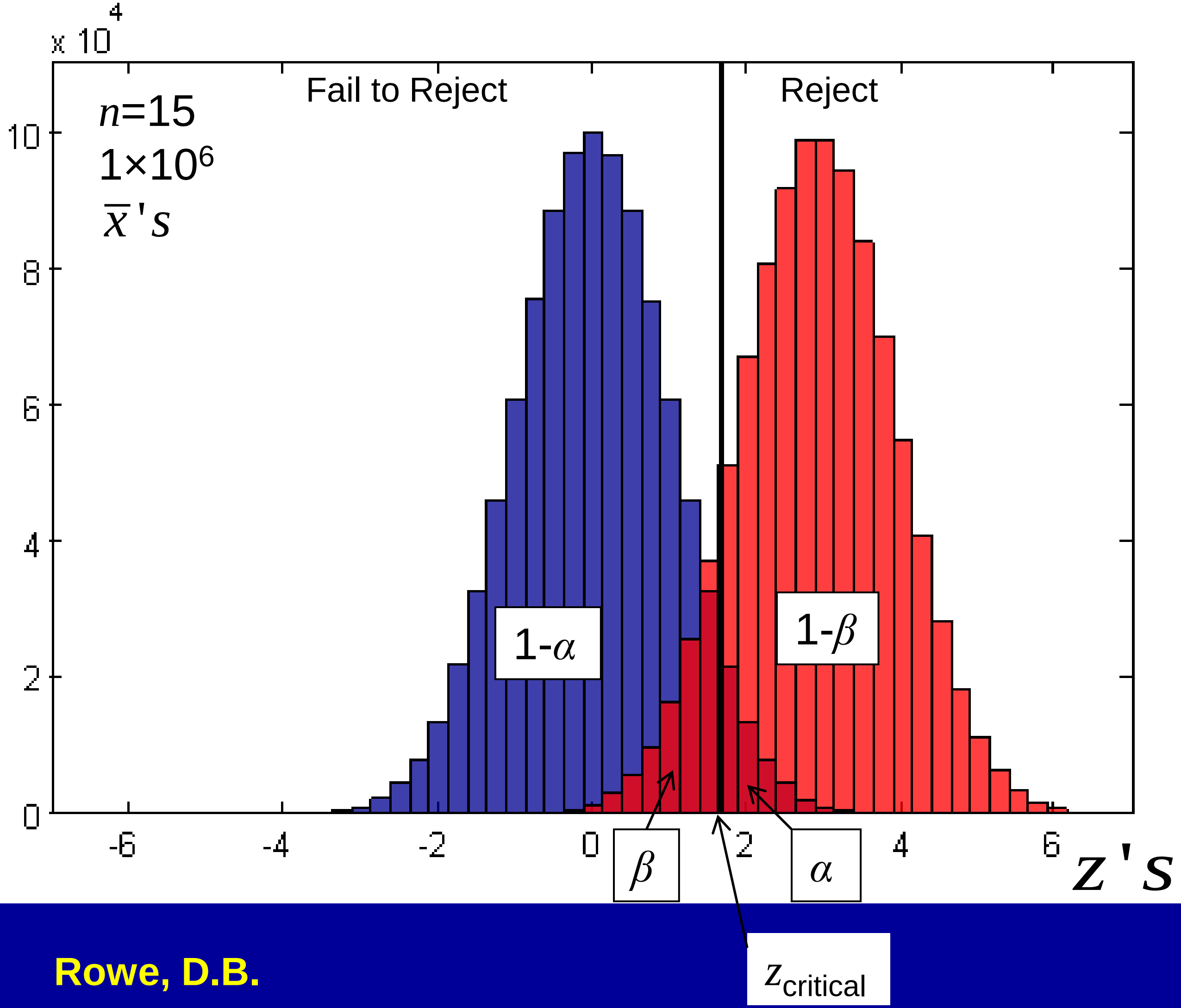
$$1 - \beta = P(\text{Reject } H_0 \mid H_0 \text{ False})$$

Discrimination ability.
Ability to detect difference.

8: Introduction to Statistical Inference

$H_0: \mu \le \mu_0$ vs. $H_a: \mu > \mu_0$

8.5 Hypothesis Test of Mean (σ Known): Classical Approach



We want our α, Prob of Type I Error to be small.

So why not just decrease α?

Decreasing α increases β.

And vice versa.

