Class 7

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Agenda:

Recap Chapter 4.3 - 4.5

Lecture Chapter 5.1 - 5.3

Recap Chapter 4.3

4: Probability

4.3 Rules of Probability – "*A* or *B*"

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

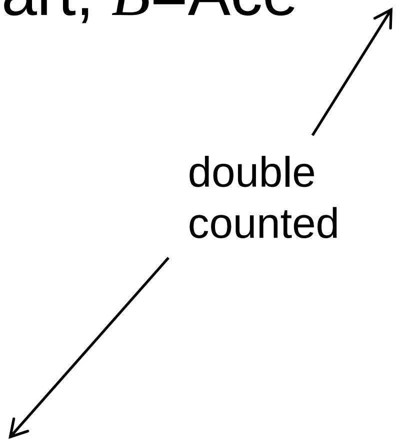
Example: Pick Card, A=Heart, B=Ace

$P(\text{Heart or Ace})$

$P(\text{Heart}) = 13/52$

$P(\text{Ace}) = 4/52$

$P(\text{Heart and Ace}) = 1/52$



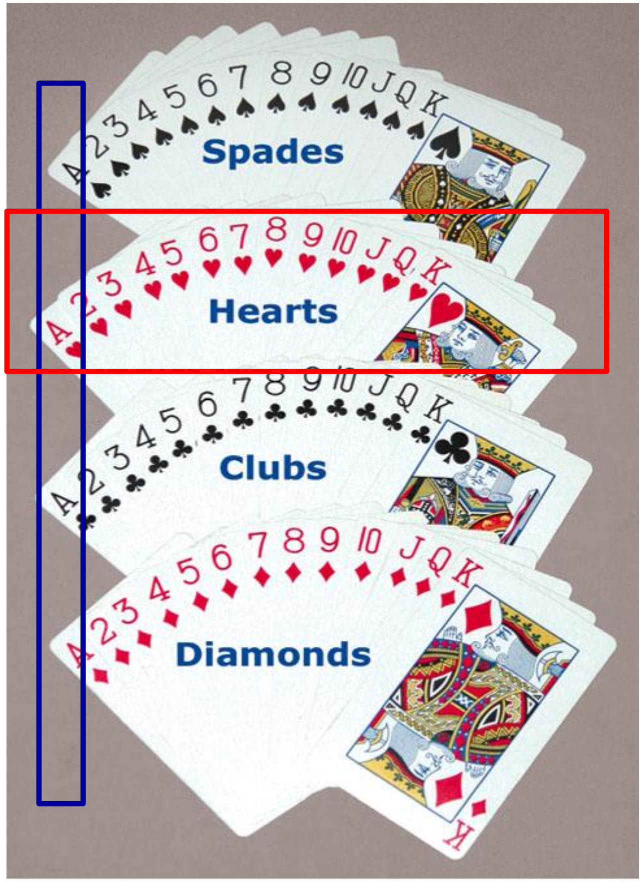


Figure from Johnson & Kuby, 2012.

$$P(\text{Heart or Ace}) = P(\text{Heart}) + P(\text{Ace}) - P(\text{Heart and Ace})$$

$$P(\text{Heart or Ace}) = 13/52 + 4/52 - 1/52 = 16/52$$

4: Probability

4.3 Rules of Probability – "*A* and *B*"

$$P(A \text{ and } B) = P(B)P(A \mid B)$$

Example: Pick Card, A=Heart, B=Ace

$P(\text{Heart and Ace})$

$P(\text{Ace}) = 4/52$

$P(\text{Heart} \mid Ace) = 1/4$

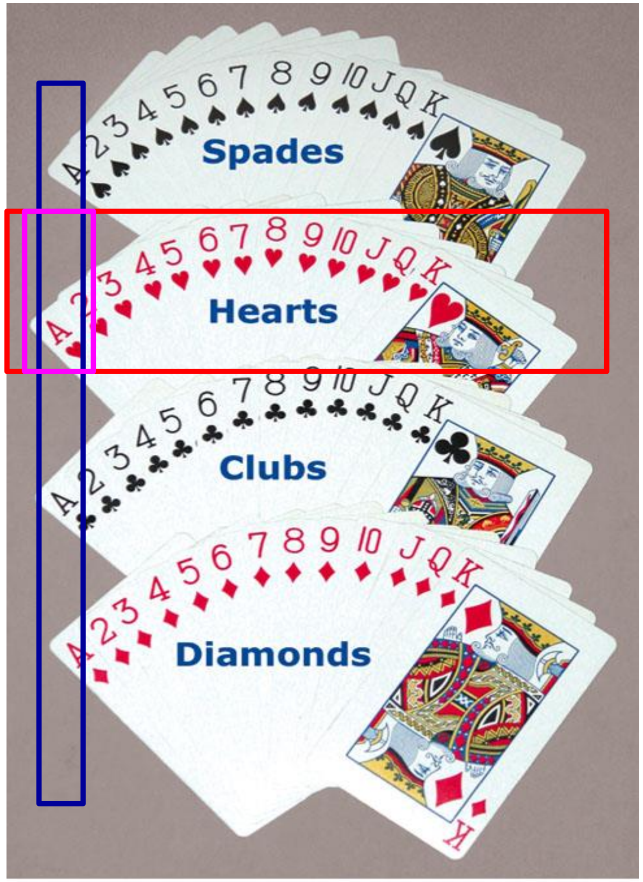


Figure from Johnson & Kuby, 2012.

