

# Biophysics 230: Nuclear Magnetic Resonance Haacke Chapter 13

Daniel B. Rowe, Ph.D.

daniel.rowe@marquette.edu  
Department of Math, Stat, Comp Sci  
Marquette University

dbrowe@mcw.edu  
Department of Biophysics  
Medical College of Wisconsin

## 13: Filtering & Resolution in FT Image Reconstruction

### 13.1: Review of Fourier Transform Image Reconstruction Fourier Encoding and Fourier Inversion

It was shown that the signal sampled in time corresponding to a 3D spatially encoded signal is the FT of the effective spin density

$$s(k_x, k_y, k_z) = \int \int \int \rho(x, y, z) e^{-i2\pi k_x x + k_y y + k_z z} dx dy dz \quad (13.1)$$

This is called *Fourier encoding*. Reconstruction involves the IFT

$$\rho(x, y, z) = \int \int \int s(k_x, k_y, k_z) e^{i2\pi k_x x + k_y y + k_z z} dx dy dz \quad (13.2)$$

Called *Fourier Inversion*.

## 13: Filtering & Resolution in FT Image Reconstruction

### Infinite Sampling and Fourier Series

The was sampled using a doubly infinite sum of  $\delta$  functions instead of continuous monitoring to yield

$$\hat{\rho}_{\infty}(x) = \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k) e^{i2\pi p\Delta k x} \quad (13.3)$$

In the absence aliasing, the central image  $\hat{\rho}_{\infty}(x)$

converges to  $\rho(x)$  when no discontinuities are present,

and to  $\frac{1}{2} (\rho(x_0^+) + \rho(x_0^-))$

when the discontinuity is at  $x_0$ .

## 13: Filtering & Resolution in FT Image Reconstruction

### Limited Fourier Imaging and Aliasing

Time constraints limit the number of data samples we can get  $N_x$  and  $N_y$ .

Reconstruction of non-infinite coverage of  $k$ -space is called *limited-Fourier inversion*

$$\hat{\rho}(x) = \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta k x} \quad (13.4)$$

with  $2n$   $k$ -space data points spaced  $\Delta k$  apart,  
the total coverage is  $W = 2n\Delta k$ . (13.5)

This reconstructed image is from the finitely (truncated) sampled  $k$ -space measurements.

To avoid aliasing need to satisfy Nyquist-Shannon  $\Delta k = \frac{1}{L} \leq \frac{1}{A}$ . (13.6)  
i.e. need  $L > A$

## 13: Filtering & Resolution in FT Image Reconstruction

### Signal Series and Spatial Resolution

The other member of the DFT pair is

$$s(k) = \sum_{q=-n}^{n-1} \hat{\rho}(q\Delta x) e^{-i2\pi kq\Delta x} \quad (13.7)$$

with step size being

$$\Delta x = \frac{L}{N} = \frac{1}{N\Delta k} = \frac{1}{W} \quad (13.8)$$

This is how we get our voxel size (resolution).

Larger  $W$  (cover more spatial frequencies), the smaller  $\Delta x$  (our voxel size)!

Don't forget that  $\Delta k = \gamma G_R \Delta t$  meaning  $\Delta x = \frac{1}{N\Delta k} = \frac{1}{2n\gamma G_R \Delta t}$ .

For fixed  $\Delta t$ , increase  $G_R$  to increase  $\Delta k$  and decrease  $\Delta x$ .

## 13.2: Filters and Point Spread Functions

Define any filter as a function  $H(k)$  that multiplies the  $k$ -space data.

Then the IFT of  $H(k)$ ,  $h(x)$  is called the point spread function of  $H(k)$ .

The reconstructed image  $\hat{\rho}(x)$  is  $\hat{\rho}(x) = \rho(x) * h(x)$ .

The point spread function is from observing the spread that occurs when we have a point source (image).

That is,  $\rho(x) = \delta(x)$ .

And thus  $\hat{\rho}(x) = \delta(x) * h(x) = h(x)$

because  $f(x) * \delta(x - x_0) = f(x - x_0)$ .

## 13.2: Filters and Point Spread Functions

### Point Spread Due to Truncation (Windowing)

Truncation is the same as multiplying  $s(k)$  by the rect function

$$H_w(k) \equiv \text{rect} \left( \frac{k + \frac{1}{2}\Delta k}{W} \right). \quad (13.9)$$

If we were only windowing so that

$$s_w(k) = s(k) \cdot H_w(k) \quad (13.10)$$

then the reconstructed image is

$$\begin{aligned} \hat{\rho}(x) &= \mathcal{F}^{-1}[s(k) \cdot H_w(k)] \\ &= \rho(x) * h_w(x) \end{aligned} \quad (13.11)$$

where

$$h_w(x) = W \text{sinc}(\pi W x) e^{-i2\pi \frac{\Delta k}{2} x} \quad (13.12)$$

This FT pair was already discussed in Chapter 12.

So the effect of windowing with  $\Delta k/2$  rect is to blur/smooth/convolve each point to other locations by (13.12).

## 13.2: Filters and Point Spread Functions

### Point Spread for Truncation and Sampled Data

Truncation and sampling is modeled as

$$\begin{aligned}
 H_{ws}(k) &\equiv \Delta k \operatorname{rect} \left( \frac{k + \frac{1}{2}\Delta k}{W} \right) \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \\
 &= \Delta k \sum_{p=-n}^{n-1} \delta(k - p\Delta k).
 \end{aligned} \tag{13.13}$$

and so the measured data is

$$\begin{aligned}
 s_m(k) &= s_{ws}(k) \\
 &= s(k) \cdot H_{ws}(k) \\
 &= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) \delta(k - p\Delta k)
 \end{aligned} \tag{13.14}$$



## 13.2: Filters and Point Spread Functions

From (13.13) and (13.14) the IFT yields the reconstructed image

$$\begin{aligned}
 \hat{\rho}(x) &\equiv \hat{\rho}_{ws}(x) \\
 &= \int_{k=-\infty}^{\infty} [s_m(k)] e^{i2\pi kx} dk \\
 &= \int_{k=-\infty}^{\infty} \left[ \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) \delta(k - p\Delta k) \right] e^{i2\pi kx} dk \\
 &= \Delta k \sum_{p=-n}^{n-1} \int_{k=-\infty}^{\infty} s(p\Delta k) \delta(k - p\Delta k) e^{i2\pi kx} dk \\
 &= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta kx} \tag{13.15}
 \end{aligned}$$

## 13.2: Filters and Point Spread Functions

Alternatively, can view as convolution

$$\hat{\rho}(x) = \rho(x) * h_{ws}(x) \quad (13.16)$$

where

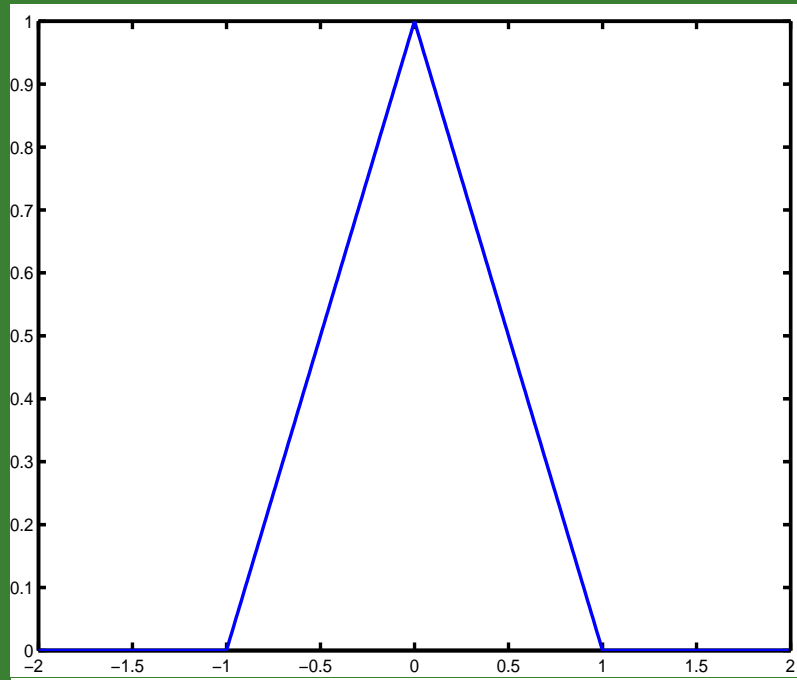
$$\begin{aligned} h_{ws}(x) &= \mathcal{F}^{-1} \{H_{ws}(k)\} \\ &= \Delta k \sum_{p=-n}^{n-1} e^{-i2\pi p \Delta k x} \end{aligned} \quad (13.17)$$