8: Introduction to Statistical Inference

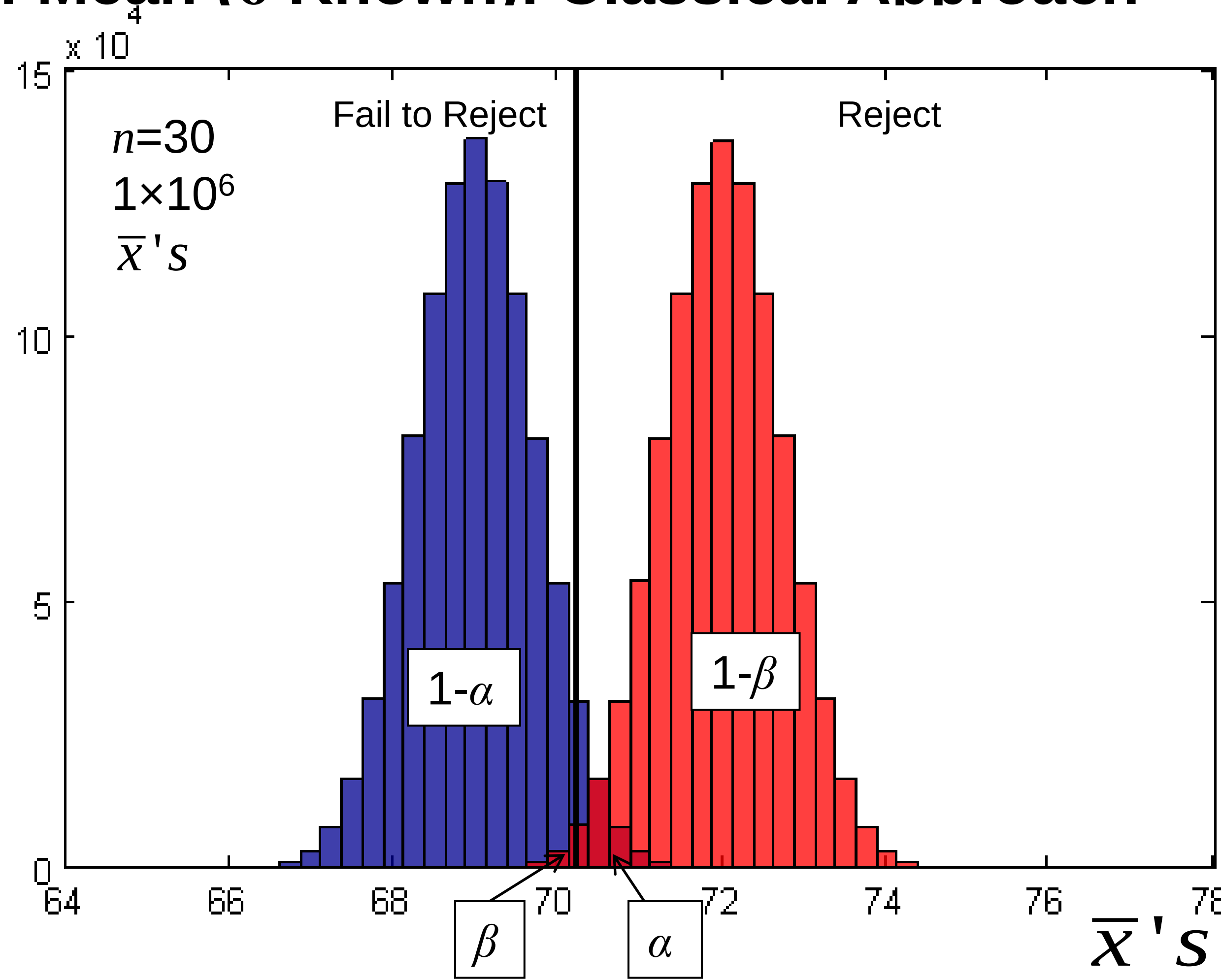
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

What is the solution?

Increase n.

Figure shows n increased to n= 30 from n =15.

Note α and β both smaller with larger n.



8: Introduction to Statistical Inference

H_0: $\mu \leq \mu_0$ vs. H_1: $\mu > \mu_0$

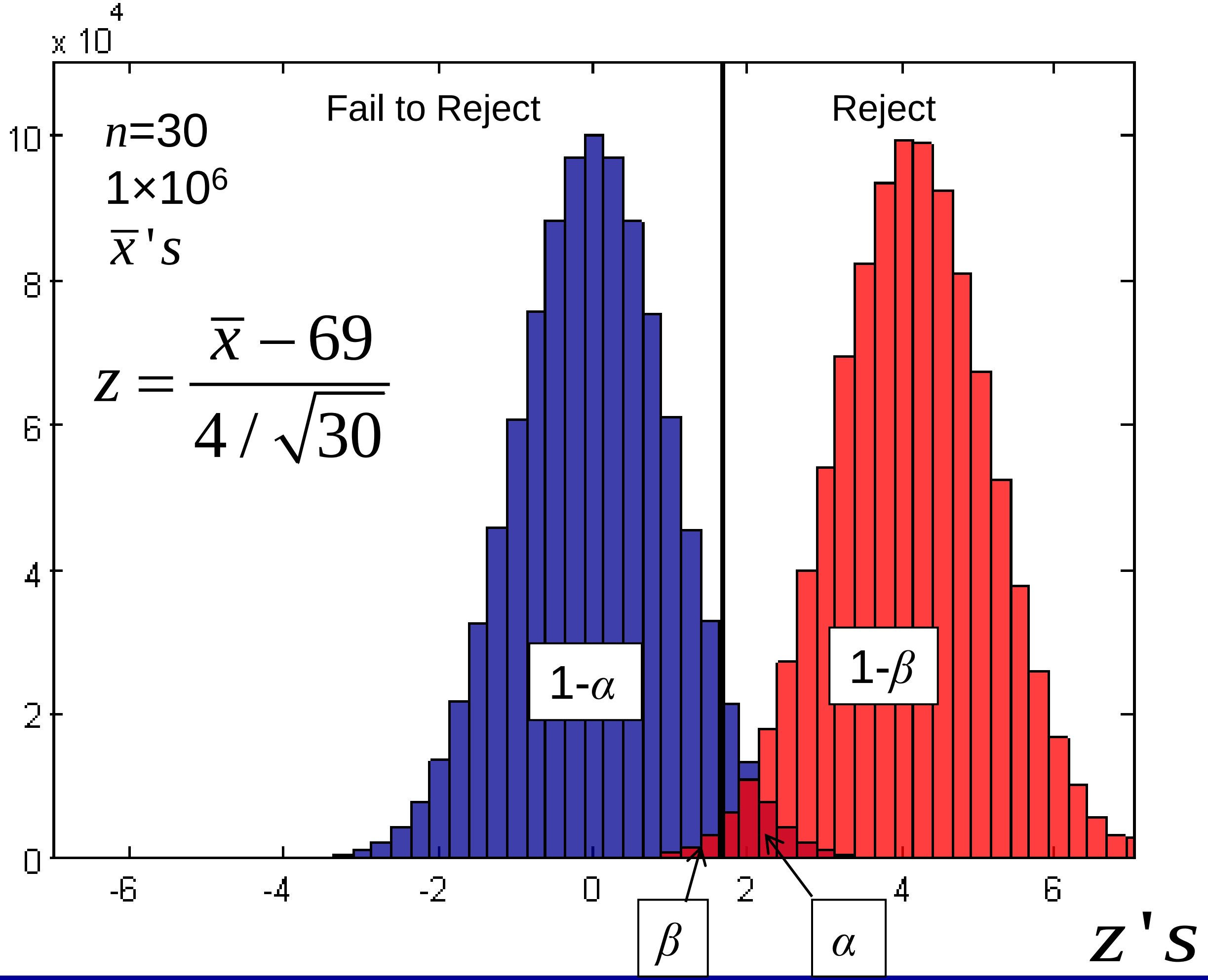
8.5 Hypothesis Test of Mean (σ Known): Classical Approach

What is the solution?