$P(\text{Heart and Ace}) = P(\text{Ace})P(\text{Heart}|\text{Ace})$

$P(\text{Heart and Ace}) = (4/52)(1/4) = 1/52$

4: Probability

Questions?

Homework: Read Chapter 4.3-4.5

Web Assign Chapter 4 # 59, 63, 65, 69, 85, 89, 91, 97, 105, 107, 113

Set one die to 4 (event B). Roll the other die 100 times. Let A be that a 3 comes up (7 is the sum of the two die). Calculate $P(A|B)$ using the empirical approach.

4: Probability

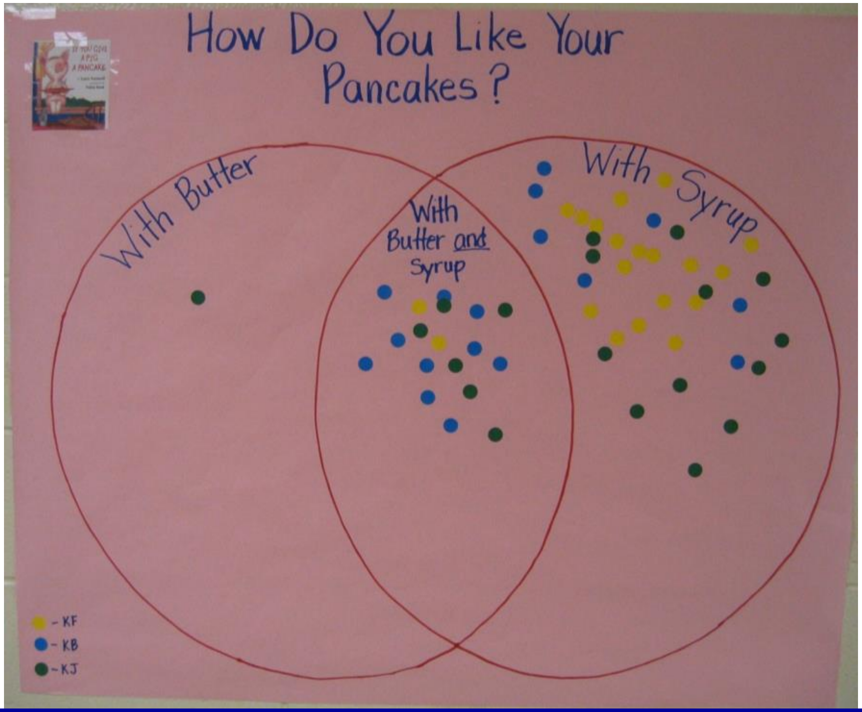
What is the probability that a random kindergartener likes syrup (B) on their pancakes given that they like butter (A) on their pancakes?

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$P(A) = \frac{n(A)}{n}$$

3 kindergarten classes



4: Probability

Homework: Watch Catch 21 Episode

https://www.youtube.com/watch?v=N73GnjXfl48

The bonus round begins at time 12:28.

After the Q♠,J♣,A♠ are drawn:

What is the probability of an A as the 4th card?

What is the probability of a “10” as the 4th card?

What is the probability of an A as the 5th card?

What is the probability of an A as the 6th card?

What is the probability of a 6 as the 7th card?

(10 Card is a 10 or a face card.)

Lecture Chapter 5.1 - 5.3

Chapter 5: Probability Distributions (Discrete Variables)

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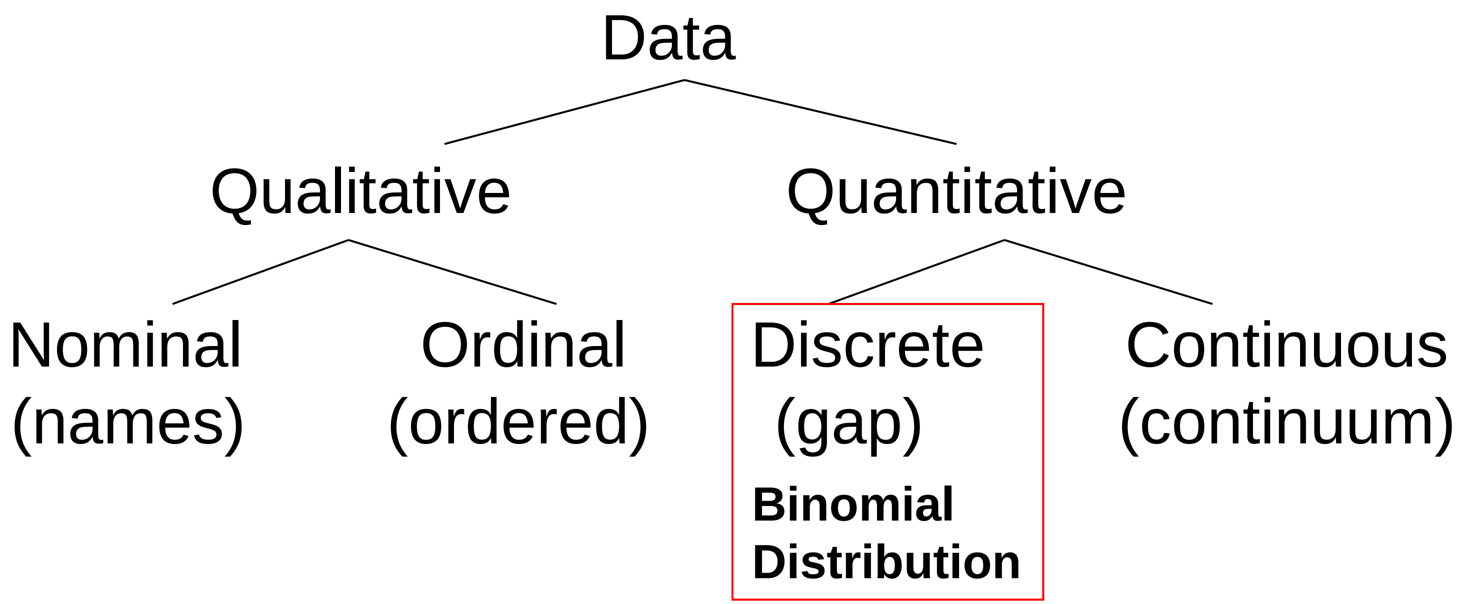
5: Probability Distributions (Discrete Variables)
5.1 Random Variables