and “it can be shown,” (see Problem 13.2) that

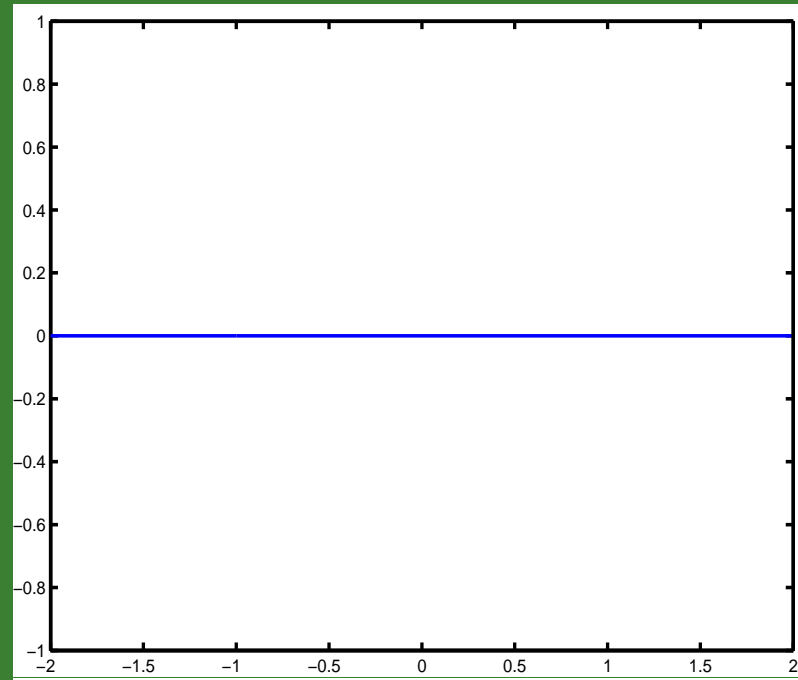
$$h_{ws}(x) = W \frac{\text{sinc}(\pi W x)}{\text{sinc}(\pi \Delta k x)} e^{-i2\pi \frac{\Delta k}{2} x} \quad (13.18)$$

So the effect of windowing with  $\Delta k/2$  rect and sampling is to blur/smooth/convolve each point to other locations by (13.18).

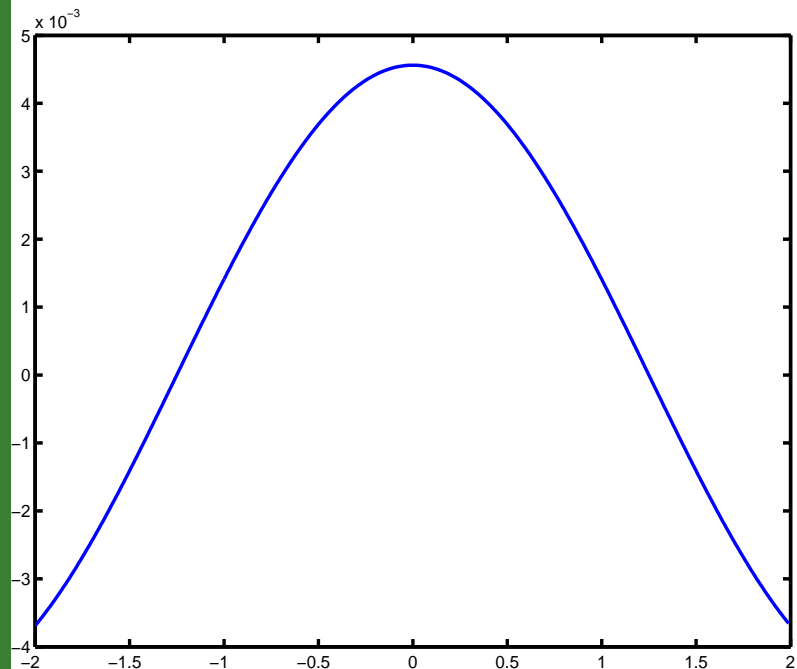
**Example: Triangle**  $A = 2$ ,  $L = 2.5$ ,  $\Delta k = 1/L = .4$ ,  $W = 64\Delta k = 25.6$



