Biophysics 230: Nuclear Magnetic Resonance
Haacke Chapter 13

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13: Filtering & Resolution in FT Image Reconstruction

13.1: Review of Fourier Transform Image Reconstruction

Fourier Encoding and Fourier Inversion

It was shown that the signal sampled in time corresponding to a 3D spatially encoded signal is the FT of the effective spin density

$$s(k_x, k_y, k_z) = \int\int\int \rho(x, y, z)e^{-i2\pi k_x x + k_y y + k_z z} dxdydz \quad (13.1)$$

This is called *Fourier encoding*. Reconstruction involves the IFT

$$\rho(x, y, z) = \int\int\int s(k_x, k_y, k_z)e^{i2\pi k_x x + k_y y + k_z z} dxdydz \quad (13.2)$$

Called *Fourier Inversion*.

13: Filtering & Resolution in FT Image Reconstruction

Infinite Sampling and Fourier Series

The was sampled using a doubly infinite sum of δ functions instead of continuous monitoring to yield

$$\hat{\rho}_{\infty}(x) = \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k) e^{i2\pi p\Delta kx} \tag{13.3}$$

In the absence aliasing, the central image $\hat{\rho}_{\infty}(x)$

converges to $\rho(x)$ when no discontinuities are present,

and to $\frac{1}{2}\left(\rho(x_0^+ + \rho(x_0^-))\right)$

when the discontinuity is at x_0.

13: Filtering & Resolution in FT Image Reconstruction

Limited Fourier Imaging and Aliasing

Time constraints limit the number of data samples we can get N_x and N_y.

Reconstruction of non-infinite coverage of k-space is called *limited-Fourier inversion*

$$\hat{\rho}(x) = \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta kx} \tag{13.4}$$

with $2n$ k-space data points spaced Δk apart,
the total coverage is $W = 2n\Delta k$. (13.5)

This reconstructed image is from the finitely (truncated) sampled k-space measurements.

To avoid aliasing need to satisfy Nyquist-Shannon $\Delta k = \frac{1}{L} \leq \frac{1}{A}$. (13.6)
i.e. need $L > A$

13: Filtering & Resolution in FT Image Reconstruction

Signal Series and Spatial Resolution

The other member of the DFT pair is

$$s(k) = \sum_{q=-n}^{n-1} \hat{\rho}(q\Delta x)e^{-i2\pi kq\Delta x} \tag{13.7}$$

with step size being

$$\Delta x = \frac{L}{N} = \frac{1}{N\Delta k} = \frac{1}{W} \tag{13.8}$$

This is how we get our voxel size (resolution).

Larger W (cover more spatial frequencies), the smaller Δx (our voxel size)!

Don't forget that $\Delta k = ⩚\, G_R\Delta t$ meaning $\Delta x = \frac{1}{N\Delta k} = \frac{1}{2n⩚ G_R\Delta t}$.

For fixed Δt, increase G_R to increase Δk and decrease Δx.

13.2: Filters and Point Spread Functions

Define any filter as a function $H(k)$ that multiplies the k-space data.

Then the IFT of $H(k)$, $h(x)$ is called the point spread function of $H(k)$.

The reconstructed image $\hat{\rho}(x)$ is $\hat{\rho}(x) = \rho(x) * h(x)$.

The point spread finction is from observing the spread that occurs when we have a point source (image).

That is, $\rho(x) = \delta(x)$.

And thus $\hat{\rho}(x) = \delta(x) * h(x) = h(x)$

because $f(x) * \delta(x - x_0) = f(x - x_0)$.

13.2: Filters and Point Spread Functions

Point Spread Due to Truncation (Windowing)

Truncation is the same as multiplying $s(k)$ by the rect function

$$H_w(k) \equiv \text{rect}\left(\frac{k + \frac{1}{2}\Delta k}{W}\right). \tag{13.9}$$

If we were only windowing so that

$$s_w(k) = s(k) \cdot H_w(k) \tag{13.10}$$

then the reconstructed image is

$$\begin{aligned} \hat{\rho}(x) &= \mathcal{F}^{-1}[s(k) \cdot H_w(k)] \\ &= \rho(x) * h_w(x) \end{aligned} \tag{13.11}$$

where

$$h_w(x) = W \text{sinc}(\pi W x) e^{-i2\pi \frac{\Delta k}{2} x} \tag{13.12}$$

This FT pair was already discussed in Chapter 12.

So the effect of windowing with $\Delta k/2$ rect is to blur/smooth/convolve each point to other locations by (13.12).

13.2: Filters and Point Spread Functions

Point Spread for Truncation and Sampled Data

Truncation and sampling is modeled as

$$\begin{aligned} H_{ws}(k) &\equiv \Delta k \mathrm{rect}\left(\frac{k+\frac{1}{2}\Delta k}{W}\right)\sum_{p=-\infty}^{\infty}\delta(k-p\Delta k) \\ &= \Delta k \sum_{p=-n}^{n-1}\delta(k-p\Delta k). \end{aligned} \tag{13.13}$$

and so the measured data is

$$\begin{aligned} s_m(k) &= s_{ws}(k) \\ &= s(k)\cdot H_{ws}(k) \\ &= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k)\delta(k-p\Delta k) \end{aligned} \tag{13.14}$$

13.2: Filters and Point Spread Functions

From (13.13) and (13.14) the IFT yields the reconstructed image

$$\begin{aligned}
\hat{\rho}(x) &\equiv \hat{\rho}_{ws}(x) \\
&= \int_{k=-\infty}^{\infty} [s_m(k)] e^{i2\pi kx} dk \\
&= \int_{k=-\infty}^{\infty} \left[\Delta k \sum_{p=-n}^{n-1} s(p\Delta k)\delta(k-p\Delta k) \right] e^{i2\pi kx} dk \\
&= \Delta k \sum_{p=-n}^{n-1} \int_{k=-\infty}^{\infty} s(p\Delta k)\delta(k-p\Delta k) e^{i2\pi kx} dk \\
&= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta k x}
\end{aligned} \tag{13.15}$$