Increase *n*.

Figure shows *n* increased to *n*= 30 from *n* =15.

Note α and β both smaller with larger *n*.



Chapter 8: Introduction to Statistical Inference

Questions?

Homework: WebAssign

Lecture Chapter 9.1

Chapter 9: Inferences Involving One Population

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9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

In Chapter 8, we performed hypothesis tests on the mean by

1) assuming that $\bar{x}$ was normally distributed (n "large"),

2) assuming the hypothesized mean μ_0 were true,

3) assuming that σ was known, so that we could form

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$ which with 1) – 3) has standard normal dist.

9: Inferences Involving One Population

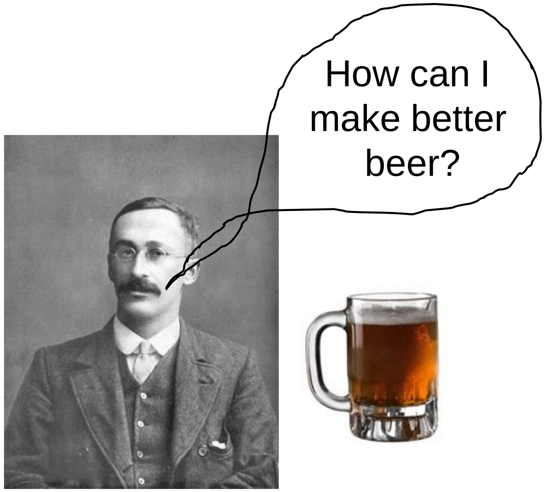
9.1 Inference about the Mean μ (σ Unknown)

However, in real life, we never know σ for

$$z^* = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

so we would like to estimate σ by s, then use

$$t^* = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} \quad .$$



But t^* does not have a standard normal distribution.

It has what is called a Student t-distribution. ← Gosset
Guinness Brewery

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

What is the Student t-distribution and how do we get it?

Background Information

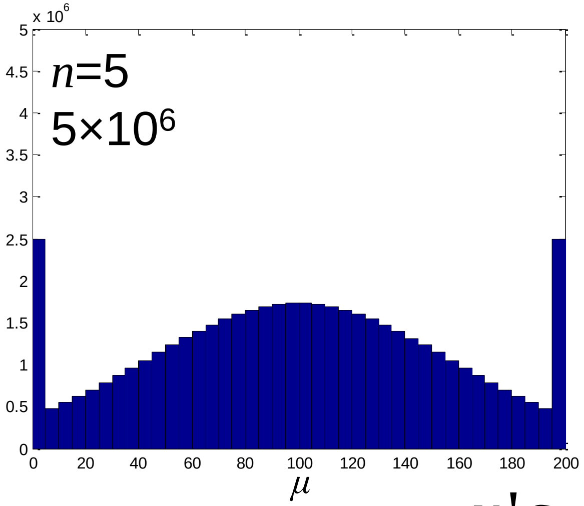
If the data comes from a normally distributed population, then

$$x \sim N(\underset{\text{mean}}{\mu},\underset{\text{variance}}{\sigma^2})$$

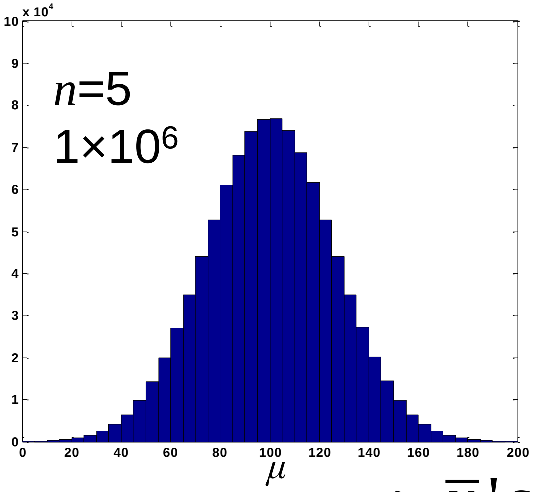
$$\bar{x} \sim N(\underset{\text{mean}}{\mu},\underset{\text{variance}}{\sigma^2 / n})$$

generate 5×10⁶ random values

$$\mu = 100$$
$$\sigma = 57.7$$
$$n = 5$$



x's



sample means from each 5 → $\bar{x}$'s

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$
$$\sigma = 57.7$$
$$n = 5$$

If we know the true population mean μ, then

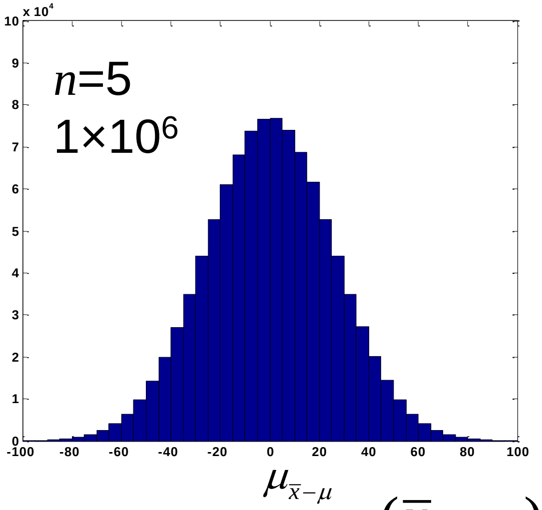
$$\bar{x} - \mu \sim N(0, \sigma^2 / n) \quad .$$

mean variance

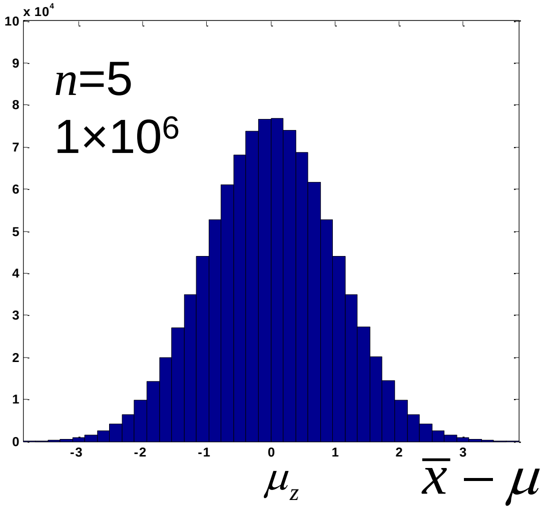
Given the variance of the mean σ^2/n, the distribution of