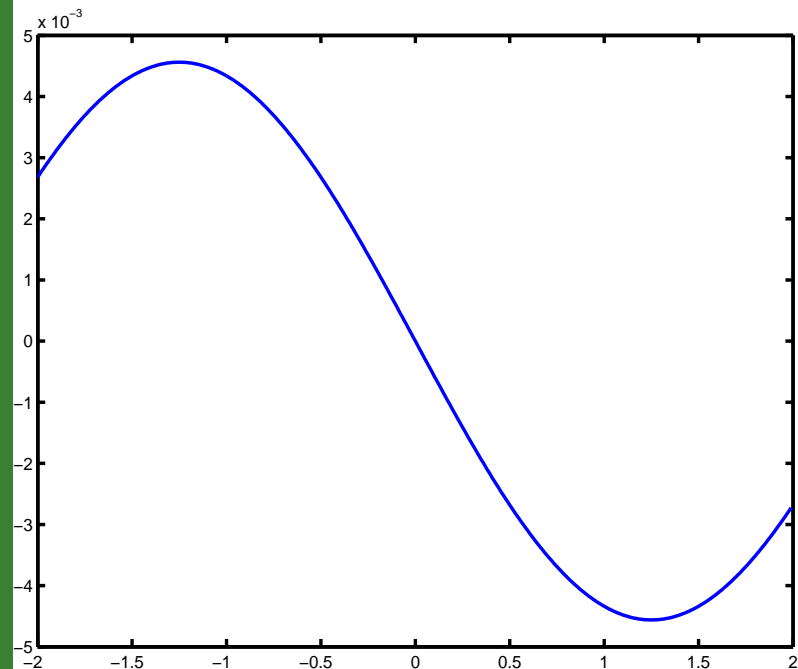
R



I

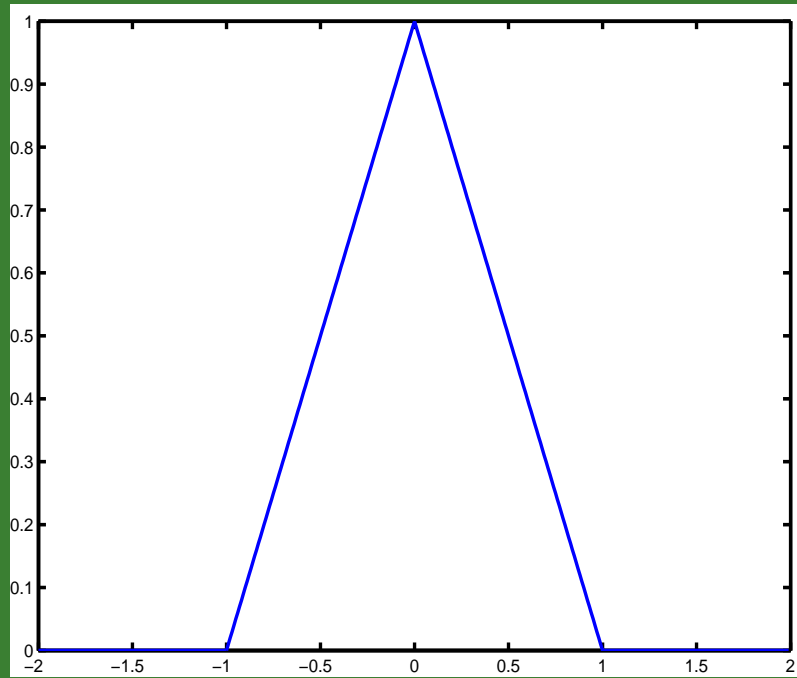


R

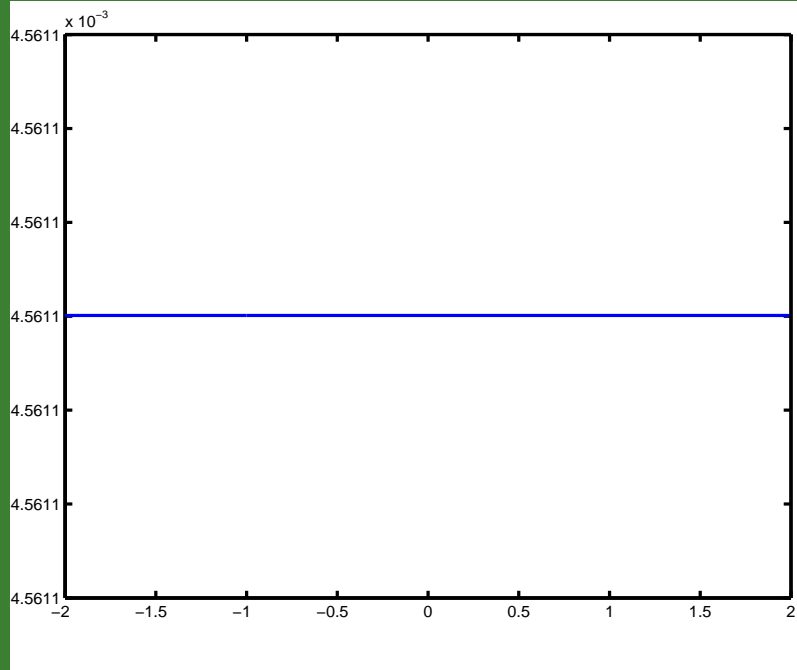


I

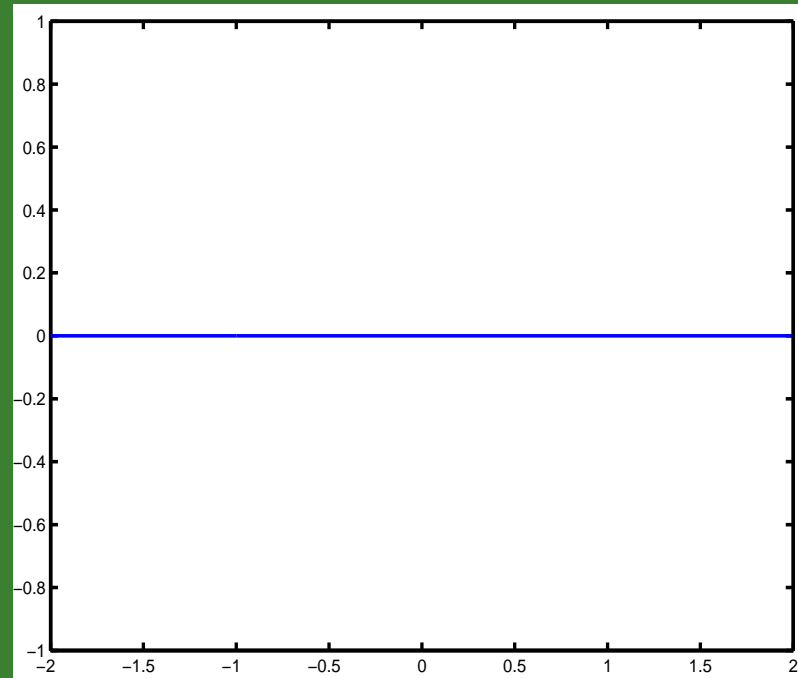
**Example: Triangle**  $A = 2$ ,  $L = 2.5$ ,  $\Delta k = 1/L = .4$ ,  $W = 64\Delta k = 25.6$



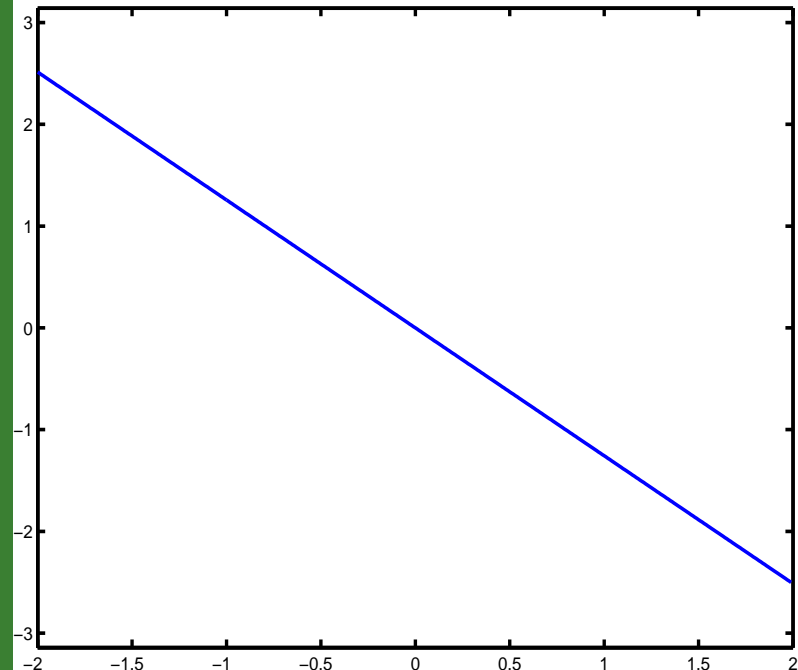
M



M

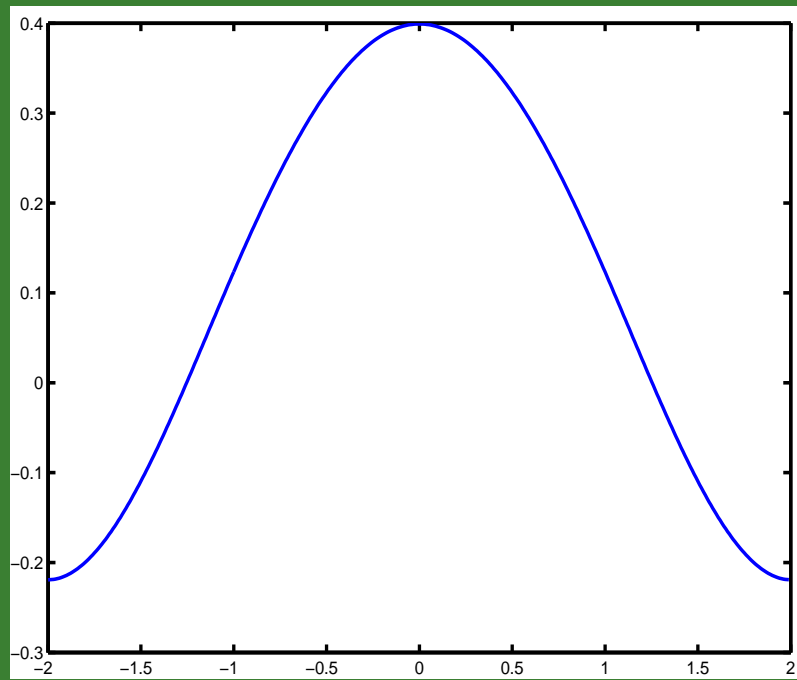


P

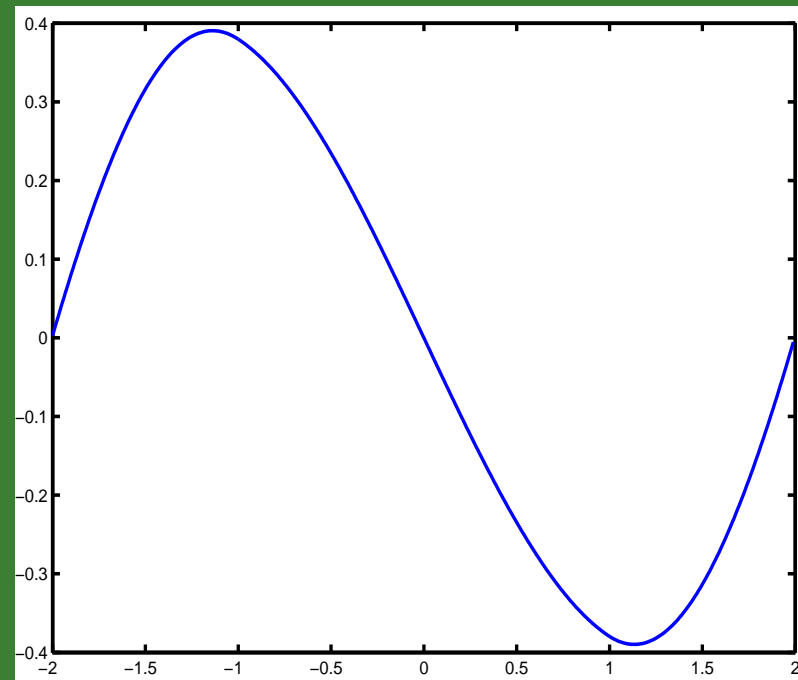


P

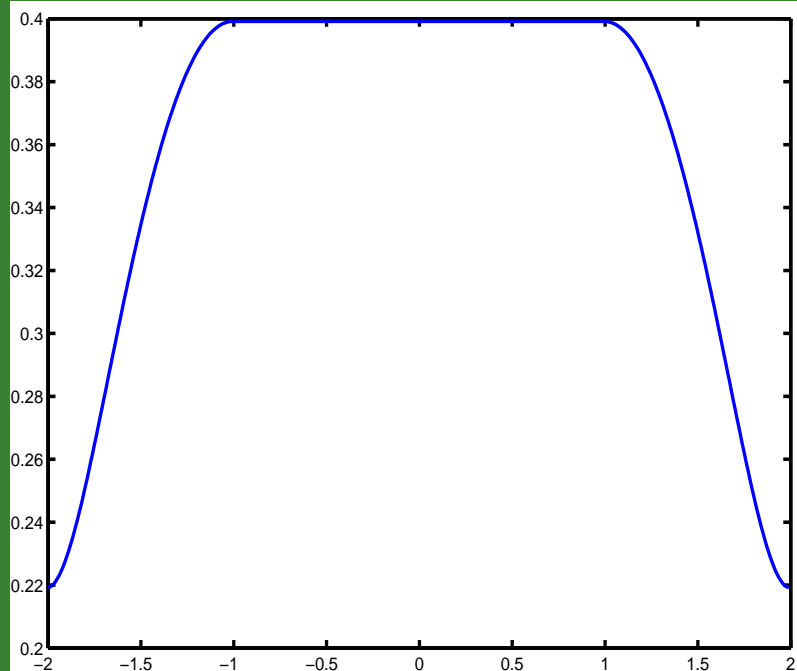
**Example: Triangle**  $A = 2$ ,  $L = 2.5$ ,  $\Delta k = 1/L = .4$ ,  $W = 64\Delta k = 25.6$