At beginning of course we talked about types of data.



5: Probability Distributions (Discrete Variables)

5.1 Random Variables

Random Variables: A variable that assumes a unique numerical value for each of the outcomes in the sample space of a probability experiment.

Example:

Let x = the number of heads when we flip a coin twice.

$x=\{0,1,2\}$ ⟵ numerical values

for

$\{TT,TH,HT,HH\}$ ⟵ outcomes in sample space

5: Probability Distributions (Discrete Variables)

5.1 Random Variables

Discrete Random Variables: A quantitative random variable that can assume a countable number of values.

Continuous Random Variable: A quantitative random variable that can assume an uncountable number of values.

Continuum of values.

Examples:

Discrete: Number of heads when we flip a coin ten times.

Continuous: Distance from earth center to sun center.

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

Probability Distribution: A distribution of the probabilities associated with each of the values of a random variable. The probability distribution is a theoretical distribution; it is used to represent populations.

Flipping a Coin

x = # H	$P(x)$
0	0.25
1	0.50
2	0.25

Rolling a Die

x = face value	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Figures from Johnson & Kuby, 2012.

Recall: **1.** each prob between 0 & 1 and **2.** sum of prob's = 1

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

Probability Function: A rule $P(x)$ that assigns probabilities to the values of the random variables, x.

Note:
1. $0 \leq P(x) \leq 1$
2. $\Sigma P(x)=1$

Example:
Let x = # of heads when we flip a coin twice.

$x=\{0,1,2\}$

$$P(x)=\frac{2!}{x!(2-x)!}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{2-x}$$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5: Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

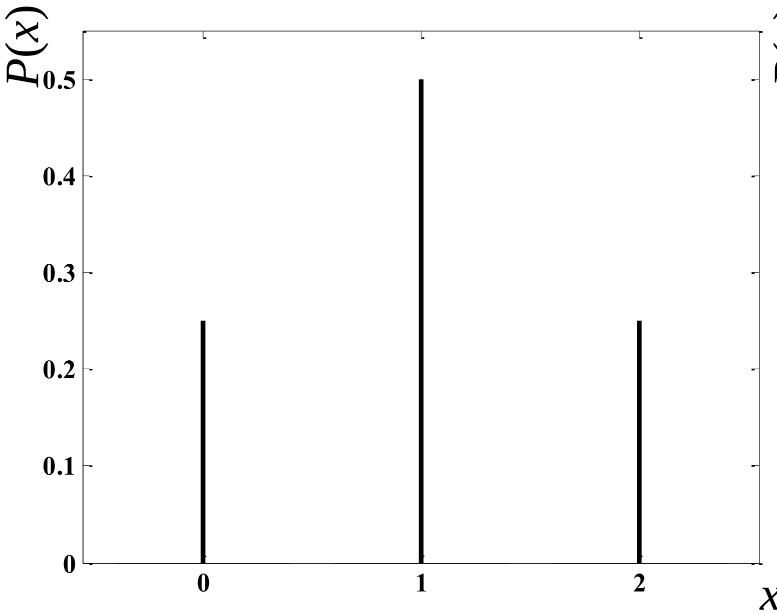
Example:

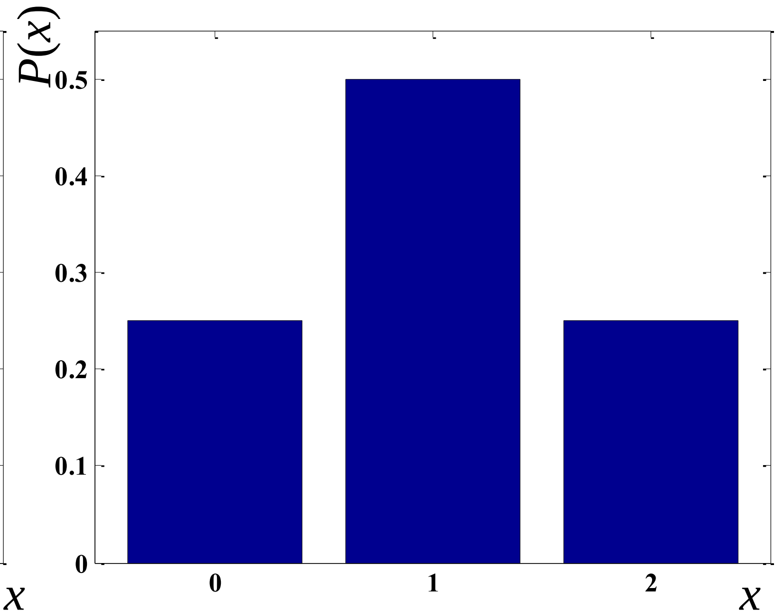
Let x = # of H when flip a coin twice.

$$P(x)=\frac{2!}{x!(2-x)!}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{2-x}$$

$$x=0,1,2$$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$





1: Statistics

1.2 What is Statistics?

Data: The set of values collected from the variable from each of the elements that belong to the sample.