$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

mean variance



$(\bar{x} - \mu)$'s



$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$'s

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$\mu = 100$

$\sigma = 57.7$

$n = 5$

It turns out that with the variance σ^2 known, the distribution of

$$\frac{(n-1)s^2}{\sigma^2}$$ has a chi-square

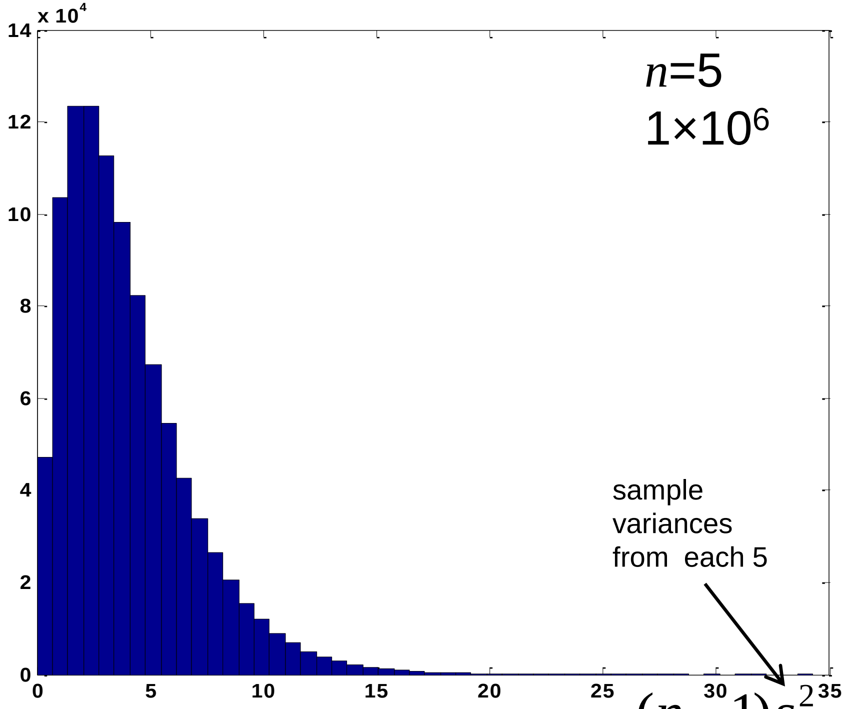
sample variance

population variance

distribution with n-1 degrees

of freedom.

(χ^2 distribution on Pages 453-454)



$$\frac{(n-1)s^2}{\sigma^2}\text{'s}$$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\mu = 100$$
$$\sigma = 57.7$$
$$n = 5$$

The ratio $\swarrow N(0,1)$

$$t = \left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right) \Big/ \sqrt{ \frac{(n-1)s^2}{\sigma^2} \Big/ (n-1) }$$

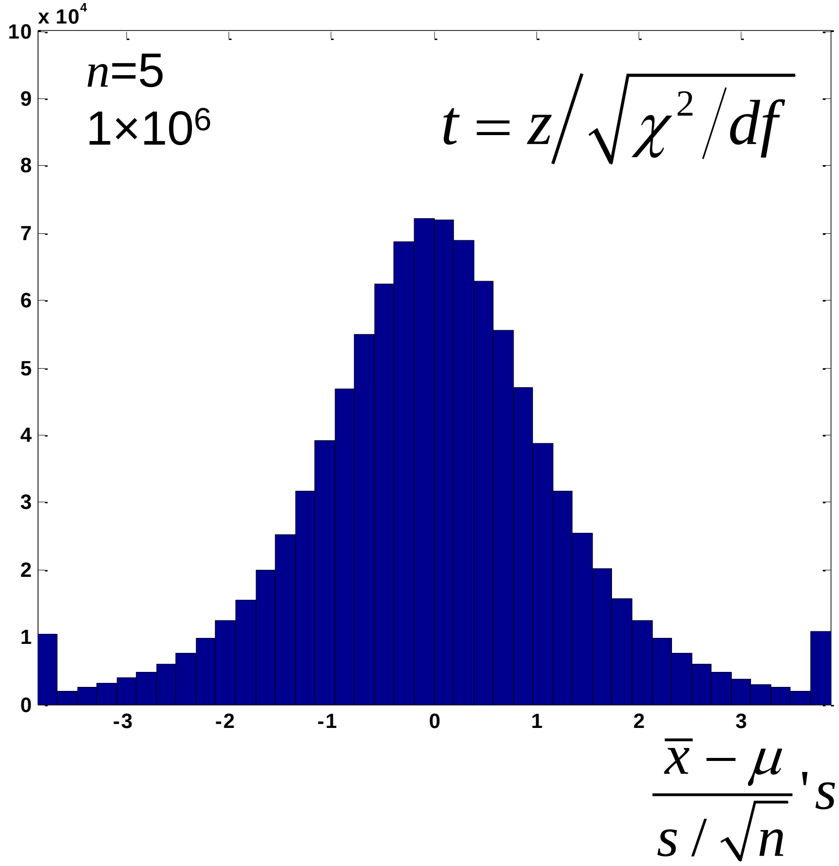
$\nwarrow \chi^2(n-1)$ $\quad \nwarrow df$

with simplification

is $\quad t = \dfrac{\bar{x} - \mu}{s / \sqrt{n}}$,

and has a Student

t-distribution with n-1 df.