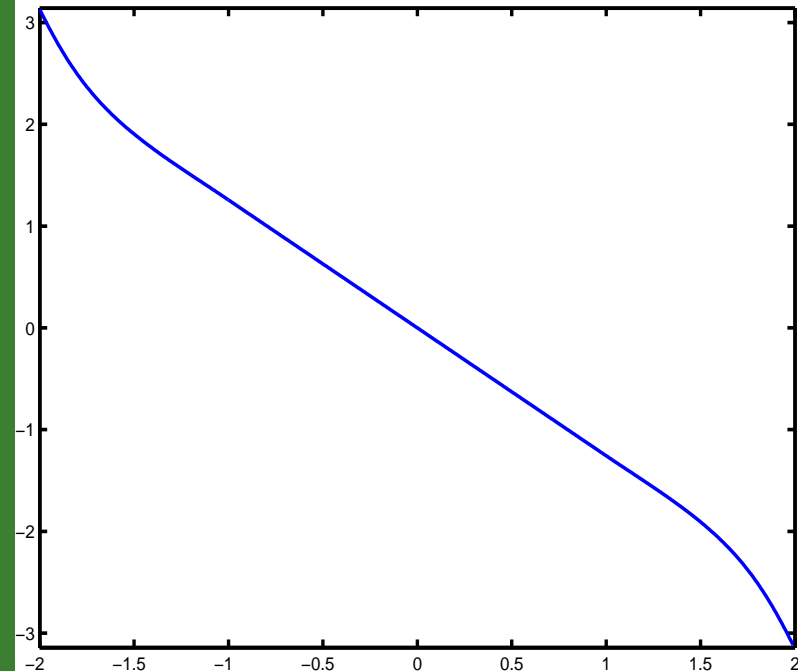
R



I



M



P

## 13.2: Filters and Point Spread Functions

### Point Spread and Additional Filters

Other factors (the Physics of relaxation  $T_2$  and  $T_2^*$ ) modify the signal  $s(k)$  and result in additional blurring.

These other factors can be modeled as filters  $H_{filter}(k)$

The more accurate windowed (with  $\Delta k/2$  shift), sampled (doubly infinite  $\delta$ 's), and other filters can be described as

$$\begin{aligned}\hat{s}_{m,filter}(k) &= s_m(k) \cdot H_{filter}(k) \\ &= s(k) \cdot H_{ws}(k) \cdot H_{filter}(k) \\ &= s(k) \cdot H_{ws,filter}(k)\end{aligned}\tag{13.20}$$

## 13.2: Filters and Point Spread Functions

The more accurate reconstructed image is

$$\begin{aligned}\hat{\rho}_{m,filter}(h) &= \hat{\rho}(x) * h_{ws}(x) * h_{filter}(x) \\ &= \rho(x) * h_{ws,filter}(h)\end{aligned}\quad (13.21)$$

So by taking into account the physical exponential decay ( $T_2$  and  $T_2^*$ )

along with the effect of windowing with  $\Delta k/2$  rect and sampling

is to add more blur/smooth/convolve of each point to other locations by (13.21).

## 13.3: Gibbs Ringing

### Gibbs Overshoot and Undershoot

We have been representing (decomposing) functions as (into) sinusoids.

When the functions are smooth this works very well.

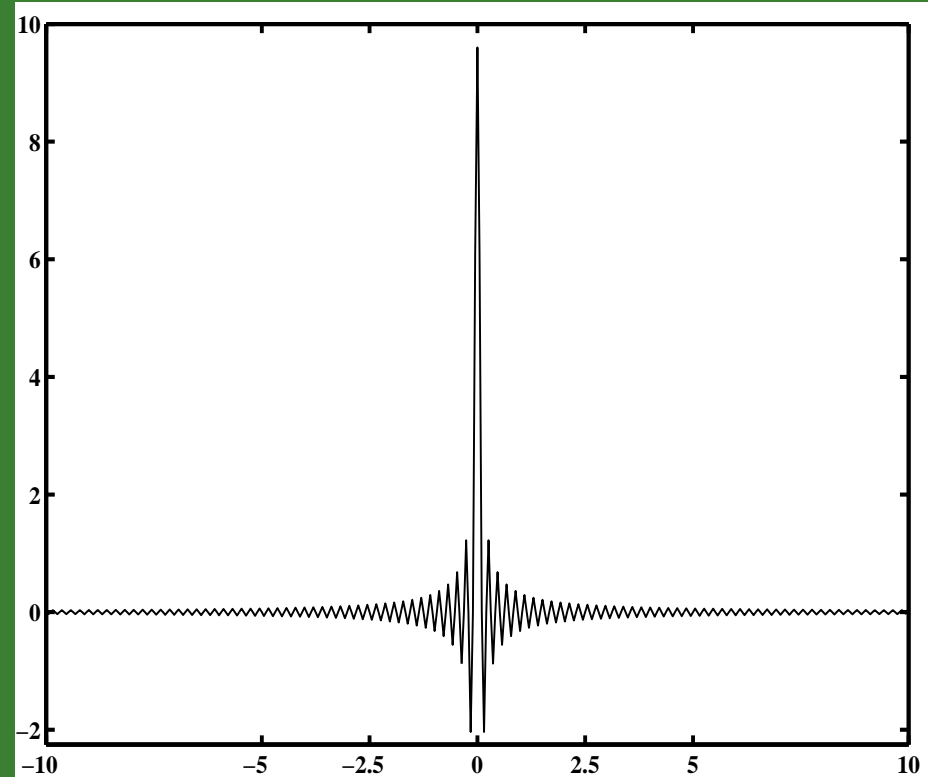
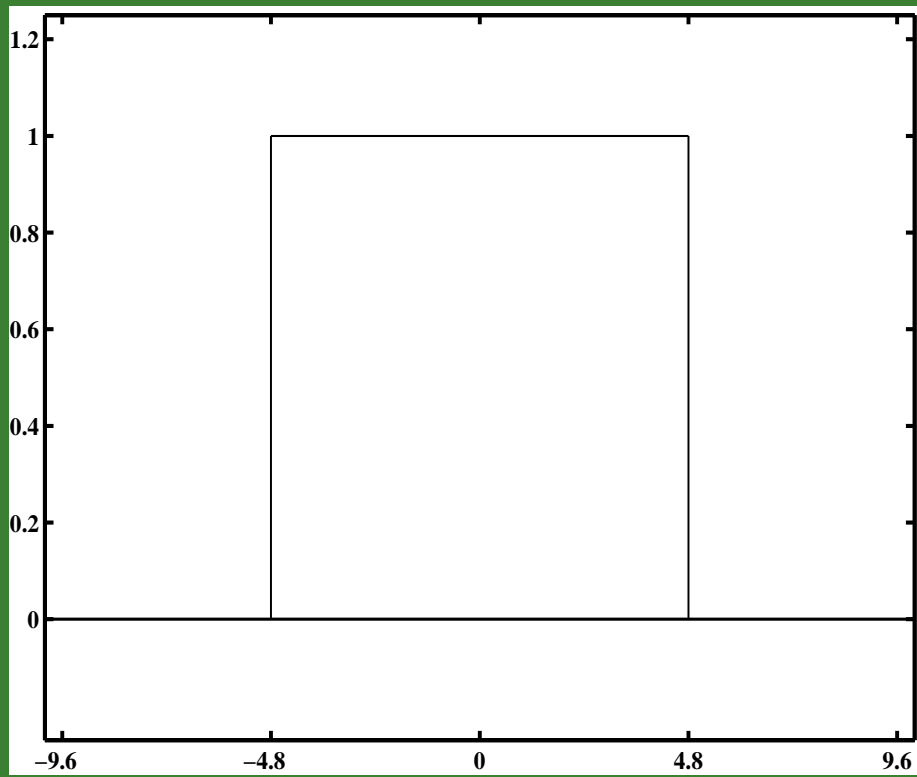
When the functions have a very sharp jump (discontinuity), then this does not work well.

The limiting over and under shoot is approx 9% of the height.

Let's look at an example of a square wave.