13.2: Filters and Point Spread Functions

Alternatively, can view as convolution

$$\hat{\rho}(x) = \rho(x) * h_{ws}(x) \tag{13.16}$$

where

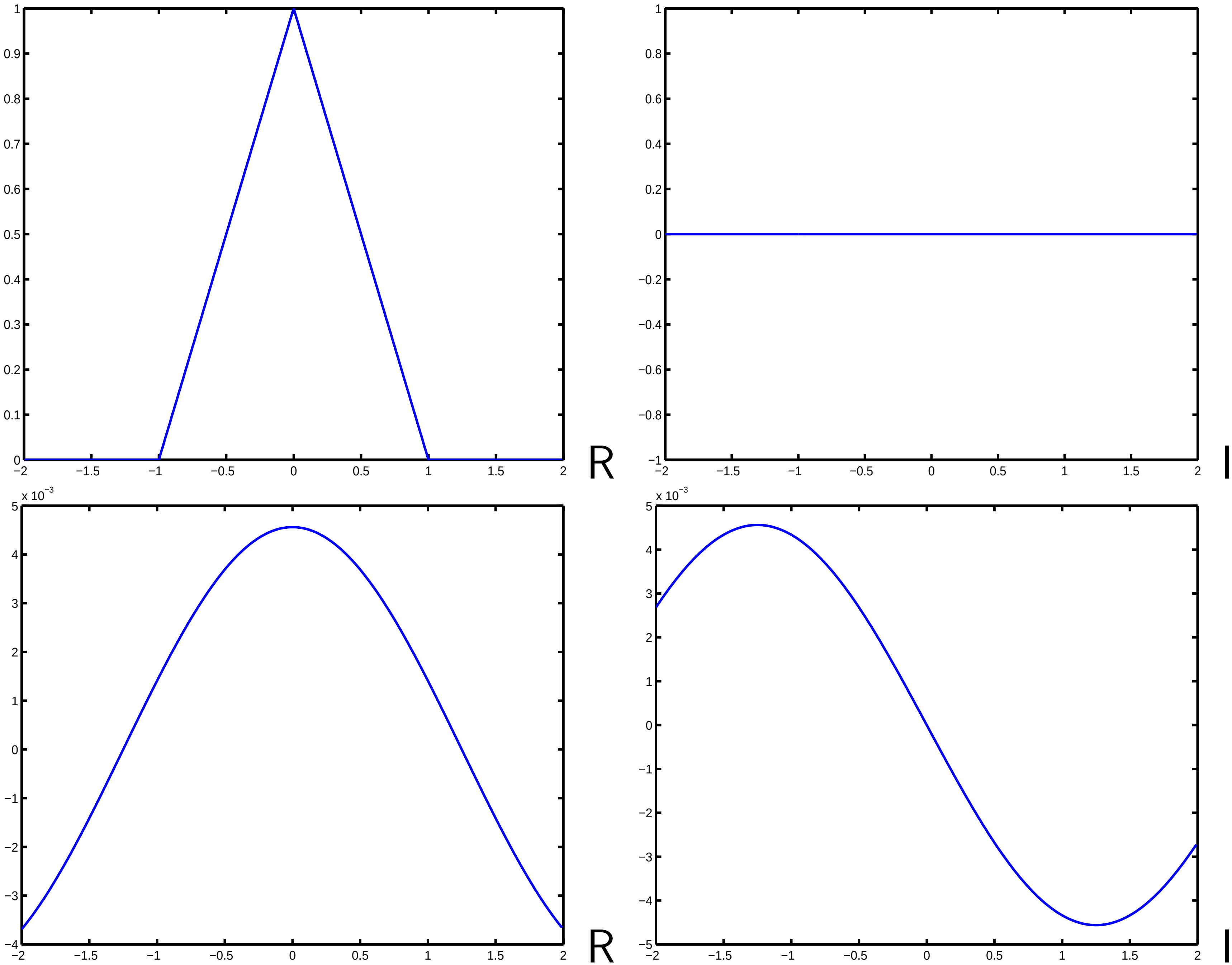
$$\begin{aligned} h_{ws}(x) &= \mathcal{F}^{-1}\{H_{ws}(k)\} \\ &= \Delta k \sum_{p=-n}^{n-1} e^{-i2\pi p \Delta k x} \end{aligned} \tag{13.17}$$

and "it can be shown," (see Problem 13.2) that

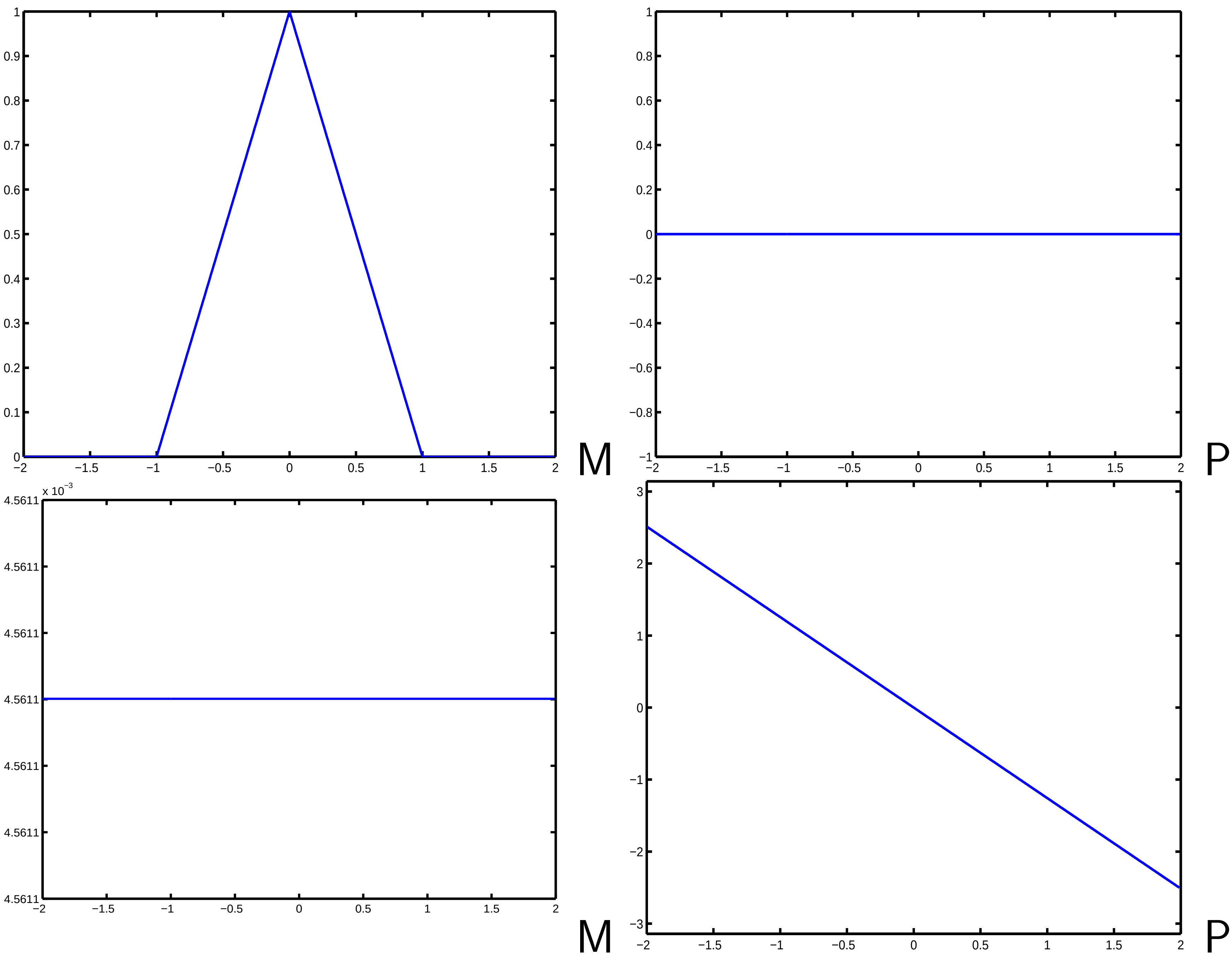
$$h_{ws}(x) = W \frac{\text{sinc}(\pi W x)}{\text{sinc}(\pi \Delta k x)} e^{-i2\pi \frac{\Delta k}{2} x} \tag{13.18}$$

So the effect of windowing with $\Delta k/2$ rect and sampling
is to blur/smooth/convolve each point to other locations by (13.18).