9: Inferences Involving One Population

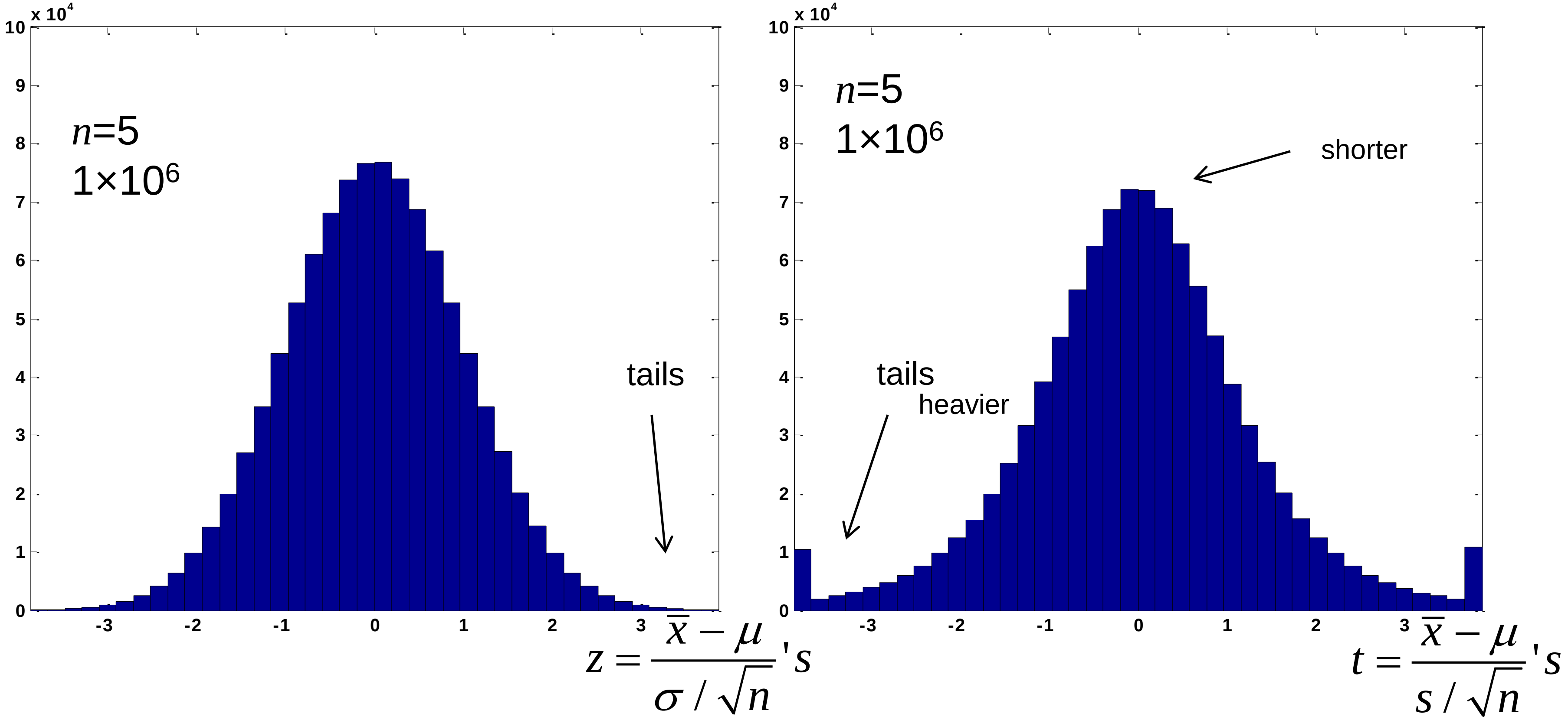
9.1 Inference about the Mean μ (σ Unknown)

$\mu = 100$

$\sigma = 57.7$

$n = 5$

Student t-distribution has heavier tails than standard normal.



$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$'s

$t = \frac{\bar{x} - \mu}{s / \sqrt{n}}$'s

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

$$\nu = df = n - 1$$

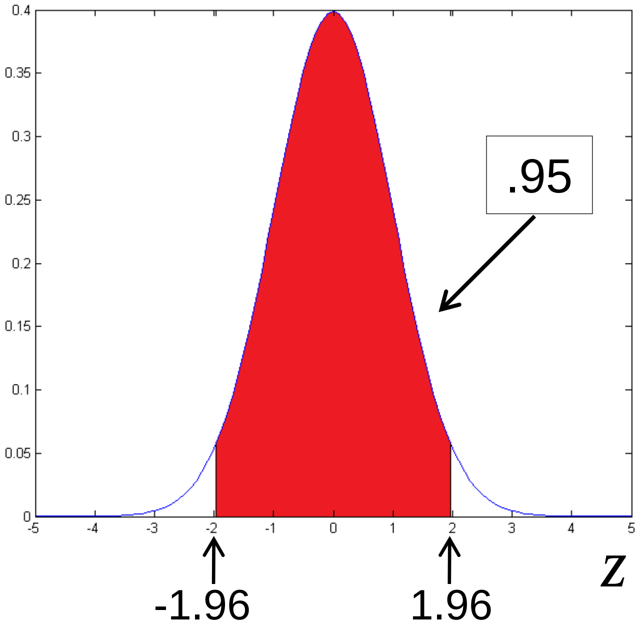
The standard normal dist is:

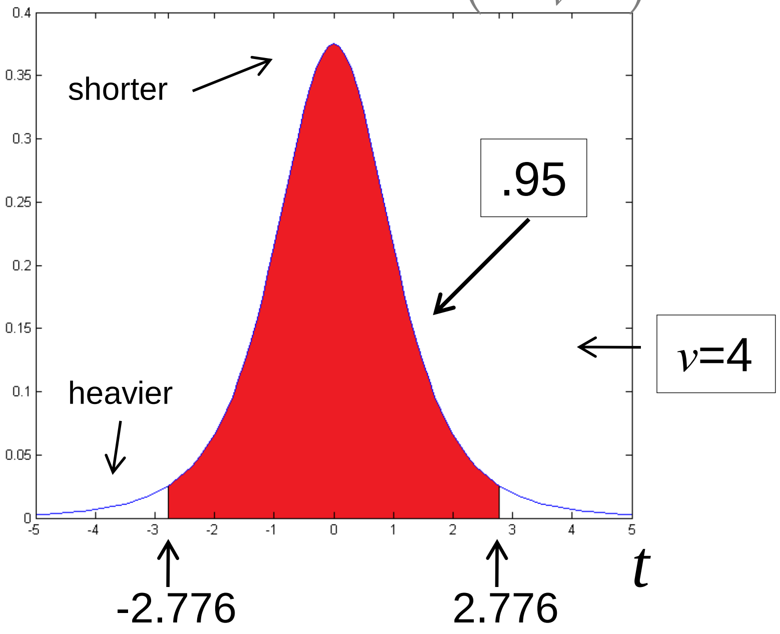
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

ignore

The Student-t distribution is:

$$f(t\,|\,\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+\frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$