## 13.3: Gibbs Ringing: Example

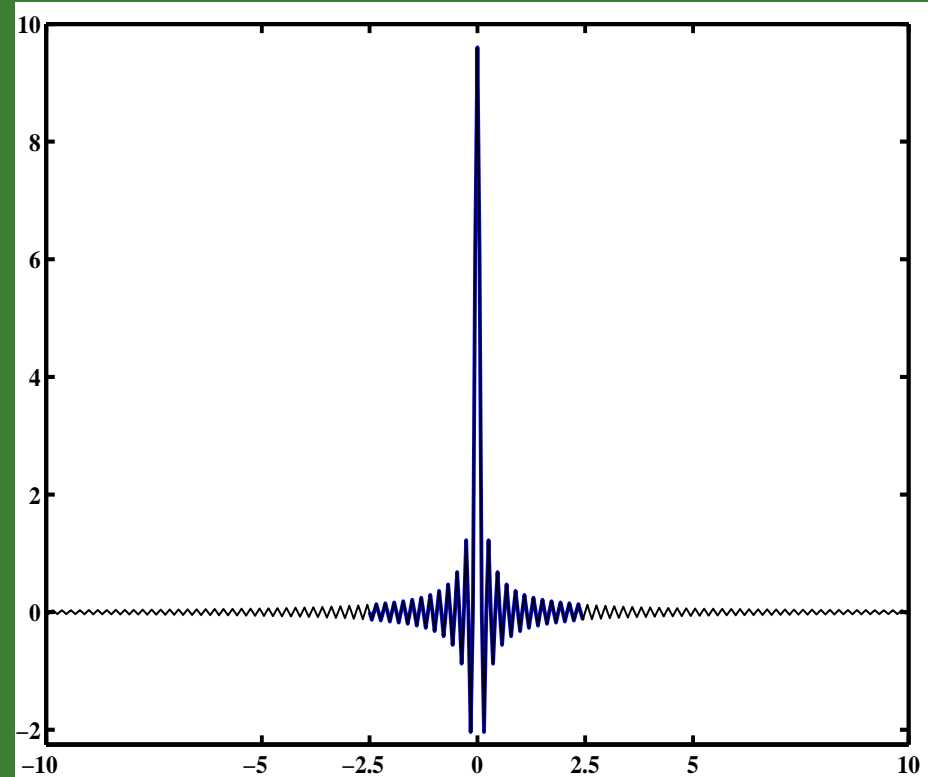
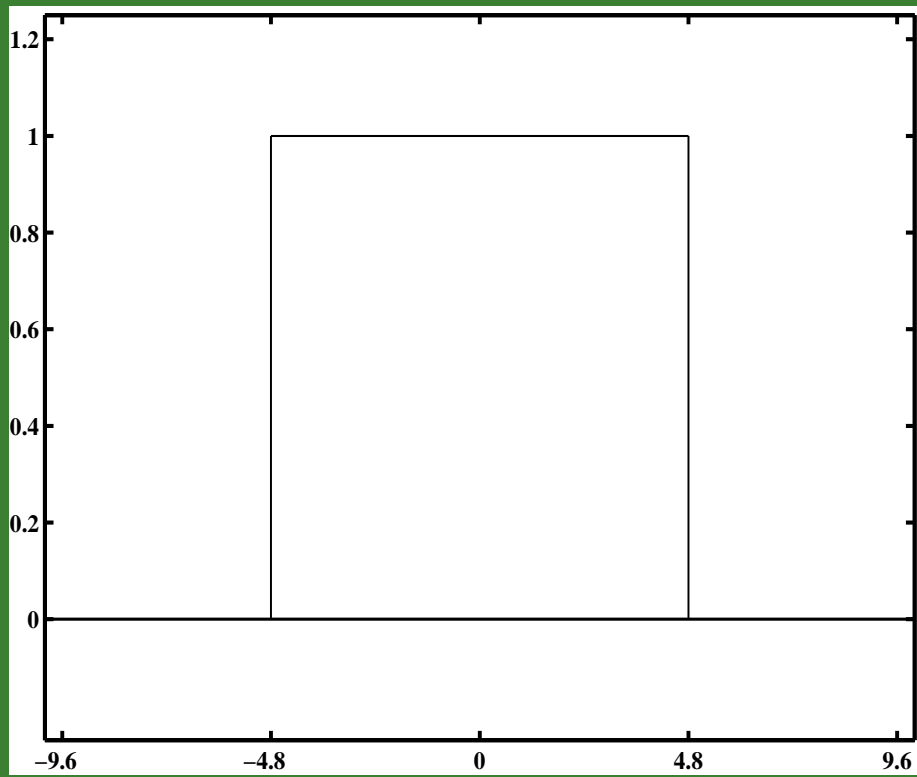


$$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right), A = 9.6\text{cm}$$

$$s(k) = 9.6 \text{ sinc}(\pi 9.6k)$$

Exact rect function on left (black). Exact sinc on right (black).

## 13.3: Gibbs Ringing: Example



$$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right), \quad A = 9.6\text{cm}$$

$$L = 19.2\text{cm}$$

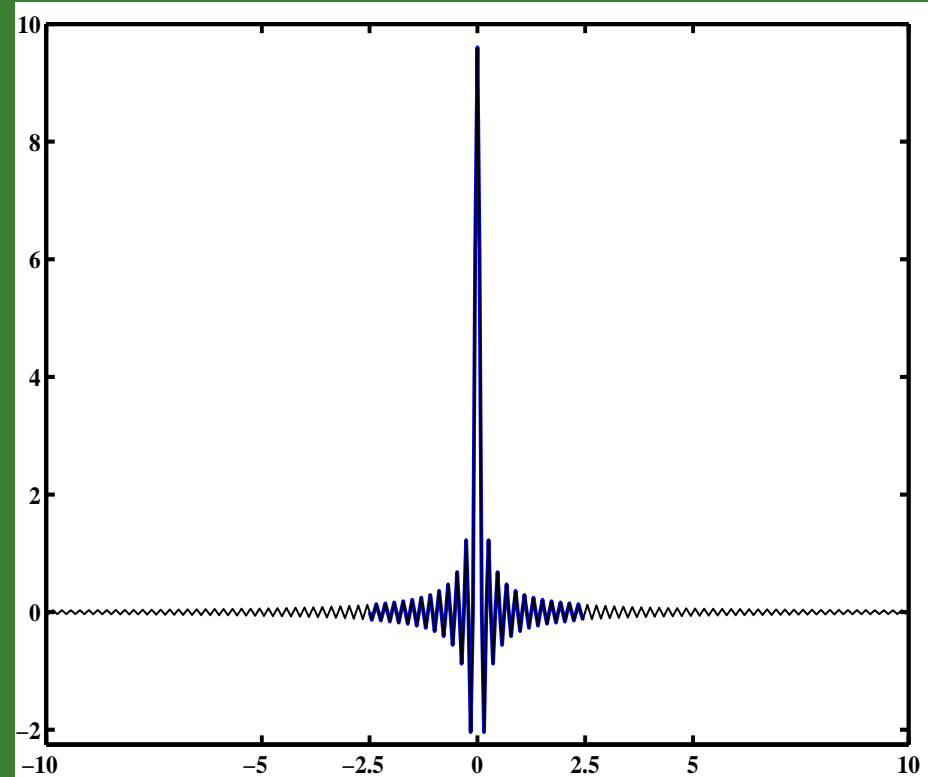
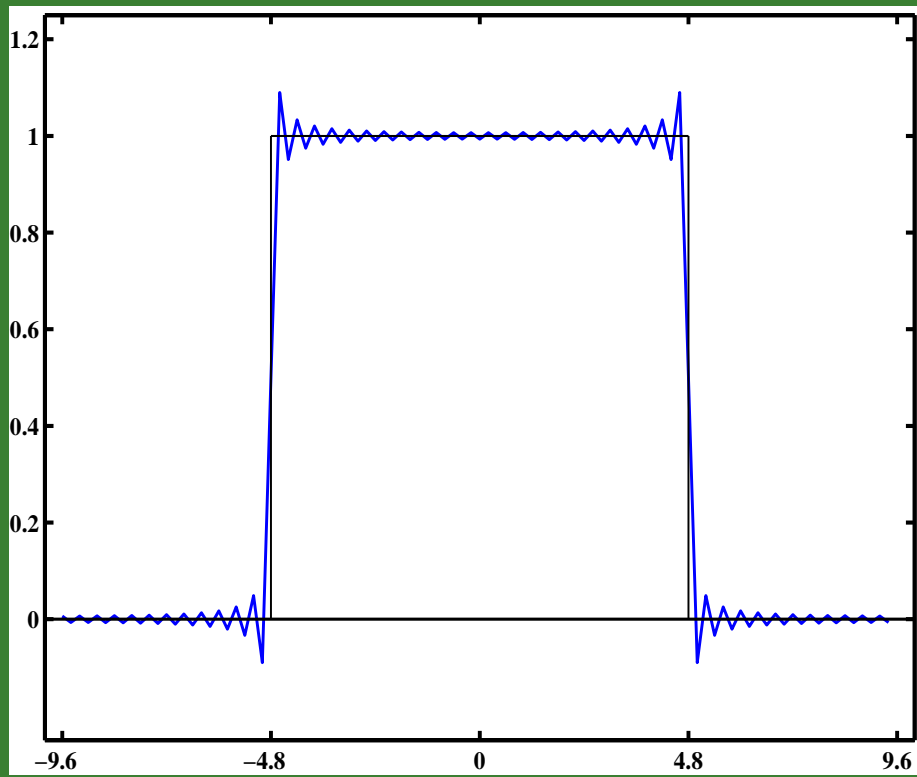
$$s(k) = 9.6 \text{ sinc}(\pi 9.6k)$$

$$\Delta k = 1/L = 1/19.2, \quad N = 96$$

Exact rect function on left (black). Exact sinc on right (black).

Truncated sinc on right (blue).