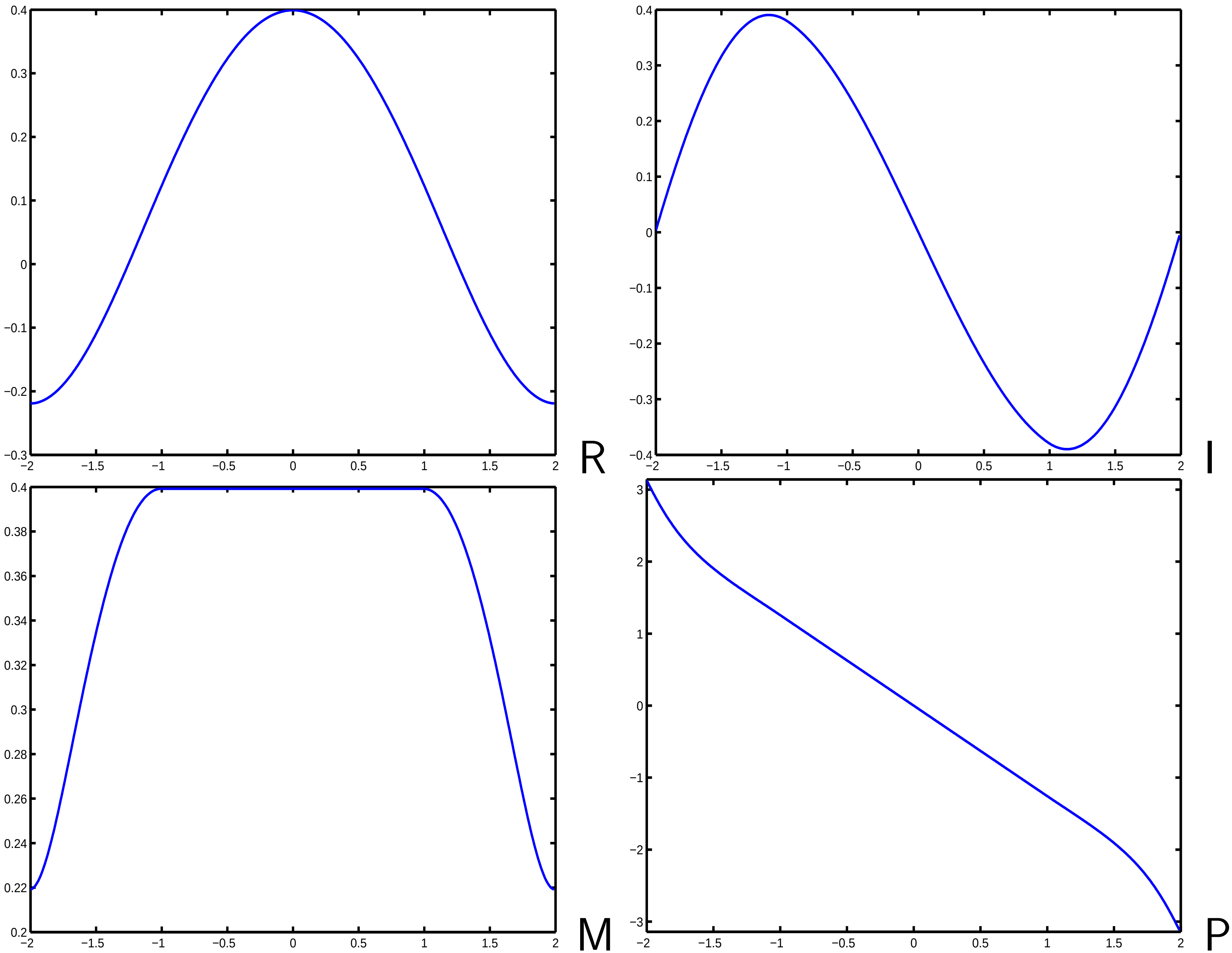
Example: Triangle $A = 2$, $L = 2.5$, $\Delta k = 1/L = .4$, $W = 64\Delta k = 25.6$



Example: Triangle $A = 2$, $L = 2.5$, $\Delta k = 1/L = .4$, $W = 64\Delta k = 25.6$



Example: Triangle $A = 2$, $L = 2.5$, $\Delta k = 1/L = .4$, $W = 64\Delta k = 25.6$



13.2: Filters and Point Spread Functions

Point Spread and Additional Filters

Other factors (the Physics of relaxation T_2 and T_2^*) modify the signal $s(k)$ and result in additional blurring.

These other factors can be modeled as filters $H_{filter}(k)$

The more accurate windowed (with $\Delta k/2$ shift),
sampled (doubly infinite δ's), and other filters can be described as

$$\begin{aligned} \hat{s}_{m,filter}(k) &= s_m(k) \cdot H_{filter}(k) \\ &= s(k) \cdot H_{ws}(k) \cdot H_{filter}(k) \\ &= s(k) \cdot H_{ws,filter}(k) \end{aligned} \tag{13.20}$$

13.2: Filters and Point Spread Functions

The more accurate reconstructed image is

$$\begin{aligned}\hat{\rho}_{m,filter}(h) &= \hat{\rho}(x) * h_{ws}(x) * h_{filter}(x) \\ &= \rho(x) * h_{ws,filter}(h)\end{aligned} \tag{13.21}$$

So by taking into account the physical exponential decay (T_2 and T_2^*)

along with the effect of windowing with $\Delta k/2$ rect and sampling

is to add more blur/smooth/convolve of each point to other locations by (13.21).

13.3: Gibbs Ringing

Gibbs Overshoot and Undershoot

We have been representing (decomposing) functions as (into) sinusoids.

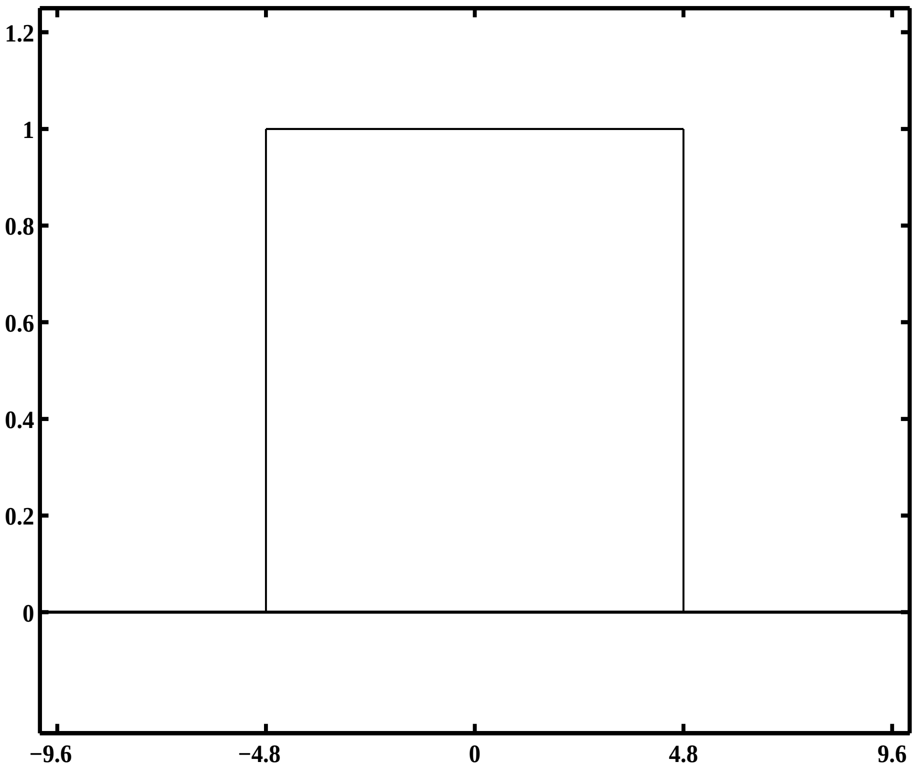
When the functions are smooth this works very well.

When the functions have a very sharp jump (discontinuity), then this does not work well.