9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

The standard normal dist. is:

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

ignore

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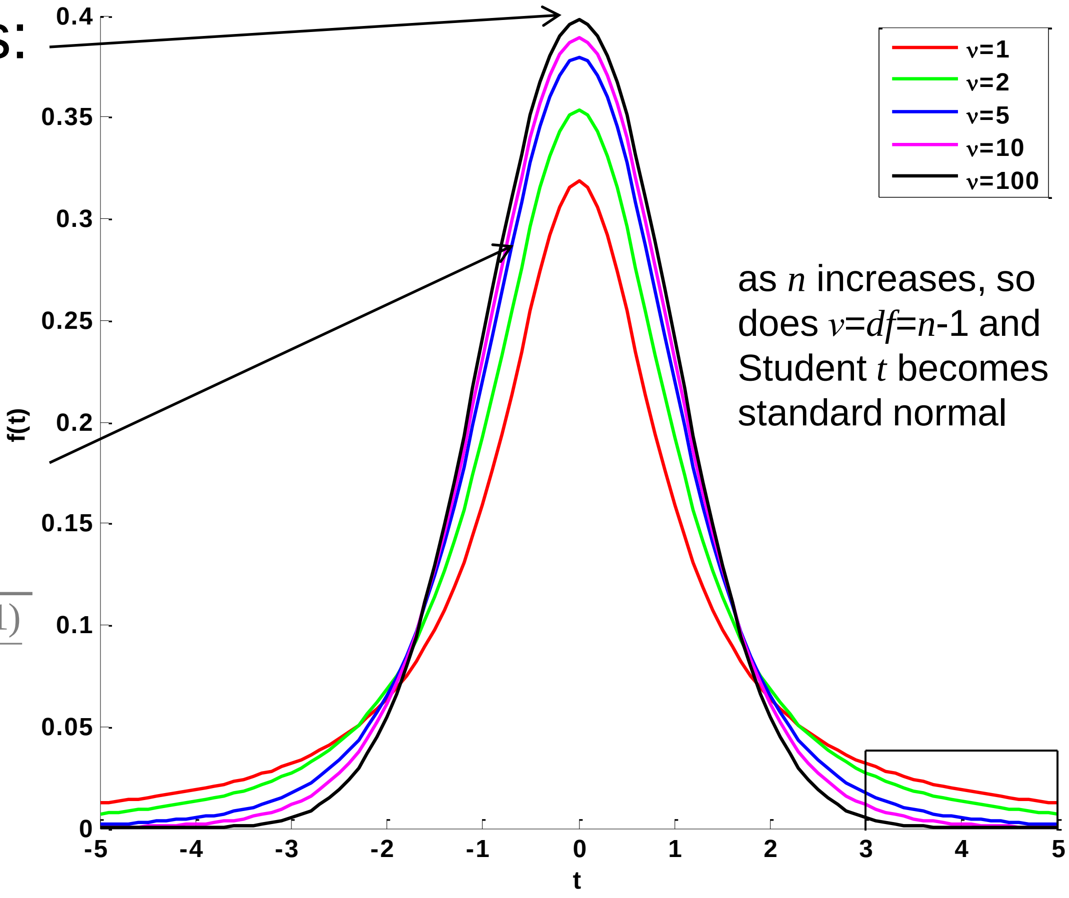
ignore



as n increases, so does $\nu=df=n$-1 and Student t becomes standard normal

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

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$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

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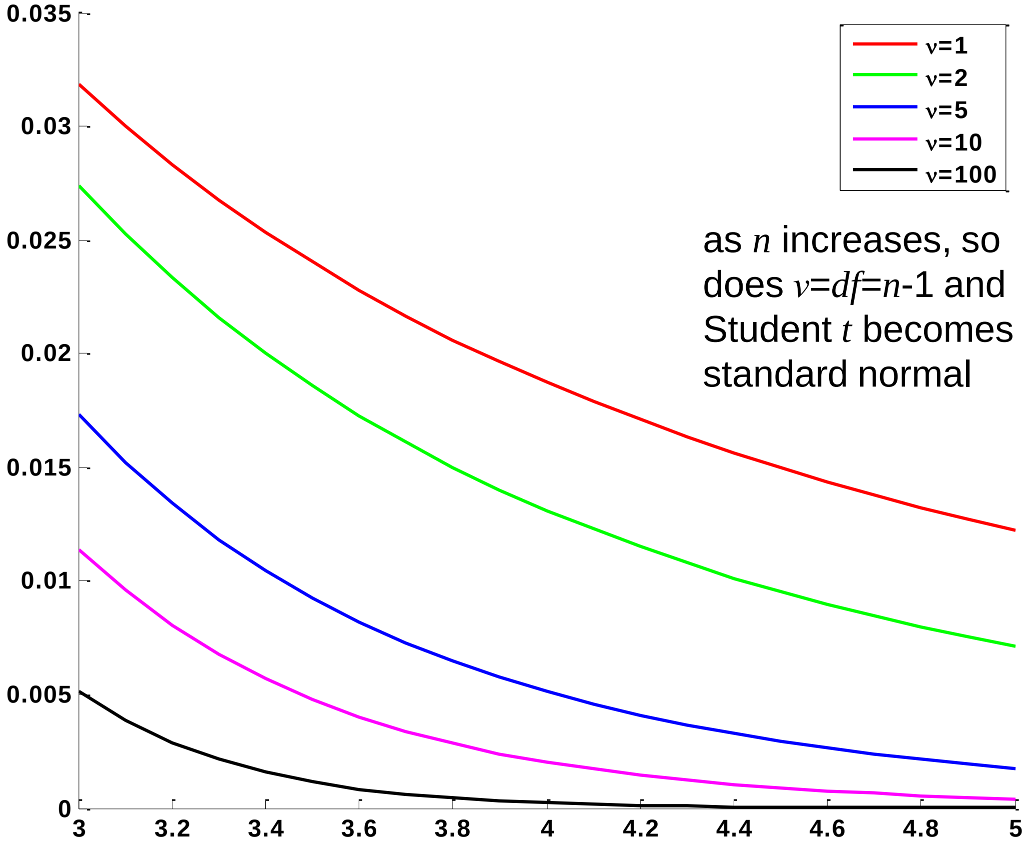
The Student-t distribution is:

$$f(t|\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\frac{1}{\nu}t^2\right)^{\frac{(\nu+1)}{2}}}$$

ignore



as n increases, so does $\nu=df=n$-1 and Student t becomes standard normal

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Using the t-Distribution Table

Finding critical value from a Student t-distribution, df=n-1

$t(df,\alpha)$, t value with α area larger than it

with df degrees of freedom

Table 6
Appendix B
Page 719.