Experiment: A planned activity whose results yield a set of data.

Sample: Subset of the population.

Parameter: A numerical value summarizing all the data of an entire population.

Statistic: A numerical value summarizing the sample data.

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

If we calculate a numerical summary from the sample of data, it is called a **statistic**. i.e. $\bar{x}$, s^2, or r.

sample mean, sample variance, sample correlation

If we calculate a numerical summary from the population of data, it is called a **parameter**. i.e. μ ,σ^2, or ρ.

population mean, population variance, population correlation

μ is the Greek letter lower case mu.

σ is the Greek letter lower case sigma.

ρ is the Greek letter lower case rho.

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Mean of a discrete random variable (expected value):
The mean, μ, of a discrete random variable x is found by multiplying each possible value of x by its own probability, $P(x)$, and then adding all of the products together:

mean of x: mu = sum of (each x multiplied by its own probability)

$$\mu = \sum_{i=1}^{n} [x_i P(x_i)] \qquad (5.1)$$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\mu = \sum_{i=1}^{n} [x_i P(x_i)] = x_1 P(x_1) + x_2 P(x_2) + \ldots + x_n P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\mu = (x_1) \cdot P(x_1) + (x_2) \cdot P(x_2) + (x_3) \cdot P(x_3)$$

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\mu = \sum_{i=1}^{n} [x_i P(x_i)] = x_1 P(x_1) + x_2 P(x_2) + \ldots + x_n P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\mu = (x_1) \cdot P(x_1) + (x_2) \cdot P(x_2) + (x_3) \cdot P(x_3)$$

$$\mu = (0) \cdot P(0) + (1) \cdot P(1) + (2) \cdot P(2)$$

$$\mu = (0) \cdot (1/4) + (1) \cdot (1/2) + (2) \cdot (1/4)$$

$$\mu = 0 + 1/2 + 1/2$$

$$\mu = 1$$

	x	$P(x)$	
x_1 →	0	$\frac{1}{4}$	← $P(x_1)$
x_2 →	1	$\frac{1}{2}$	← $P(x_2)$
x_3 →	2	$\frac{1}{4}$	← $P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Variance of a discrete random variable: The variance, σ^2, of a discrete random variable x is found by multiplying each possible value of the squared deviation, $(x-\mu)^2$, by its own probability, $P(x)$, and then adding all of the products together:

variance of x: sigma squared

= sum of (squared deviation times probability)

$$\sigma^2 = \sum_{i=1}^{n}[(x_i-\mu)^2 P(x_i)] \qquad (5.2)$$

equivalent formula $$\sigma^2 = \sum_{i=1}^{n}[x_i^{\,2} P(x_i)] - \mu^2 \qquad (5.3b)$$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\sigma^2 = \sum_{i=1}^{n}[(x_i-\mu)^2 P(x_i)] = (x_1-\mu)^2 P(x_1) + (x_2-\mu)^2 P(x_2) + \ldots + (x_n-\mu)^2 P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$$\sigma^2 = (x_1-\mu)^2 \cdot P(x_1) + (x_2-\mu)^2 \cdot P(x_2) + (x_3-\mu)^2 \cdot P(x_3)$$

$\mu = 1$

	x	$P(x)$	
x_1 →	0	$\frac{1}{4}$	← $P(x_1)$
x_2 →	1	$\frac{1}{2}$	← $P(x_2)$
x_3 →	2	$\frac{1}{4}$	← $P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

$$\sigma^2 = \sum_{i=1}^{n} [(x_i - \mu)^2 P(x_i)] = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \ldots + (x_n - \mu)^2 P(x_n)$$

For the # of H when we flip a coin twice discrete distribution:

$\sigma^2 = (x_1-\mu)^2 \cdot P(x_1) + (x_2-\mu)^2 \cdot P(x_2) + (x_3-\mu)^2 \cdot P(x_3)$

$\sigma^2 = (0-1)^2 \cdot P(0) + (1-1)^2 \cdot P(1) + (2-1)^2 \cdot P(2)$

$\sigma^2 = (-1)^2 \cdot (1/4) + (0)^2 \cdot (1/2) + (1)^2 \cdot (1/4)$

$\sigma^2 = 1/4 + 0 + 1/4$

$\sigma^2 = 1/2$

$\mu = 1$

	x	$P(x)$	
x_1	0	$\frac{1}{4}$	$P(x_1)$
x_2	1	$\frac{1}{2}$	$P(x_2)$
x_3	2	$\frac{1}{4}$	$P(x_3)$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Standard deviation of a discrete variable: The positive square root of the variance.

$$\sigma = \sqrt{\sigma^2} \quad (5.4)$$

$$\sigma = \sqrt{\sum_{i=1}^{n} [(x_i - \mu)^2 P(x_i)]}$$

$\sigma^2 =$

$\sigma =$

5: Probability Distributions (Discrete Variables)

5.2 Mean and Variance of a Discrete Random Variable

Standard deviation of a discrete variable: The positive square root of the variance.

$$\sigma = \sqrt{\sigma^2} \tag{5.4}$$

$$\sigma = \sqrt{\sum_{i=1}^{n} [(x_i - \mu)^2 P(x_i)]}$$

$\sigma^2 = 1/2$

$\sigma = 1/\sqrt{2}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Let's assume we have two independent events E_1 and E_2.