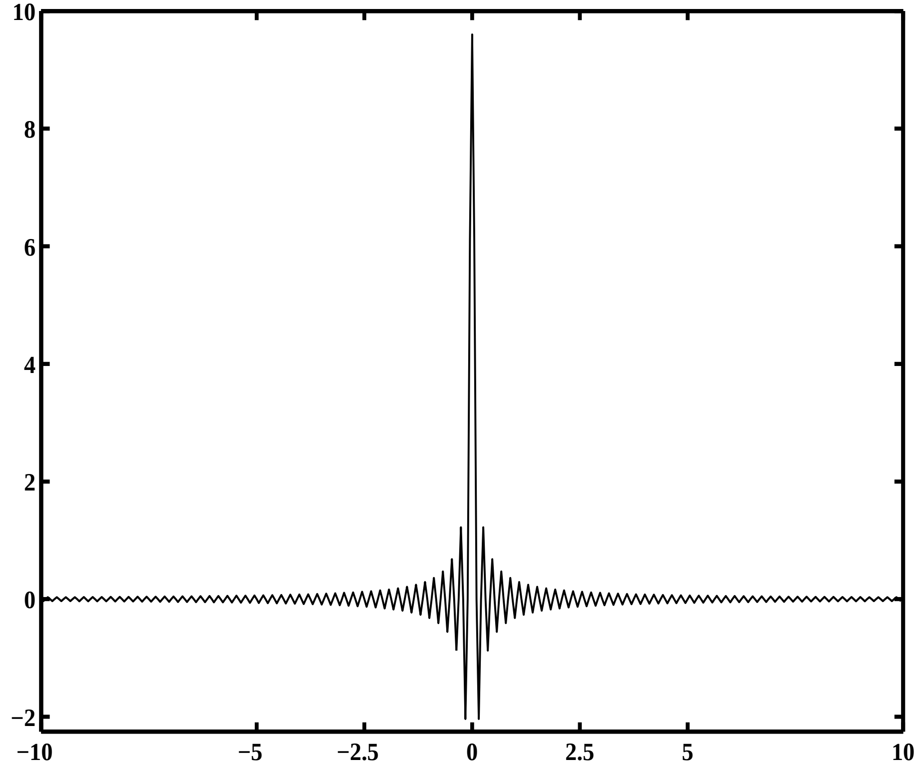
The limiting over and under shoot is approx 9% of the height.

Let's look at an example of a square wave.

13.3: Gibbs Ringing: Example



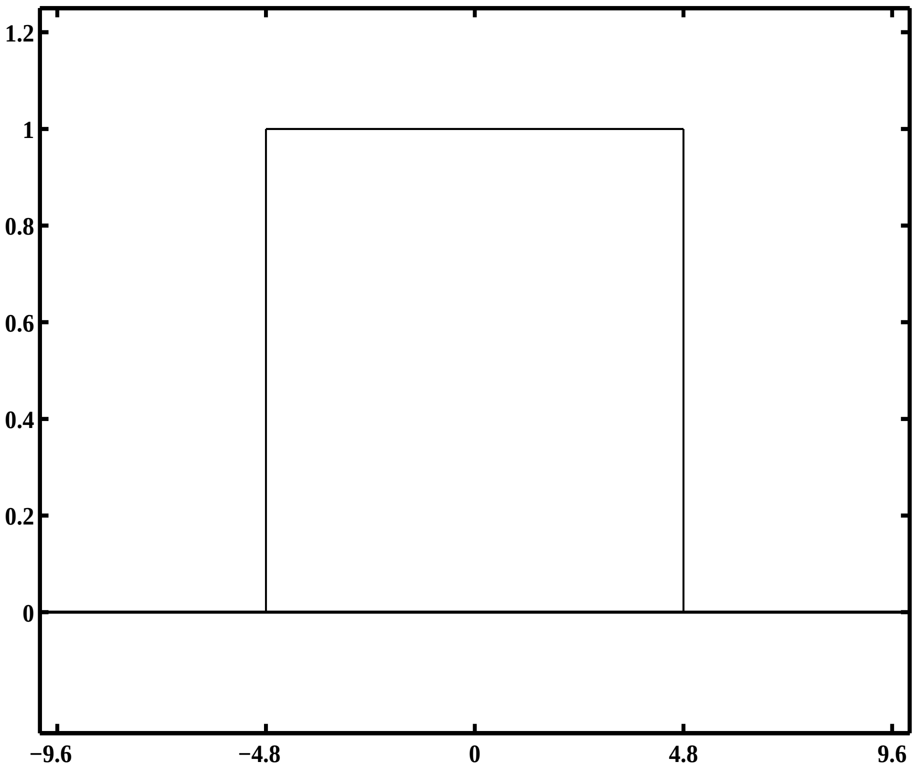
$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right)$, $A = 9.6cm$

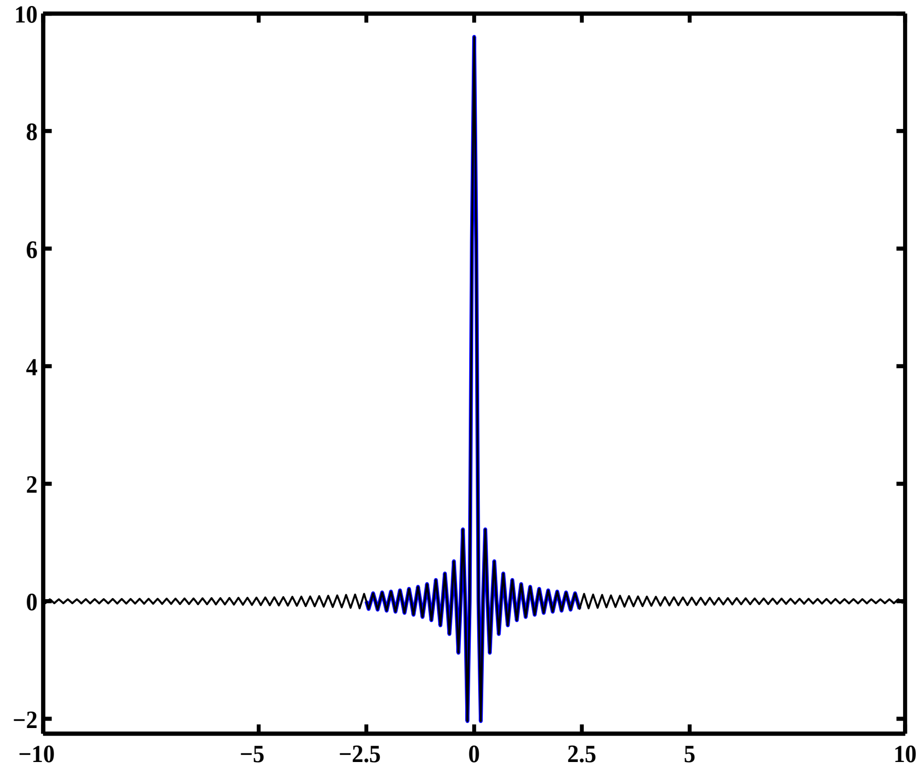


$s(k) = 9.6 \,\text{sinc}\,(\pi 9.6 k)$

Exact rect function on left (black). Exact sinc on right (black).