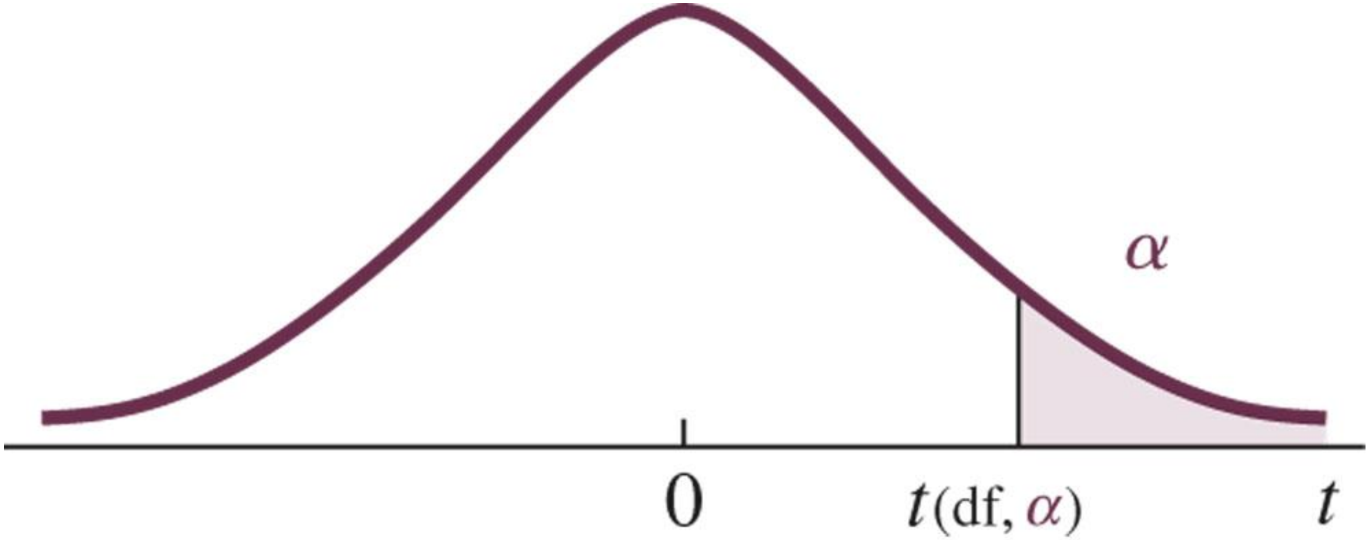
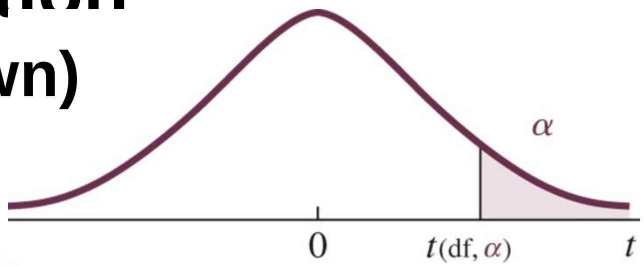


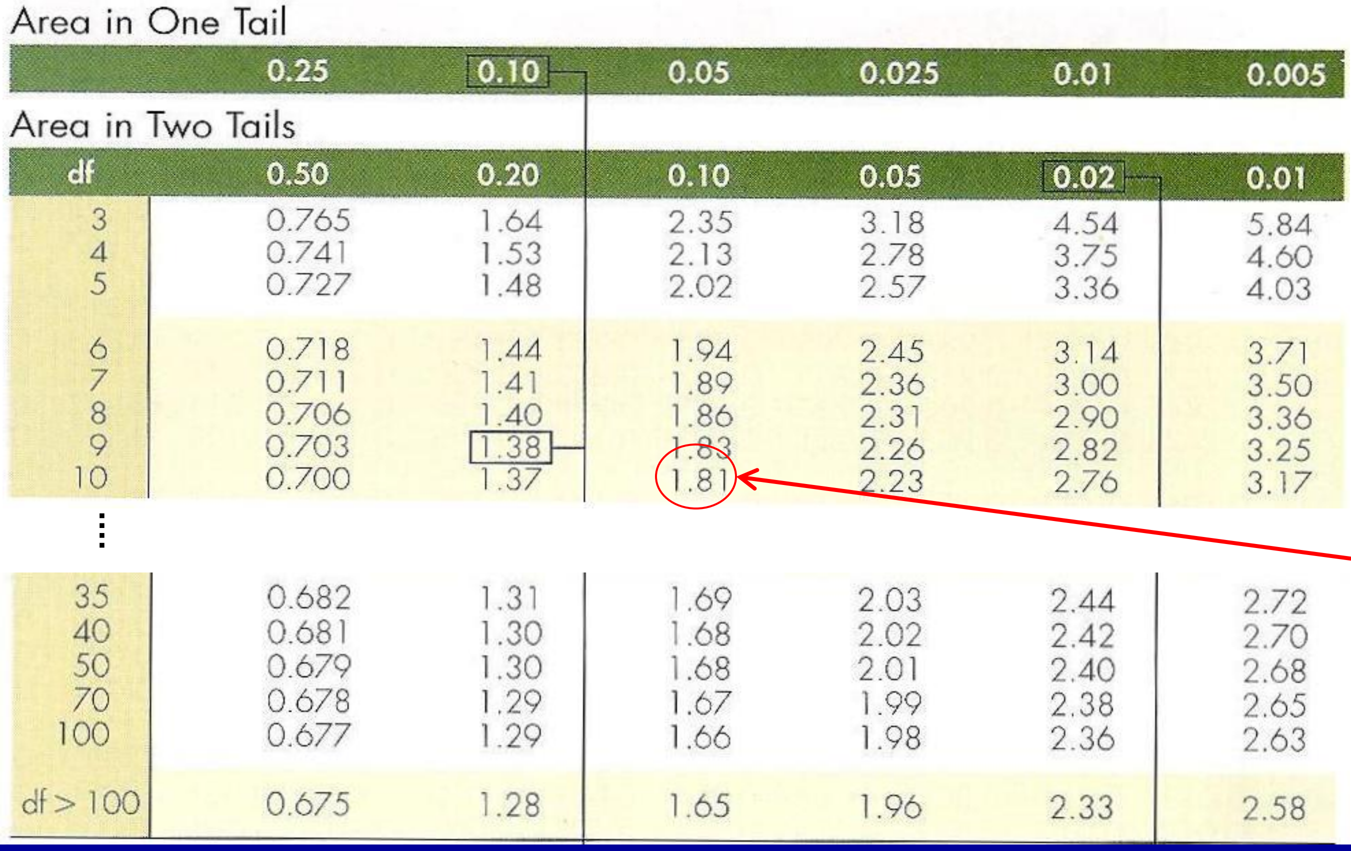
Figure from Johnson & Kuby, 2012.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Example: Find the value of $t(10,0.05)$, $df=10$, $\alpha=0.05$.