We know that $P(E_1 \text{ and } E_2) = P(E_1)P(E_2)$. Page 211.

More generally, if we have n independent events $\mathrm{E}_1,\ldots,\mathrm{E}_n$.

We know that $P(E_1 \text{ and } E_2 \cdots \text{ and } E_n) = P(E_1)P(E_2)\cdots P(E_n)$.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Let's assume we are flipping a coin twice.
E_1=Head on first flip, E_2=Tail on second flip.

The probability of heads on any given flip is $p = P(H)$.
The probability of tails (not heads) on any given flip is $q = (1-p)$.

Then $P(HT)=P(H)P(T)$
$=p(1-p)$.

Similarly $P(TH) = P(T)P(H)$
$= (1-p)p$.

Let x = # of heads in two flips of a coin.

$P(x=1) = P(HT)+P(TH)$
$= p(1-p)+(1-p)p = 2p(1-p)$.

2 ways to get one H and one T
2 ways to get x=1 heads

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Bi means two like bicycle

An experiment with only two outcomes is called a Binomial exp.
Call one outcome *Success* and the other *Failure*.
Each performance of expt. is called a trial and are independent.

$$P(x) = \underbrace{\frac{n!}{x!(n-x)!}}_{\text{num}(x \text{ successes})} \underbrace{p^x (1-p)^{n-x}}_{P(x \text{ successes and } n-x \text{ failures})} \qquad x = 0, \ldots, n \qquad (5.5)$$

Prob of exactly x successes

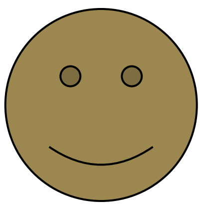
n = number of trials or times we repeat the experiment.
x = the number of successes out of n trials.
p = the probability of success on an individual trial.

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once.



O	$P(O)$
H	$1/2$
T	$1/2$

x= # of Heads

x	$P(x)$
0	$1/2$
1	$1/2$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	1

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

num flips ⟶ n=1
num succ. ⟶ x=1
prob succ ⟶ p=1/2

$$P(0)=$$

$$P(1)=$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once.

O	$P(O)$
H	$1/2$
T	$1/2$

x= # of Heads

x	$P(x)$
0	$1/2$
1	$1/2$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	1

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

num flips ⟶ n=1

num succ. ⟶ x=1

prob succ ⟶ p=1/2

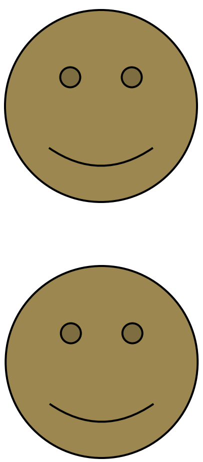
$$P(0)=\frac{1!}{0!(1-0)!}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{1-0}=\frac{1}{1\times 1}\left(\frac{1}{2}\right)^1=\frac{1}{2}$$

$$P(1)=\frac{1!}{1!(1-1)!}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{1-1}=\frac{1}{1\times 1}\left(\frac{1}{2}\right)^1=\frac{1}{2}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice.



O	$P(O)$
HH	$1/4$
HT	$1/4$
TH	$1/4$
TT	$1/4$

x= # of Heads

x	$P(x)$
0	$1/4$
1	$2/4$
2	$1/4$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	2
2	1

num flips ⟶ n=2

num succ. ⟶ x=1

prob succ ⟶ p=1/2

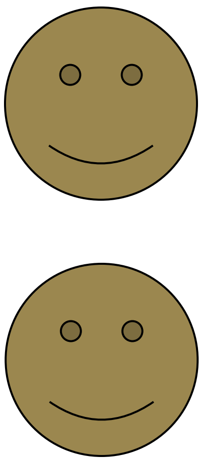
$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$P(1)=$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice.



O	$P(O)$
HH	$1/4$
HT	$1/4$
TH	$1/4$
TT	$1/4$

x= # of Heads

x	$P(x)$
0	$1/4$
1	$2/4$
2	$1/4$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	2
2	1

num flips ⟶ n=2
num succ. ⟶ x=1
prob succ ⟶ p=1/2

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$P(1)=\frac{2!}{1!(2-1)!}(1/2)^1(1-1/2)^{2-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times.

O	$P(O)$
HHH	$1/8$
HHT	$1/8$
HTH	$1/8$
HTT	$1/8$
THH	$1/8$
THT	$1/8$
TTH	$1/8$
TTT	$1/8$

x= # of Heads

x	$P(x)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	3
2	3
3	1

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

n=3
x=1
p=1/2

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times.