13.3: Gibbs Ringing: Example



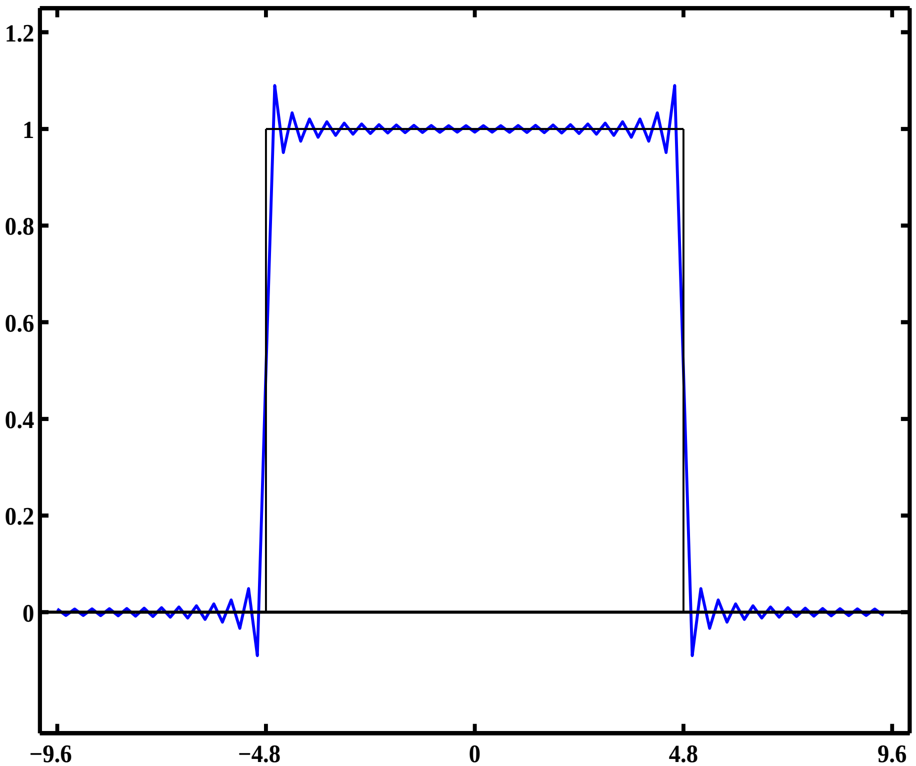


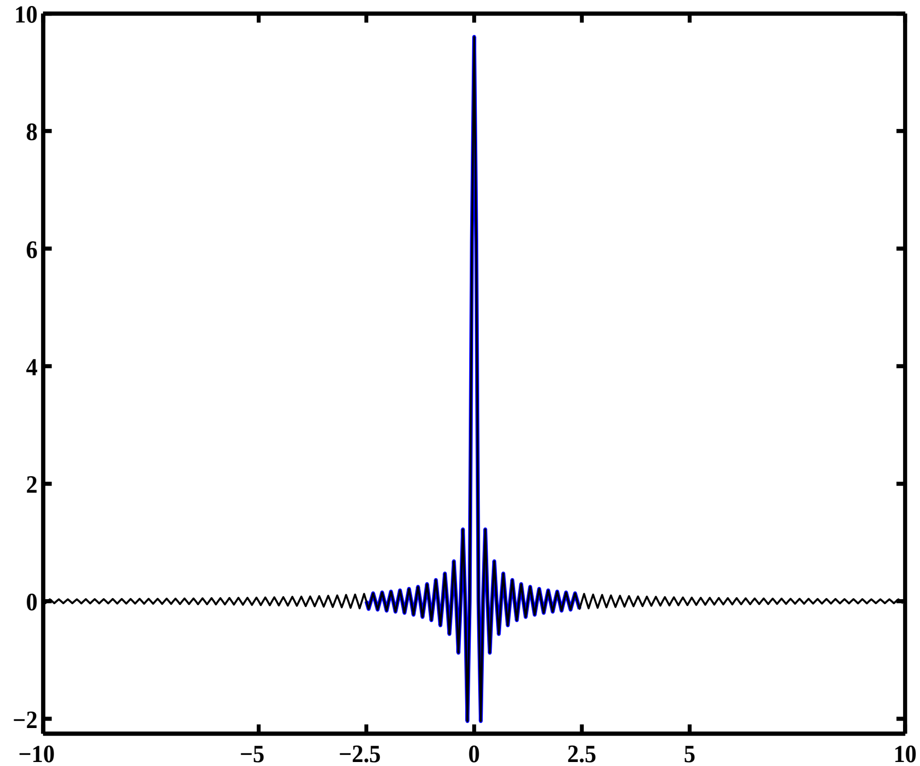
$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right)$, $A = 9.6cm$
$L = 19.2cm$

$s(k) = 9.6 \text{ sinc}\left(\pi 9.6 k\right)$
$\Delta k = 1/L = 1/19.2, \quad N = 96$

Exact rect function on left (black). Exact sinc on right (black).
Truncated sinc on right (blue).

13.3: Gibbs Ringing: Example





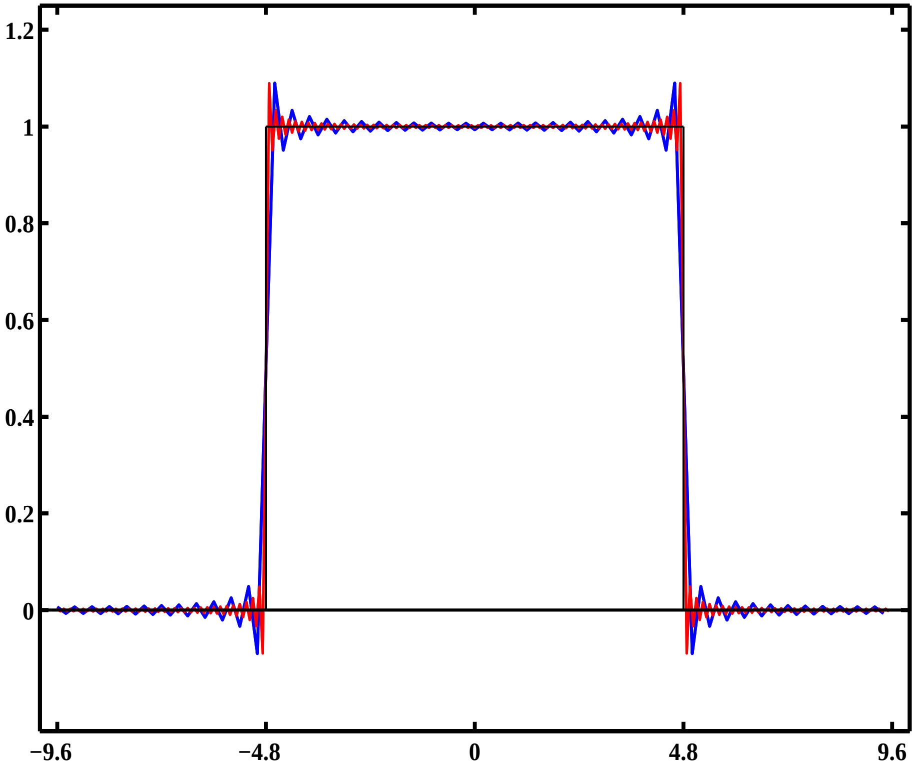
$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right)$, $A = 9.6cm$ $\quad$ $s(k) = 9.6\ \text{sinc}\left(\pi 9.6 k\right)$

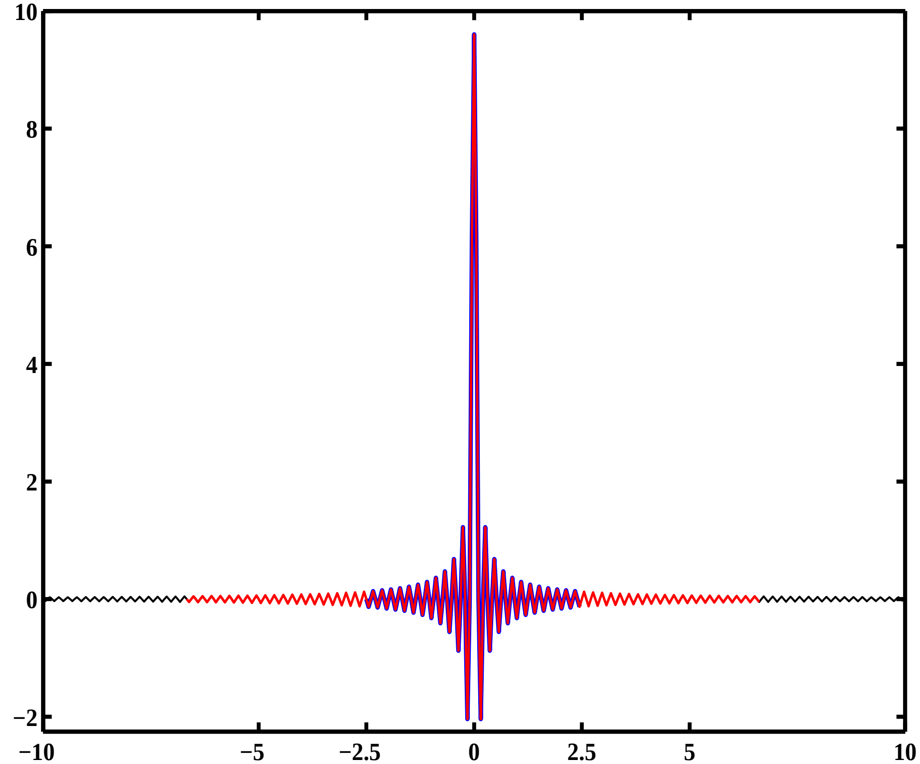
$L = 19.2cm$, $\Delta x = L/N = .2cm$ $\quad$ $\Delta k = 1/L = 1/19.2, \quad N = 96$

Exact rect function on left (black). Exact sinc on right (black).
Rect from truncated sinc left (blue). Truncated sinc on right (blue).