Area in One Tail

Area in Two Tails

	0.25	0.10	0.05	0.025	0.01	0.005
df	0.50	0.20	0.10	0.05	0.02	0.01
3	0.765	1.64	2.35	3.18	4.54	5.84
4	0.741	1.53	2.13	2.78	3.75	4.60
5	0.727	1.48	2.02	2.57	3.36	4.03
6	0.718	1.44	1.94	2.45	3.14	3.71
7	0.711	1.41	1.89	2.36	3.00	3.50
8	0.706	1.40	1.86	2.31	2.90	3.36
9	0.703	1.38	1.83	2.26	2.82	3.25
10	0.700	1.37	1.81	2.23	2.76	3.17
⋮						
35	0.682	1.31	1.69	2.03	2.44	2.72
40	0.681	1.30	1.68	2.02	2.42	2.70
50	0.679	1.30	1.68	2.01	2.40	2.68
70	0.678	1.29	1.67	1.99	2.38	2.65
100	0.677	1.29	1.66	1.98	2.36	2.63
df > 100	0.675	1.28	1.65	1.96	2.33	2.58

Table 6
Appendix B
Page 719.

Go to 0.05 One Tail column and down to 10 *df* row.

Figures from Johnson & Kuby, 2012.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

When making a confidence interval for μ when σ unknown,

we assume that the population is normal, not just mean,

but when n is “large,” can often use for nonnormal distributions.

The assumption for inferences about the mean μ when σ is unknown: The sampled population is normally distributed.

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Discussed a confidence interval for the μ when σ was known,

Confidence Interval for Mean:

$$\bar{x} - z(\alpha/2)\frac{\sigma}{\sqrt{n}} \quad \text{to} \quad \bar{x} + z(\alpha/2)\frac{\sigma}{\sqrt{n}} \qquad \textbf{(8.1)}$$

now, with σ unknown, the CI for the mean is

Confidence Interval for Mean:

$$\bar{x} - t(df,\alpha/2)\frac{s}{\sqrt{n}} \quad \text{to} \quad \bar{x} + t(df,\alpha/2)\frac{s}{\sqrt{n}} \qquad \textbf{(9.1)}$$

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Example: A random sample of n=15 math1700 student heights yielded $\bar{x} = 67.2$. Assume σ=4.0. Construct a 95% CI for μ.

Fill In

$$\bar{x} \pm z(\alpha / 2)\frac{\sigma}{\sqrt{n}}$$

Example: A random sample of n=15 math1700 student heights yielded $\bar{x} = 67.2$ and s=3.5. Construct a 95% CI for μ

$$\bar{x} - t(df, \frac{\alpha}{2})\frac{s}{\sqrt{n}}$$

Fill In

9: Inferences Involving One Population

9.1 Inference about the Mean μ (σ Unknown)

Confidence Interval Procedure

Example: A random sample of n=15 math1700 student heights yielded $\bar{x} = 67.2$. Assume σ=4.0. Construct a 95% CI for μ.

$$\bar{x} \pm z(\alpha / 2)\frac{\sigma}{\sqrt{n}} \longrightarrow 67.2 - 1.96\frac{4.0}{\sqrt{15}} \quad \text{to} \quad 67.2 + 1.96\frac{4.0}{\sqrt{15}}$$

$$\boxed{65.2 \quad \text{to} \quad 69.2}$$

9: Inferences Involving One Population

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$vs.$

$$\boxed{65.2 \quad \text{to} \quad 69.2}$$

Example: A random sample of n=15 math1700 student heights yielded $\bar{x} = 67.2$ and s=3.5. Construct a 95% CI for μ.

$$\bar{x} - t(df, \frac{\alpha}{2})\frac{s}{\sqrt{n}}$$

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Confidence Interval Procedure

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$$\bar{x} \pm z(\alpha/2)\frac{\sigma}{\sqrt{n}} \longrightarrow 67.2 - 1.96\frac{4.0}{\sqrt{15}} \text{ to } 67.2 + 1.96\frac{4.0}{\sqrt{15}}$$

$vs.$

$$\boxed{65.2 \text{ to } 69.2}$$

Example: A random sample of n=15 math1700 student heights yielded $\bar{x} = 67.2$ and s=3.5. Construct a 95% CI for μ.

$$\bar{x} - t(df, \frac{\alpha}{2})\frac{s}{\sqrt{n}} \longrightarrow 67.2 - 2.14\frac{3.5}{\sqrt{15}} \text{ to } 67.2 + 2.14\frac{3.5}{\sqrt{15}}$$

$$\boxed{65.3 \text{ to } 69.1}$$

Chapter 9: Inferences Involving One Population

Questions?

Homework: Read Chapter 9.1

WebAssign

Chapter 9 # 9, 21, 23, 45, 55