O	$P(O)$
HHH	$1/8$
HHT	$1/8$
HTH	$1/8$
HTT	$1/8$
THH	$1/8$
THT	$1/8$
TTH	$1/8$
TTT	$1/8$

x= # of Heads

x	$P(x)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

$n(x)$= ways to get x Heads

x	$n(x)$
0	1
1	3
2	3
3	1

n=3
x=1
p=1/2

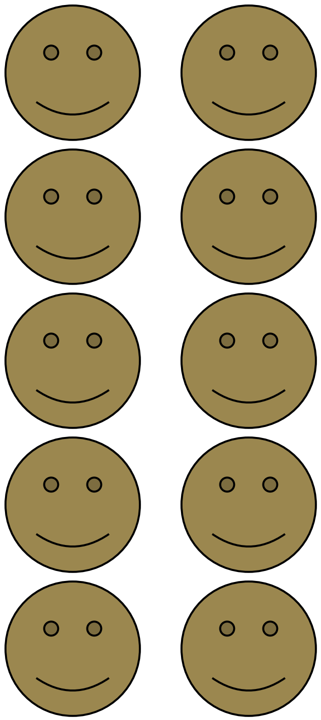
$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) = \frac{3!}{1!(3-1)!} (1/2)^1 (1-1/2)^{3-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin ten times. x= # of Heads $n(x)$= ways to get x Heads



x	0	1	2	3	4	5	6	7	8	9	10
$n(x)$											
$P(x)$											

n=10
x=0,…,10
p=1/2

$$n(x) = \frac{n!}{x!(n-x)!}$$

$$p^x(1-p)^{n-x} =$$

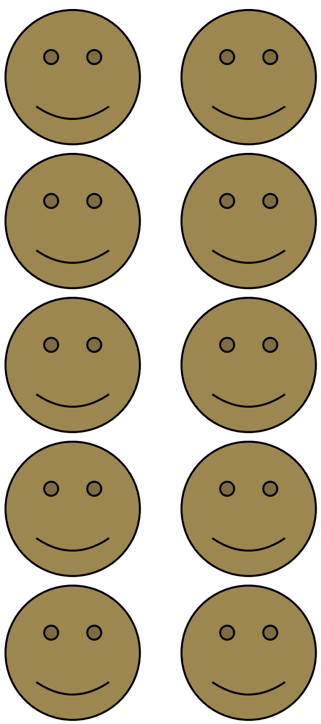
$$P(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

Note:
1. 0 ≤ $P(x)$≤ 1
2. $\Sigma P(x)$=1

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin ten times. x= # of Heads $n(x)$= ways to get x Heads



x	0	1	2	3	4	5	6	7	8	9	10
$n(x)$	1	10	45	120	210	252	210	120	45	10	1
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

n=10
x=0,…,10
p=1/2

$$n(x) = \frac{n!}{x!(n-x)!}$$

$$p^x(1-p)^{n-x} = 1/1024$$

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Note:
1. $0 \le P(x) \le 1$
2. $\Sigma P(x) = 1$

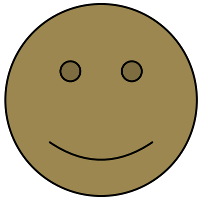
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once. p=2/3

x= # of Heads

$n(x)$= ways to get x Heads



O	$P(O)$
H	$2/3$
T	$1/3$

x	$P(x)$
0	$1/3$
1	$2/3$

x	$n(x)$
0	1
1	1

num flips ⟶ n=1

num succ. ⟶ x=1

prob succ ⟶ p=2/3

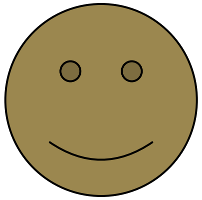
$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(1) =$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin once. p=2/3



x= # of Heads

$n(x)$= ways to get x Heads

O	$P(O)$
H	$2/3$
T	$1/3$

x	$P(x)$
0	$1/3$
1	$2/3$

x	$n(x)$
0	1
1	1

num flips ⟶ n=1
num succ. ⟶ x=1
prob succ ⟶ p=2/3

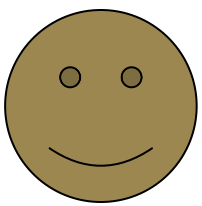
$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$P(1)=\frac{1!}{1!(1-1)!}\left(2/3\right)^1(1-2/3)^{1-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice. p=2/3 x= # of Heads $n(x)$= ways to get x Heads

O	$P(O)$
HH	$4/9$
HT	$2/9$
TH	$2/9$
TT	$1/9$

x	$P(x)$
0	$1/9$
1	$4/9$
2	$4/9$

x	$n(x)$
0	1
1	2
2	1

num flips ⟶ n=2
num succ. ⟶ x=1
prob succ ⟶ p=2/3

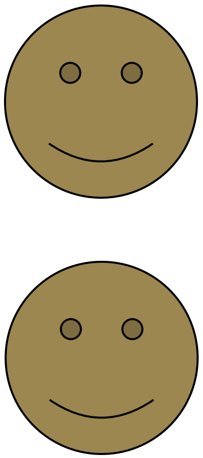
$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$P(1)=$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin twice. p=2/3 x= # of Heads $n(x)$= ways to get x Heads