When we have a sharp discontinuity in MRI from head to space or between brain structures we can get Gibbs ringing!

13.3: Gibbs Ringing: Example





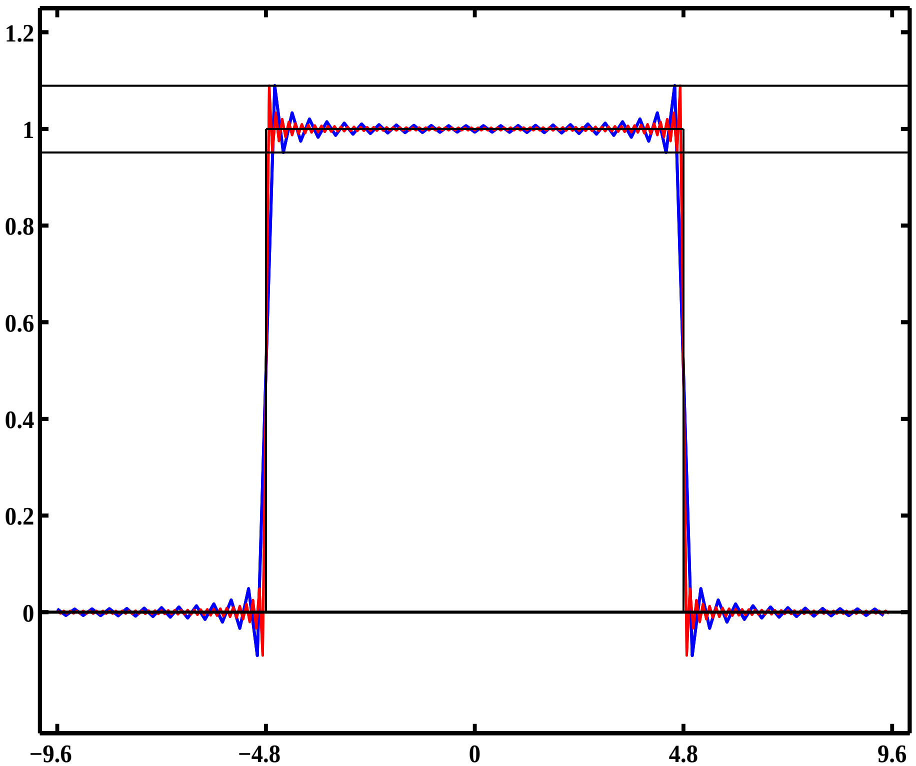
$\rho(x) = \mathrm{rect}\left(\frac{x}{9.6}\right)$, $A = 9.6cm$ $\quad s(k) = 9.6 \ \mathrm{sinc}\,(\pi 9.6k)$,

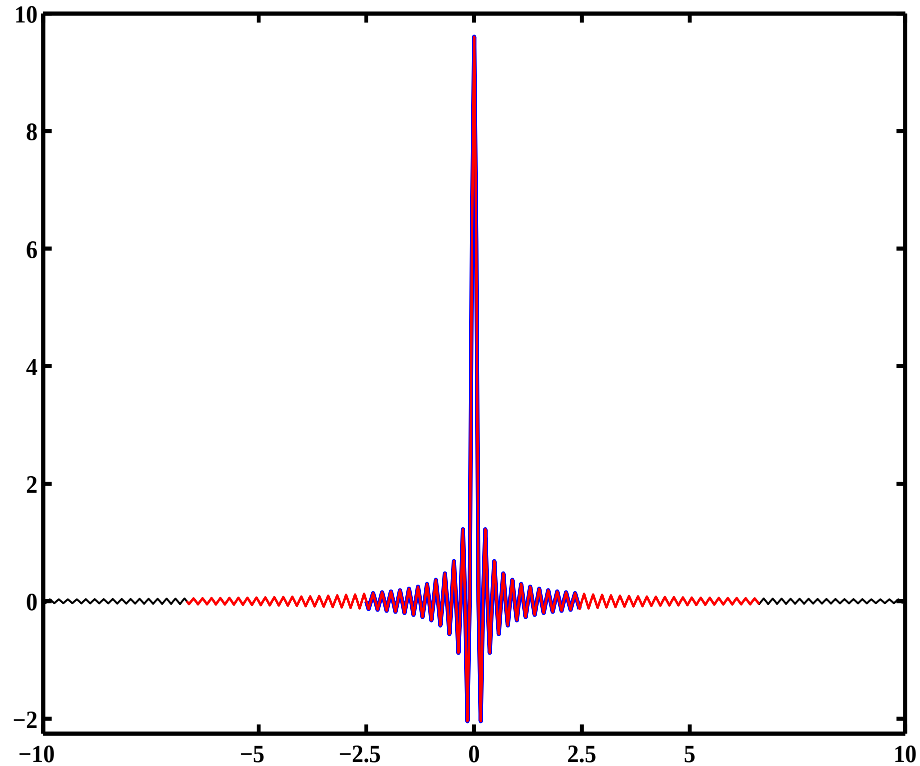
$L = 19.2cm$, $\Delta x = L/N = .075cm$ $\quad \Delta k = 1/L = 1/19.2$, $\quad N = 256$

Exact rect function on left (black). Exact sinc on right (black).
Rect from truncated sinc left (blue). Truncated sinc on right (blue).
Rect from less truncated sinc left (red). Less truncated sinc on right (red).

Less Gibbs ringing the more frequencies we sample!

13.3: Gibbs Ringing: Example





$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right)$, $A = 9.6cm$ $s(k) = 9.6 \text{ sinc}(\pi 9.6k)$,

$L = 19.2cm$, $\Delta x = L/N = .075cm$ $\Delta k = 1/L = 1/19.2$, $N = 256$

Note that the red reconstructed image is sharper to the discontinuity

But the height is still the same. Red image numbers.

-4.8000	0.5000
-4.7250	1.0895
-4.6500	0.9514
-4.5750	1.0331
-4.5000	0.9749

$(1.0895 - 1)/1 \cdot 100\% \approx 9\%$ overshoot!

13.3: Gibbs Ringing

Gibbs Oscillation Frequency

As we sample at same minimum Δk but further out,
the over/under shoot gets closer to the discontinuity.

Going further out does not affect the height.

If we sample twice as far out in k-space,
overshoot is half as far from discontinuity.

13.3: Gibbs Ringing

Reducing Gibbs Ringing by Filtering

If the data is multiplied by a function that vanishes, along with its first derivative at $k = \pm k_{max}$, it is said to have been *apodized*.