## 13.3: Gibbs Ringing: Example



$$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right), \quad A = 9.6\text{cm}$$

$$L = 19.2\text{cm}, \quad \Delta x = L/N = .2\text{cm}$$

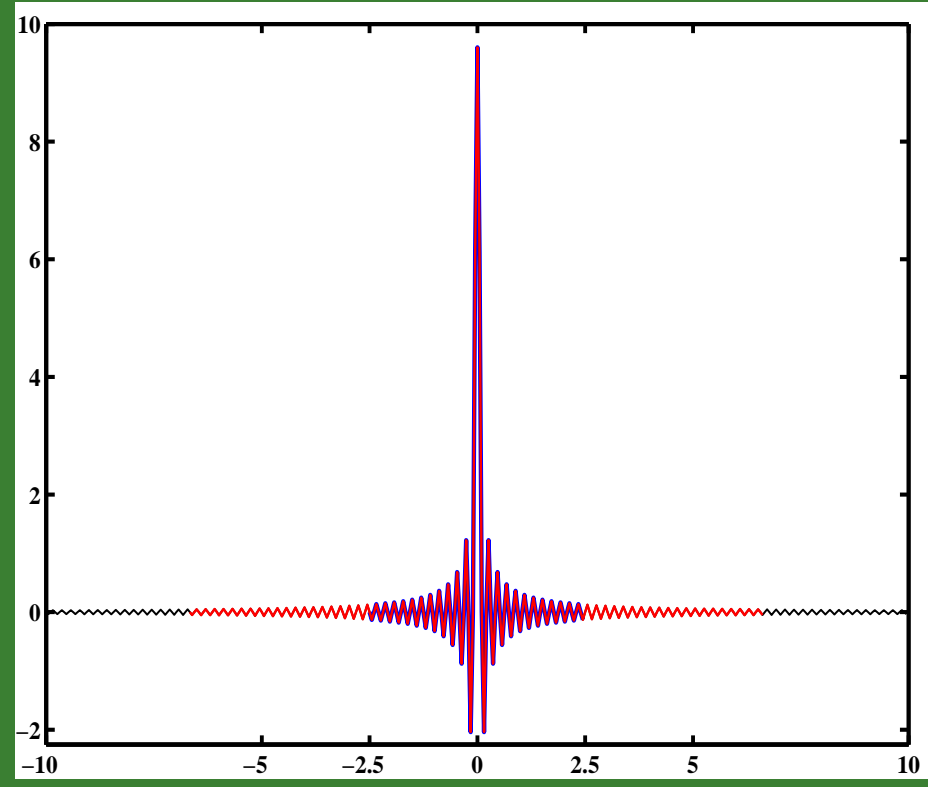
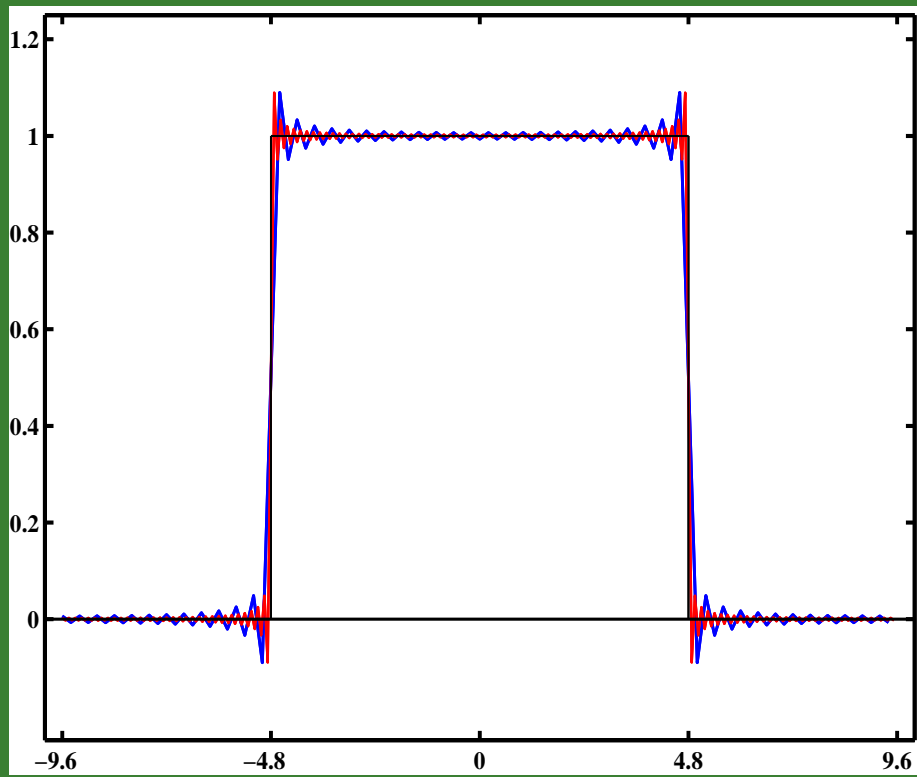
$$s(k) = 9.6 \text{ sinc}(\pi 9.6k)$$

$$\Delta k = 1/L = 1/19.2, \quad N = 96$$

Exact rect function on left (black). Exact sinc on right (black).  
 Rect from truncated sinc left (blue). Truncated sinc on right (blue).

When we have a sharp discontinuity in MRI from head to space or between brain structures we can get Gibbs ringing!

## 13.3: Gibbs Ringing: Example



$$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right), \quad A = 9.6\text{cm}$$

$$s(k) = 9.6 \text{ sinc}(\pi 9.6k),$$

$$L = 19.2\text{cm}, \quad \Delta x = L/N = .075\text{cm} \quad \Delta k = 1/L = 1/19.2, \quad N = 256$$

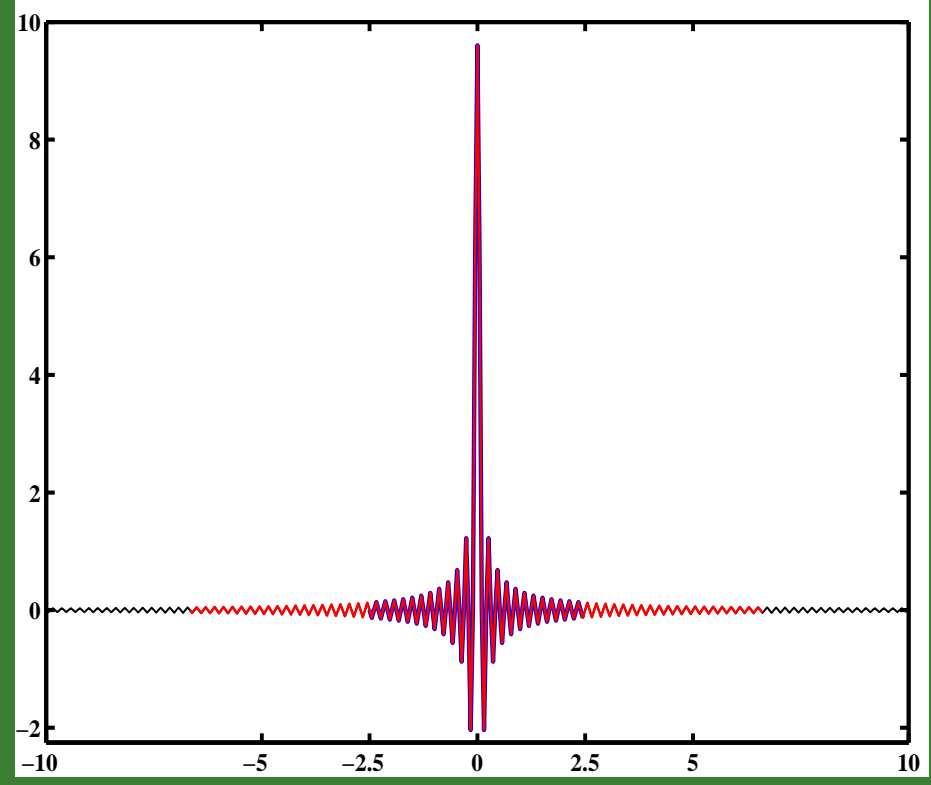
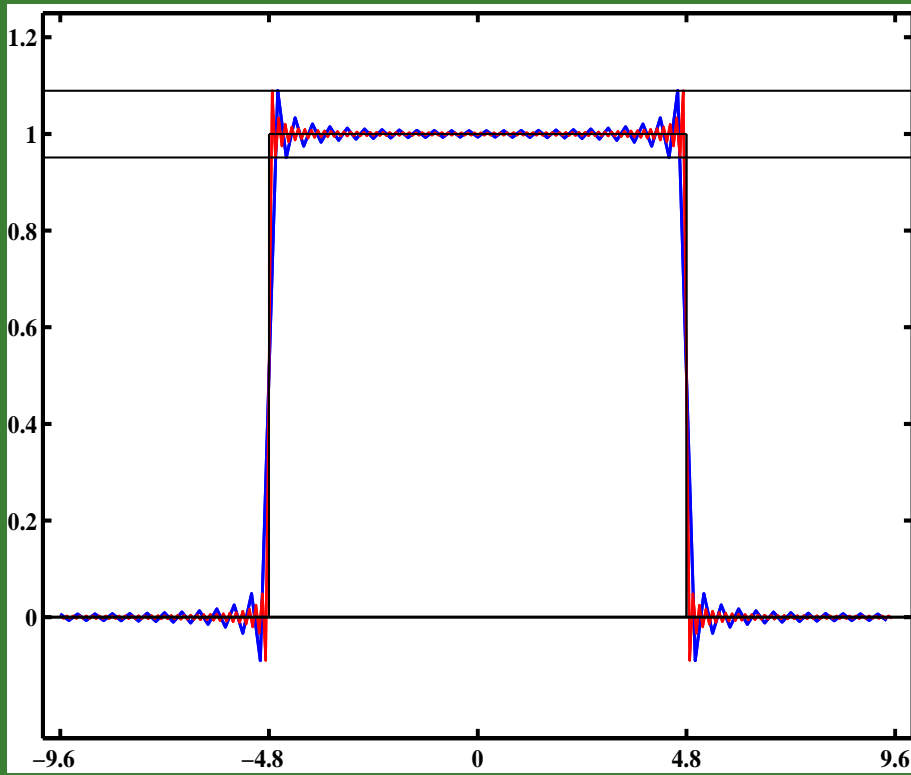
Exact rect function on left (black). Exact sinc on right (black).