O	$P(O)$
HH	$4/9$
HT	$2/9$
TH	$2/9$
TT	$1/9$

x	$P(x)$
0	$1/9$
1	$4/9$
2	$4/9$

x	$n(x)$
0	1
1	2
2	1

num flips ⟶ n=2
num succ. ⟶ x=1
prob succ ⟶ p=2/3

$$P(x)=\frac{n!}{x!(n-x)!}p^{x}(1-p)^{n-x}$$

$$P(1)=\frac{2!}{1!(2-1)!}(2/3)^{1}(1-2/3)^{2-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times. p=2/3 x= # of Heads $n(x)$= ways to get x Heads

O	$P(O)$
HHH	$8/27$
HHT	$4/27$
HTH	$4/27$
HTT	$2/27$
THH	$4/27$
THT	$2/27$
TTH	$2/27$
TTT	$1/27$

x	$P(x)$
0	$1/27$
1	$6/27$
2	$12/27$
3	$8/27$

x	$n(x)$
0	1
1	3
2	3
3	1

n=3
x=1
p=2/3

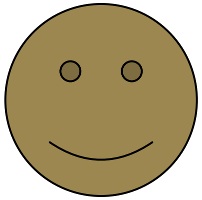
$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

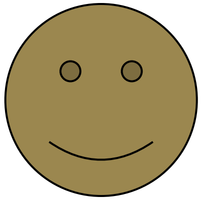
$$P(1) =$$

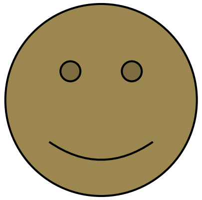
5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Flip coin three times. p=2/3 x= # of Heads $n(x)$= ways to get x Heads







O	$P(O)$
HHH	$8/27$
HHT	$4/27$
HTH	$4/27$
HTT	$2/27$
THH	$4/27$
THT	$2/27$
TTH	$2/27$
TTT	$1/27$

x	$P(x)$
0	$1/27$
1	$6/27$
2	$12/27$
3	$8/27$

x	$n(x)$
0	1
1	3
2	3
3	1

n=3
x=1
p=2/3

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$P(1)=\frac{3!}{1!(3-1)!}\left(2/3\right)^1(1-2/3)^{3-1}$$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Example: n=10, p=0.5

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

These Binomial probabilities can also be found in the back of the book. Table 2 in Appendix B

Please turn to page 713

TABLE 2 **page 713**

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x}\right]$

number of success in n trials

number of trials

prob of success on a trial

n	x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	x
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2
3	0	.970	.857	.729	.512	.343	.216	.125	.064	.027	.008	.001	0+	0+	0
	1	.029	.135	.243	.384	.441	.432	.375	.288	.189	.096	.027	.007	0+	1
	2	0+	.007	.027	.096	.189	.288	.375	.432	.441	.384	.243	.135	.029	2
	3	0+	0+	.001	.008	.027	.064	.125	.216	.343	.512	.729	.857	.970	3
4	0	.961	.815	.656	.410	.240	.130	.062	.026	.008	.002	0+	0+	0+	0
	1	.039	.171	.292	.410	.412	.346	.250	.154	.076	.026	.004	0+	0+	1
	2	.001	.014	.049	.154	.265	.346	.375	.346	.265	.154	.049	.014	.001	2
	3	0+	0+	.004	.026	.076	.154	.250	.346	.412	.410	.292	.171	.039	3
	4	0+	0+	0+	.002	.008	.026	.062	.130	.240	.410	.656	.815	.961	4
⋮															
8	0	.923	.663	.430	.168	.058	.017	.004	.001	0+	0+	0+	0+	0+	0
	1	.075	.279	.383	.336	.198	.090	.031	.008	.001	0+	0+	0+	0+	1
	2	.003	.051	.149	.294	.296	.209	.109	.041	.010	.001	0+	0+	0+	2
	3	0+	.005	.033	.147	.254	.279	.219	.124	.047	.009	0+	0+	0+	3
	4	0+	0+	.005	.046	.136	.232	.273	.232	.136	.046	.005	0+	0+	4
	5	0+	0+	0+	.009	.047	.124	.219	.279	.254	.147	.033	.005	0+	5
	6	0+	0+	0+	.001	.010	.041	.109	.209	.296	.294	.149	.051	.003	6
	7	0+	0+	0+	0+	.001	.008	.031	.090	.198	.336	.383	.279	.075	7
	8	0+	0+	0+	0+	0+	.001	.004	.017	.058	.168	.430	.663	.923	8

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Figure from Johnson & Kuby, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2 page 713

Binomial Probabilities $\left[\binom{n}{x}\cdot p^x \cdot q^{n-x}\right]$

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

		p													
n	x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	x
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2

n=2, *p*=1/2

x	$P(x)$
0	$1/4$
1	$2/4$
2	$1/4$

$$P(0)=\frac{2!}{0!(2-0)!}(1/2)^0(1-1/2)^{2-0}$$

$$P(1)=\frac{2!}{1!(2-1)!}(1/2)^1(1-1/2)^{2-1}$$

$$P(2)=\frac{2!}{2!(2-2)!}(1/2)^2(1-1/2)^{2-2}$$

Figure from Johnson & Kuby, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2 **page 713**

Binomial Probabilities $\left[\binom{n}{x} \cdot p^x \cdot q^{n-x}\right]$ (continued)

n=10, x=7, p=1/2

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Figure from Johnson & Kuby, 2012.