One apodizing filter that is used is the *Hanning filter*.

$$H_{Hanning}(k) = \frac{1 + \cos\left(\frac{2\pi k}{W}\right)}{2} = \cos^2\left(\frac{\pi k}{W}\right) \qquad (13.33)$$

The IFT of the Hanning is

$$h_{Hanning}(x) = \frac{1}{4}\delta(x - \Delta x) + \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x + \Delta x) \qquad (13.34)$$

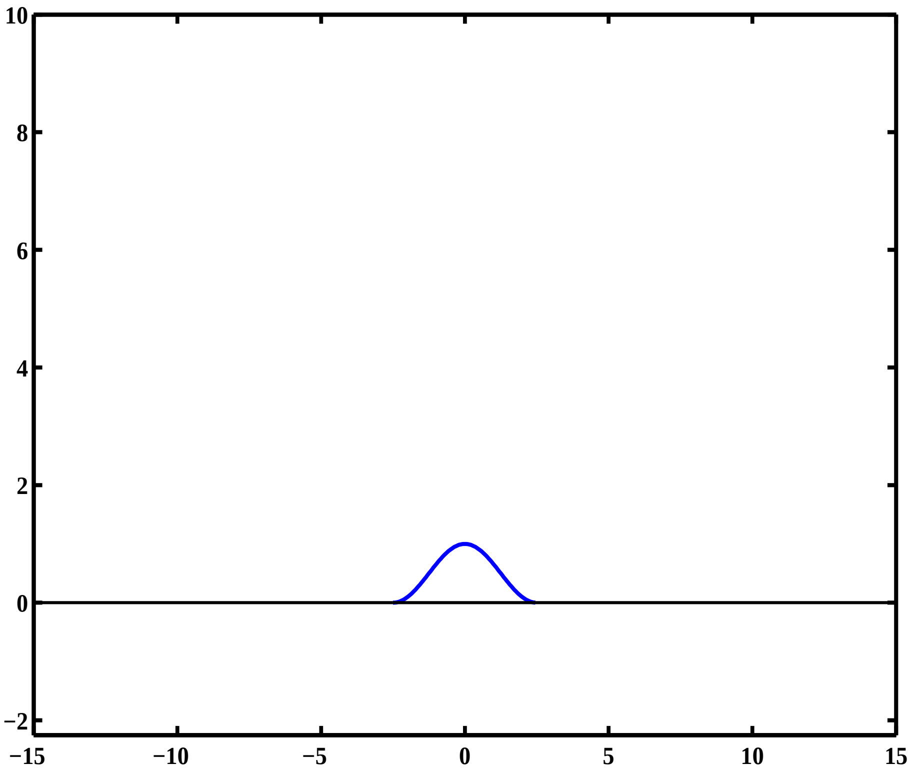
13.3: Gibbs Ringing

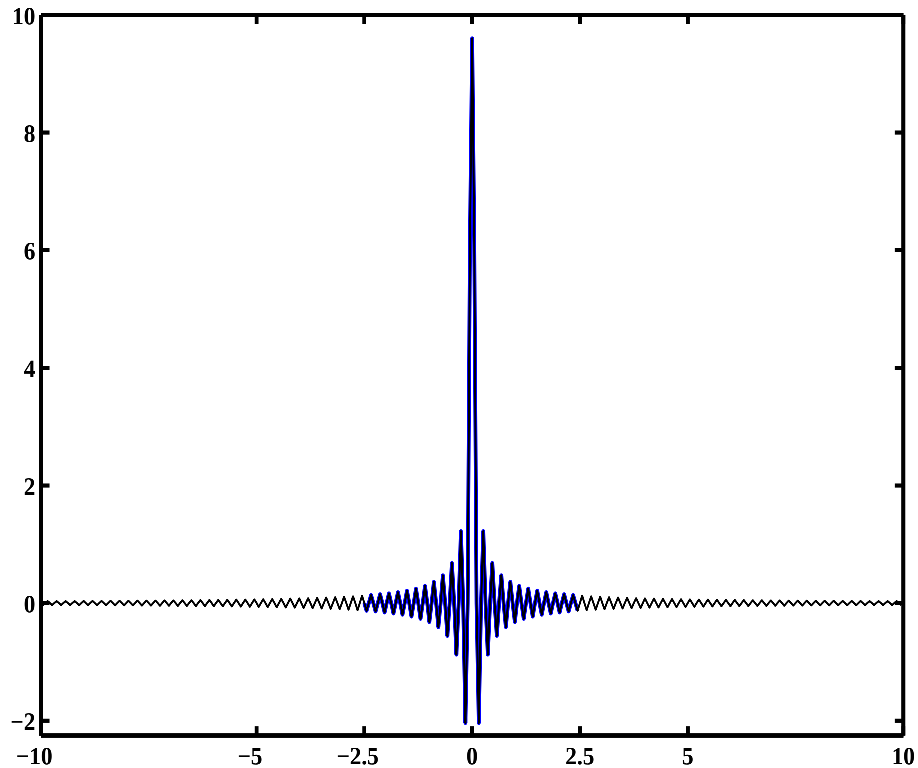
The corresponding reconstructed spin density is The IFT of the Hanning is

$$\begin{aligned}\hat{\rho}_{Hanning}(x) &= h_{Hanning}(x) * \hat{\rho}(x) \\ &= \frac{1}{4}\hat{\rho}(x-\Delta x) + \frac{1}{2}\hat{\rho}(x) + \frac{1}{4}\hat{\rho}(x+\Delta x) \qquad (13.35)\end{aligned}$$

The k-space hanning filter corresponds to an averaging in image space.