Rect from truncated sinc left (blue). Truncated sinc on right (blue).

Rect from less truncated sinc left (red). Less truncated sinc on right (red).

Less Gibbs ringing the more frequencies we sample!

# 13.3: Gibbs Ringing: Example



$$\rho(x) = \text{rect}\left(\frac{x}{9.6}\right), \quad A = 9.6\text{cm} \quad s(k) = 9.6 \text{ sinc}(\pi 9.6k),$$

$$L = 19.2\text{cm}, \quad \Delta x = L/N = .075\text{cm} \quad \Delta k = 1/L = 1/19.2, \quad N = 256$$

Note that the red reconstructed image is sharper to the discontinuity  
 But the height is still the same. Red image numbers.

-4.8000	0.5000
-4.7250	1.0895
-4.6500	0.9514
-4.5750	1.0331
-4.5000	0.9749

$$(1.0895 - 1)/1 \cdot 100\% \approx 9\% \text{ overshoot!}$$

## 13.3: Gibbs Ringing

### Gibbs Oscillation Frequency

As we sample at same minimum  $\Delta k$  but further out, the over/under shoot gets closer to the discontinuity.

Going further out does not affect the height.

If we sample twice as far out in  $k$ -space, overshoot is half as far from discontinuity.

## 13.3: Gibbs Ringing

### Reducing Gibbs Ringing by Filtering

If the data is multiplied by a function that vanishes, along with its first derivative at  $k = \pm k_{max}$ , it is said to have been *apodized*.

One apodizing filter that is used is the *Hanning filter*.

$$H_{Hanning}(k) = \frac{1 + \cos\left(\frac{2\pi k}{W}\right)}{2} = \cos^2\left(\frac{\pi k}{W}\right) \quad (13.33)$$

The IFT of the Hanning is

$$h_{Hanning}(x) = \frac{1}{4}\delta(x - \Delta x) + \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x + \Delta x) \quad (13.34)$$

## 13.3: Gibbs Ringing

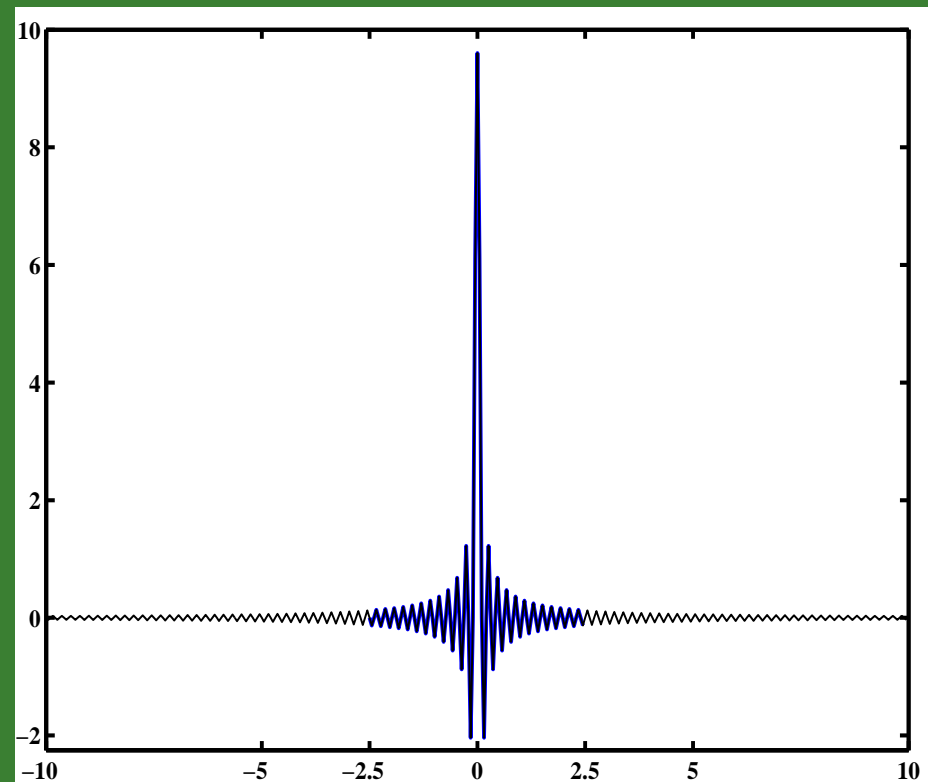
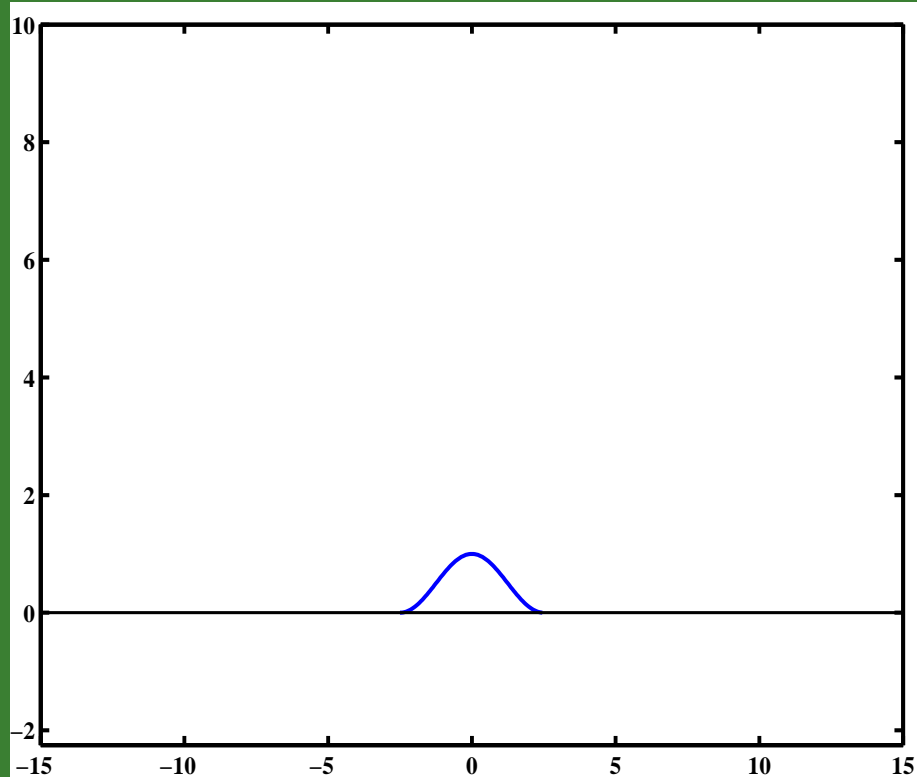
The corresponding reconstructed spin density is The IFT of the Hanning is

$$\begin{aligned}\hat{\rho}_{Hanning}(x) &= h_{Hanning}(x) * \hat{\rho}(x) \\ &= \frac{1}{4}\hat{\rho}(x - \Delta x) + \frac{1}{2}\hat{\rho}(x) + \frac{1}{4}\hat{\rho}(x + \Delta x) \quad (13.35)\end{aligned}$$

The  $k$ -space hanning filter corresponds to an averaging in image space.