n	x	p: 0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	x
10	0	.904	.599	.349	.107	.028	.006	.001	0+	0+	0+	0+	0+	0+	0
	1	.091	.315	.387	.268	.121	.040	.010	.002	0+	0+	0+	0+	0+	1
	2	.004	.075	.194	.302	.233	.121	.044	.011	.001	0+	0+	0+	0+	2
	3	0+	.010	.057	.201	.267	.215	.117	.042	.009	.001	0+	0+	0+	3
	4	0+	.001	.011	.088	.200	.251	.205	.111	.037	.006	0+	0+	0+	4
	5	0+	0+	.001	.026	.103	.201	.246	.201	.103	.026	.001	0+	0+	5
	6	0+	0+	0+	.006	.037	.111	.205	.251	.200	.088	.011	.001	0+	6
	7	0+	0+	0+	.001	.009	.042	.117	.215	.267	.201	.057	.010	0+	7
	8	0+	0+	0+	0+	.001	.011	.044	.121	.233	.302	.194	.075	.004	8
	9	0+	0+	0+	0+	0+	.002	.010	.040	.121	.268	.387	.315	.091	9
	10	0+	0+	0+	0+	0+	0+	.001	.006	.028	.107	.349	.599	.904	10

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

TABLE 2

Binomial Probabilities $\left[\binom{n}{x}\cdot p^x\cdot q^{n-x}\right]$

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

		p													
n	x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	x
2	0	.980	.902	.810	.640	.490	.360	.250	.160	.090	.040	.010	.002	0+	0
	1	.020	.095	.180	.320	.420	.480	.500	.480	.420	.320	.180	.095	.020	1
	2	0+	.002	.010	.040	.090	.160	.250	.360	.490	.640	.810	.902	.980	2

n=2, p=2/3

x	$P(x)$
0	1/9
1	4/9
2	4/9

p=2/3
.111
.444
.444

Not every p is on the table.

Figure from Johnson & Kuby, 2012.

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2

What is the probability of getting 4, 5, or 6 heads?

$P(4 \le x \le 6) = P(4) + P(5) + P(6)$

$P(4 \le x \le 6) =$

$P(4 \le x \le 6) =$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2
What is the probability of getting 4, 5, or 6 heads?

$P(4\leq x\leq 6)=P(4)+P(5)+P(6)$
$P(4\leq x\leq 6)=210/1024+252/1024+210/1024$
$P(4\leq x\leq 6)=672/1024\approx 0.6123$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2

What is the probability of getting 7 or fewer heads?

$P(x \leq 7) = P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2

What is the probability of getting 7 or fewer heads?

$P(x\leq7)=P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)$

$P(x\leq7)=(1+10+45+120+210+252+210+120)/1024$

$P(x\leq7)=968/1024$

$P(x\leq7)\approx0.9453$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2
What is the probability of getting 7 or fewer heads?

$P(x \leq 7) = 1 - P(x \geq 8)$
$P(x \leq 7) =$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 The Binomial Probability Distribution

Example: n=10, p=1/2
What is the probability of getting 7 or fewer heads?

$P(x \le 7) = 1 - P(x \ge 8)$
$P(x \le 7) = 1 - (45+10+1)/1024$
$P(x \le 7) = 968/1024$
$P(x \le 7) \approx 0.9453$

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

An experiment with two outcomes does not have to be H and T.

More general than H and T, call one *Success* and other *Failure*.

Generally call the one we're interested in the *Success*.

Now that we've established that the formula works.

We can determine the theoretical mean number of heads, μ, and the theoretical variance for the number of heads, σ^2.

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical mean μ of a discrete distribution is:

x_1=0, x_2=1, x_3=2, …

$$\mu = \sum_{i=1}^{n} [x_i P(x_i)]$$

And upon insertion of the Binomial distribution

$$\mu = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \quad (5.7)$$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

The general formula for determining the theoretical variance σ^2 of a discrete distribution is:

x_1=0, x_2=1, x_3=2, …

$$\sigma^2 = \sum_x (x-\mu)^2 P(x)$$

And upon insertion of the Binomial distribution

$$\sigma^2 = \sum_{x=0}^{n} (x-\mu)^2 \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np(1-p) \longrightarrow \sigma = \sqrt{np(1-p)} \qquad (5.8)$$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example:

Before, using $\mu = \sum_{x=0}^{n} [xP(x)]$, we found $\mu = 1$.

n=2
x=1
p=1/2

Now using $\mu = np$, we get $\mu = (2) \cdot (1/2) = 1$.

Before, using $\sigma^2 = \sum_{x=0}^{n} [(x-\mu)^2 P(x)]$, we found $\sigma^2 = 1/2$.

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Now using $\sigma^2 = np(1-p)$,

we get $\sigma^2 = (2) \cdot (1/2) \cdot (1/2) = 1/2$.

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example: n=10, p=1/2

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

What are μ and σ^2?

$$\mu = np =$$

$$\sigma^2 = np(1-p) =$$

5: Probability Distributions (Discrete Variables)

5.3 Mean and Standard Deviation of the Binomial Distribution

Example: n=10, p=1/2

x	0	1	2	3	4	5	6	7	8	9	10
$P(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

What are μ and σ^2?

$$\mu = np = 5$$

$$\sigma^2 = np(1-p) = 2.5$$

5: Probability Distributions (Discrete Variables)

Questions?

Homework: Read Chapter 5.1-5.3
WebAssign
Chapter 5 # 15, 17, 19, 29, 31, 43,
55a,b, 63, 77, 85, 89