13.3: Gibbs Ringing



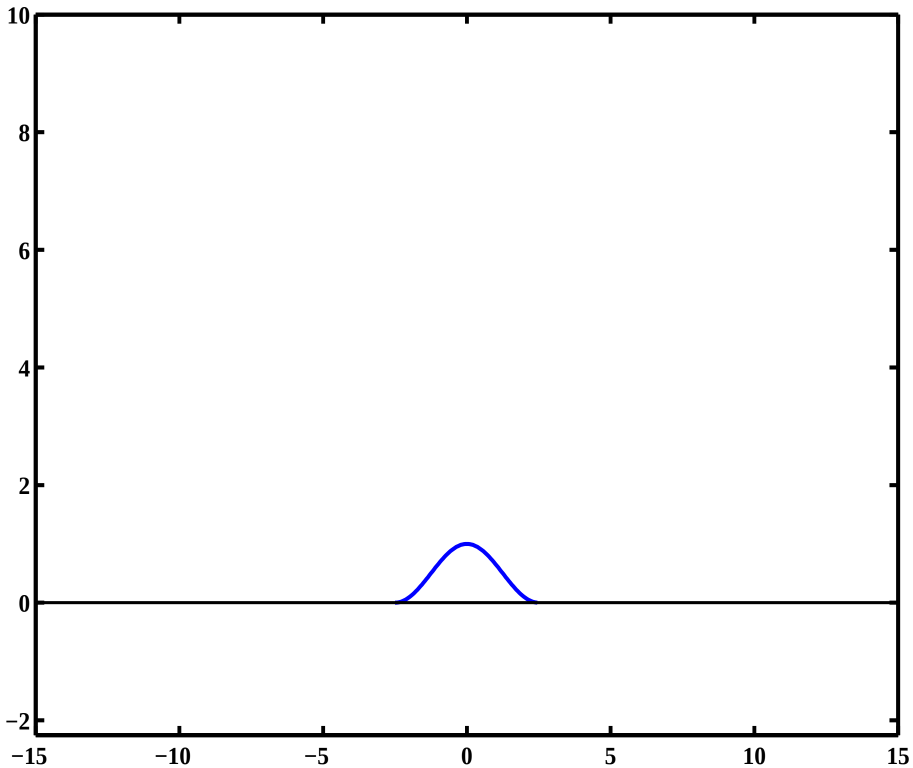


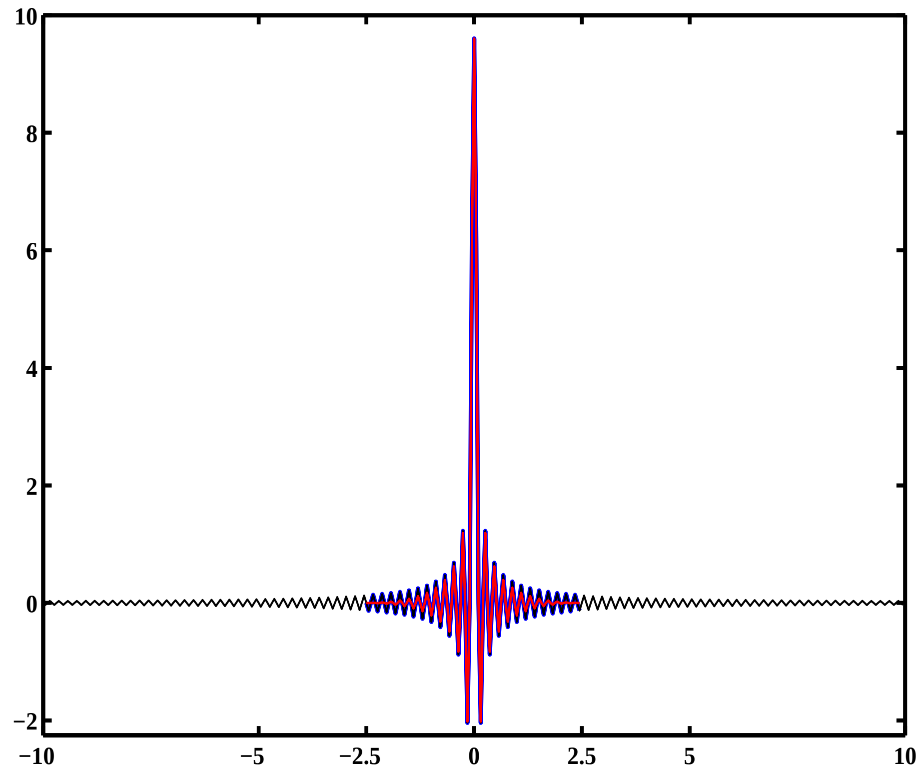
$$H_{Hanning}(k) = \cos^2(\pi k/W)$$
$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

$$s(k) = 9.6 \,\text{sinc}\,(\pi 9.6 k),$$
$$\Delta k = 1/L = 1/19.2$$

Hanning filter on left (blue). Exact sinc on right (black).
Truncated sinc on right (blue).

13.3: Gibbs Ringing



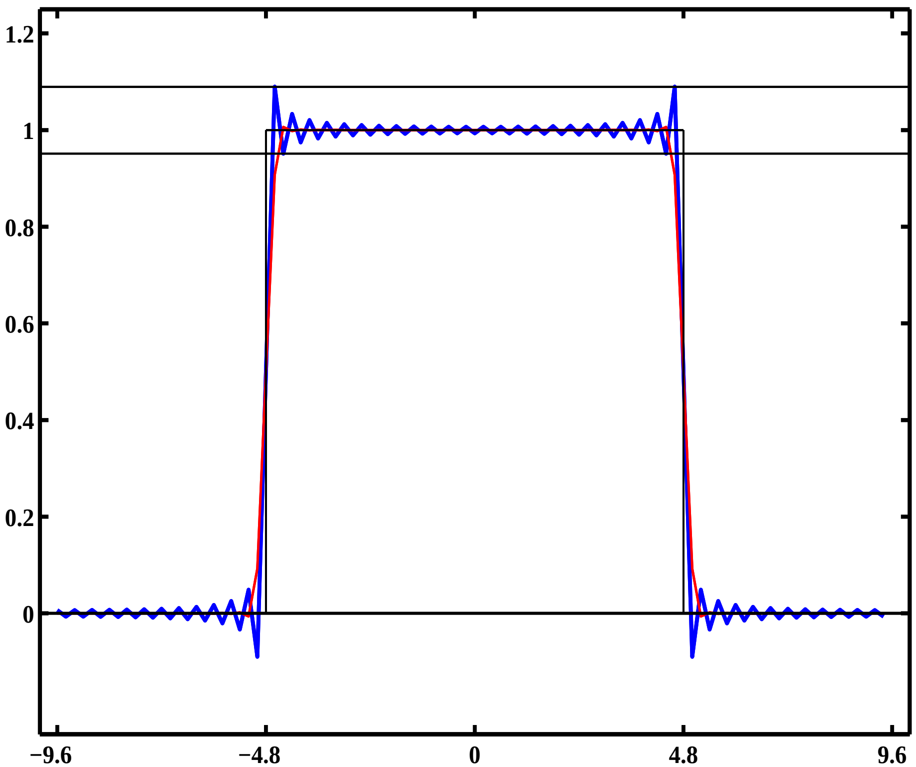


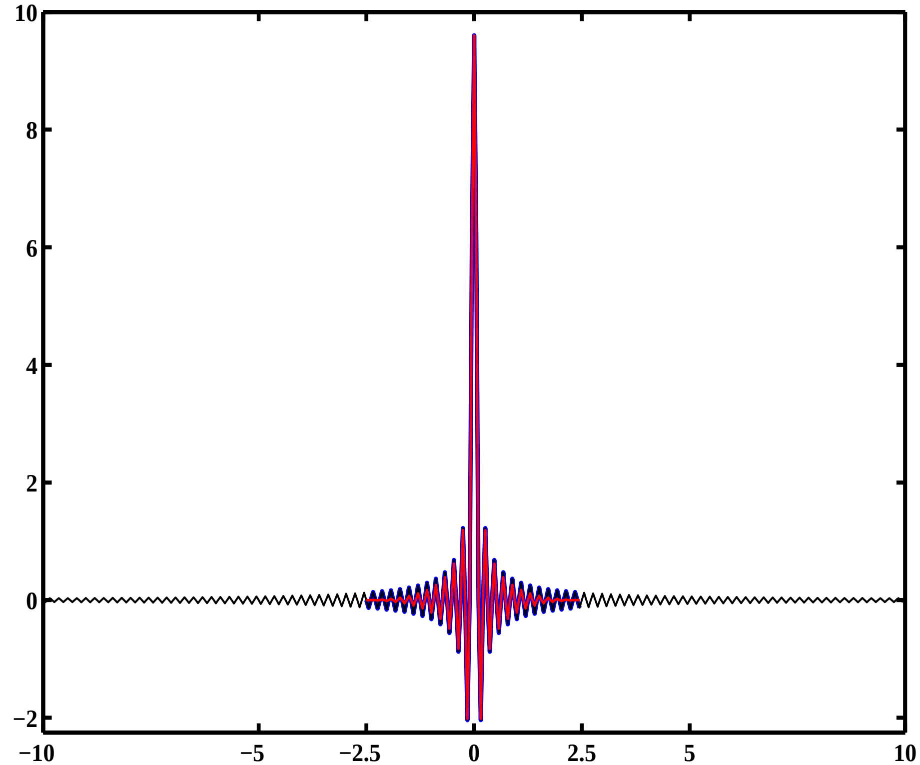
$$H_{Hanning}(k) = \cos^2(\pi k/W)$$
$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

$$s(k) = 9.6\ \mathrm{sinc}\,(\pi 9.6 k),$$
$$\Delta k = 1/L = 1/19.2$$

Hanning filter on left (blue). Exact sinc on right (black).
Truncated sinc on right (blue).
Truncated sinc multiplied by
Hanning filter on right (red).

13.3: Gibbs Ringing





$$H_{Hanning}(k) = \cos^2(\pi k/W)$$
$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

$$s(k) = 9.6 \text{ sinc}(\pi 9.6 k),$$
$$\Delta k = 1/L = 1/19.2$$

Exact rect on left (black).
Rect from truncated sinc on left (blue).
Rect from truncated sinc multiplied by Hanning filter on left (red).

Exact sinc on right (black).
Truncated sinc on right (blue).
Truncated sinc multiplied by Hanning filter on right (red).

Note reduced Gibbs ringing from hanning smoothing, called apodization.

Read Sections

13.4: Spatial Resolution in MRI

13.5: Filtering Due to T_2 and T_2^* Decay

13.6: Zero Filled Interpolation, Sub-Voxel Fourier Transform Shift Concepts and Point Spread Function Effects

13.7: Partial Fourier Imaging and Reconstruction

13.8: Digital Truncation

Homework

Do 13.1, 13.2, 13.3, 13.4