## 13.3: Gibbs Ringing



$$H_{\text{Hanning}}(k) = \cos^2(\pi k/W)$$

$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

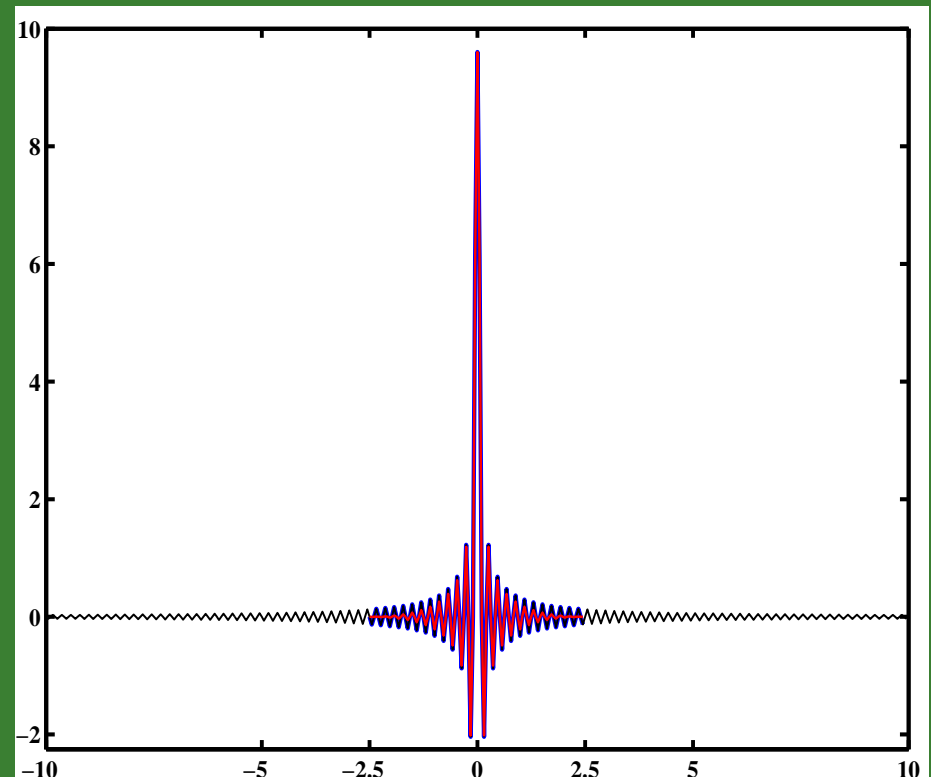
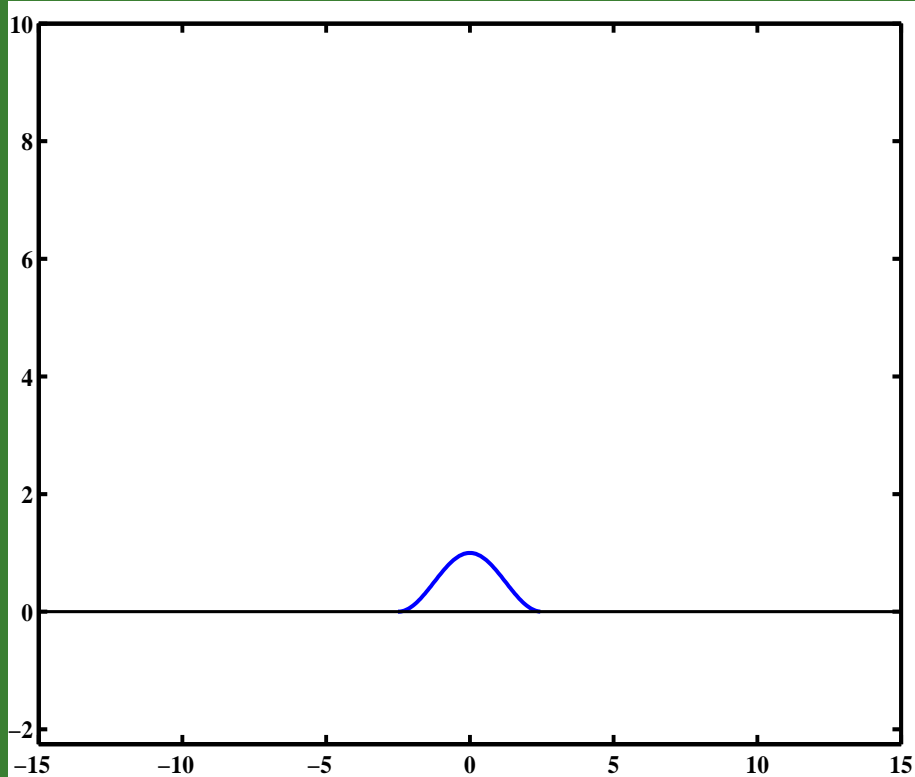
$$s(k) = 9.6 \operatorname{sinc}(\pi 9.6k),$$

$$\Delta k = 1/L = 1/19.2$$

Hanning filter on left (blue). Exact sinc on right (black).

Truncated sinc on right (blue).

## 13.3: Gibbs Ringing



$$H_{\text{Hanning}}(k) = \cos^2(\pi k/W)$$

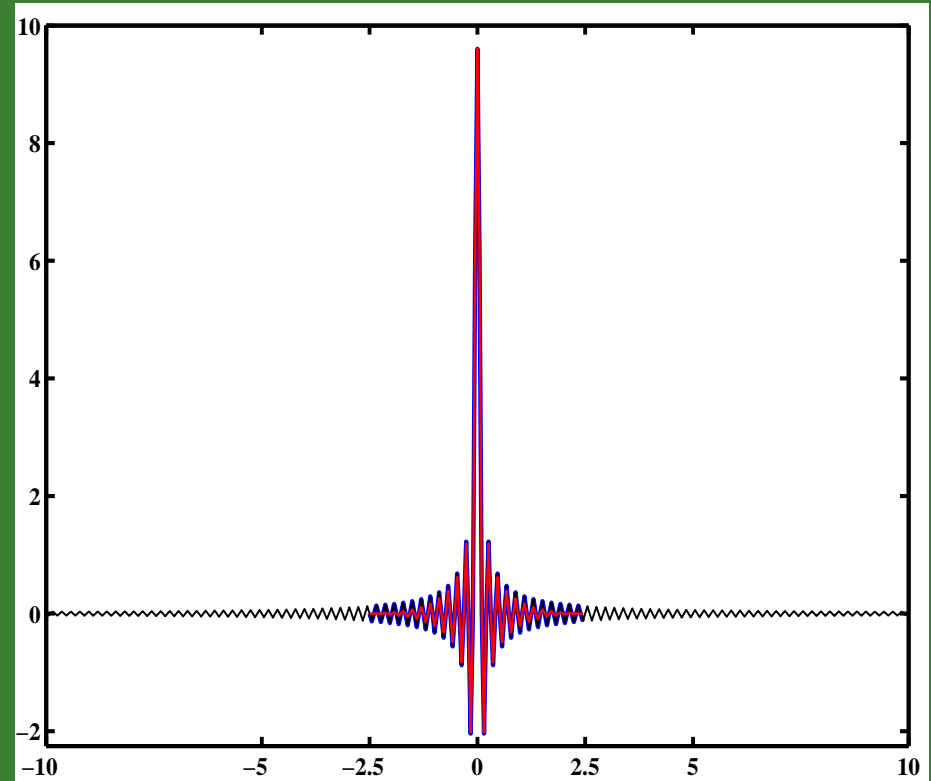
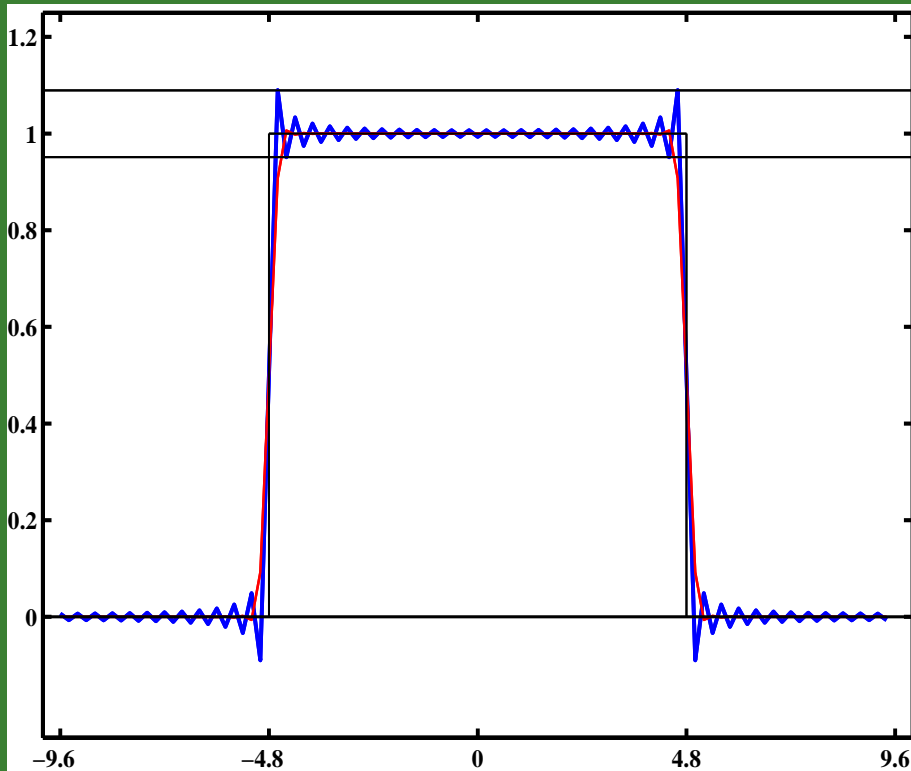
$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

$$s(k) = 9.6 \operatorname{sinc}(\pi 9.6k),$$

$$\Delta k = 1/L = 1/19.2$$

Hanning filter on left (blue). Exact sinc on right (black).  
 Truncated sinc on right (blue).  
 Truncated sinc multiplied by  
 Hanning filter on right (red).

# 13.3: Gibbs Ringing



$$H_{Hanning}(k) = \cos^2(\pi k/W)$$

$$W = 2 \cdot N/2 \cdot \Delta k = 5$$

$$s(k) = 9.6 \text{ sinc}(\pi 9.6k),$$

$$\Delta k = 1/L = 1/19.2$$

Exact rect on left (black).

Exact sinc on right (black).

Rect from truncated sinc on left (blue).

Truncated sinc on right (blue).

Rect from truncated sinc multiplied by Hanning filter on left (red).

Truncated sinc multiplied by Hanning filter on right (red).

Note reduced Gibbs ringing from hanning smoothing, called apodization. 27

## Read Sections

13.4: Spatial Resolution in MRI

13.5: Filtering Due to  $T_2$  and  $T_2^*$  Decay

13.6: Zero Filled Interpolation, Sub-Voxel Fourier Transform Shift Concepts and Point Spread Function Effects

13.7: Partial Fourier Imaging and Reconstruction

13.8: Digital Truncation

## Homework

Do 13.1, 13.2, 13.3, 13.4