Biophysics 230: Nuclear Magnetic Resonance
Haacke Chapter 12

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12: Sampling and Aliasing in Image Reconstruction

1) We talked about the FT of a continuous function $\rho(x)$.

2) We talked about the IFT of a continuous function $s(k)$.

3) We talked about a rect function rect(k) and its IFT sinc(x).

4) We talked about a sampling function $u(k)$ and its IFT $U(x)$.

5) We talked about the convolution of two or three functions.

6) Now talk about taking the IFT of $s(k) \cdot u(k) \cdot \mathrm{rect}(k)$.

7) What effect does multiplying $s(k)$ by $u(k)$ and rect(k) have on $\hat{\rho}(x)$.

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Infinite Sampling

The sampling or comb function

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \tag{12.2}$$

where with constant x gradient G_R, $\Delta k = \gamma\!\!\!- G_R \Delta t$. (12.1)

Let's sample $s(k)$ by multiplying it by $u(k)$.

$$s_\infty(k) \equiv s(k) \cdot u(k) = \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k)\delta(k - p\Delta k) \tag{12.3}$$

Note that $s(k)$ is nonzero only at $p\Delta k$ so it has been moved past the sum.

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Because we have now (infinitely) sampled $s(k)$,
what does the reconstructed 1D image $\hat{\rho}(x)$ look like?

$$\begin{aligned}
\hat{\rho}_{\infty}(x) &= \int_{-\infty}^{\infty} [s_{\infty}(k)] e^{i2\pi kx} dk \\
&= \int_{-\infty}^{\infty} [s(k) \cdot u(k)] e^{i2\pi kx} dk \\
&= \int_{-\infty}^{\infty} \left[\Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k) \delta(k - p\Delta k) \right] e^{i2\pi kx} dk \\
&= \Delta k \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} s(p\Delta k) \delta(k - p\Delta k) e^{i2\pi kx} dk \\
&= \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k) e^{i2\pi p\Delta kx} \qquad (12.4)
\end{aligned}$$

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Note that

$$\begin{aligned} \hat{\rho}_\infty(x) &= \mathcal{F}^{-1}\{s(k)\cdot u(k)\} \\ &= \rho(x) * U(x) \end{aligned} \tag{12.5}$$

and since

$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - q/\Delta k) \tag{12.6}$$

the convolution of $\rho(x)$ with δ function is

$$\rho(x) * \delta(x - x_0) = \int \rho(x')\delta(x - x' - x_0)dx' = \rho(x - x_0) \tag{12.7}$$

and thus

$$\hat{\rho}_\infty(x) = \sum_{q=-\infty}^{\infty} \rho(x - q/\Delta k). \tag{12.8}$$

That is, a copy of $\rho(x)$ placed every $1/\Delta k$.

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Nyquist Sampling Criterion

In signal processing, $f(t)$ is said to be "band-limited" if $F(\nu)$ contains no frequencies beyond a frequency ν_0.

That is,

$$F(\nu) = \begin{cases} F(\nu) & -\nu_0 \leq \nu \leq \nu_0 \\ 0 & \text{otherwise} \end{cases} .$$

This is the whole principle that took us from analog signals

vinyl LPs, 8-tracks, and cassette to CDs, DVDs and digital music!

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Nyquist Sampling Criterion

In MRI, the frequencies $s(k)$ are said to be "band-limited" if $\rho(x)$ contains no object beyond a location $A/2$. Object of length A.

That is,

$$\rho(x) = \begin{cases} \rho(x) & -A/2 \leq x \leq A/2 \\ 0 & \text{otherwise} \end{cases} .$$

Band-limiting is critical in MRI because of the multiple object copies.

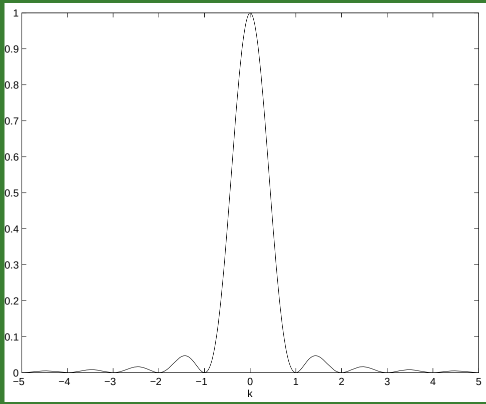
We need to cover enough k-space to capture the objects spatial frequencies.

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

Example: Triangle Function

One example of a band-limited function is the square of a sinc function

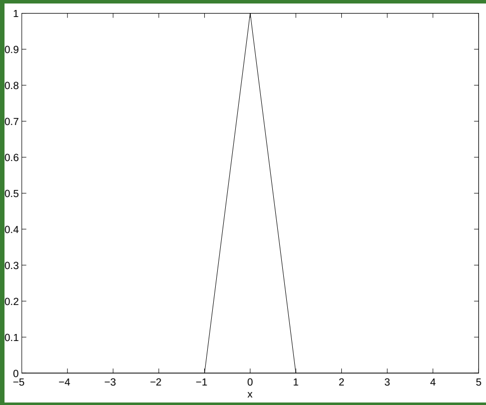
$$s(k) = \mathrm{sinc}^2(\pi A k)$$



shown in Figure 12.1b and here with $A = 2$.

The inverse Fourier Transform of a sinc^2 function is an isosceles triangle

$$\rho(x) = \begin{cases} 1 - \frac{2}{A} \mid x \mid & -A/2 \leq x \leq A/2 \\ 0 & \text{otherwise} \end{cases} .$$



12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

When $s(k)$ is sampled with $u(k)$, δ's of spacing and 'height' Δk.

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k)$$

The IFT of this is

$$\begin{aligned}
\hat{\rho}_{\infty}(x) &= \mathcal{F}^{-1}\{\text{sinc}^2(k) \cdot u(k)\} \\
&= \rho_{tri}(x) * U(x) \\
&= \sum_{q=-\infty}^{\infty} \int \rho_{tri}(x')\delta(x - x' - q/\Delta k)dx' \\
&= \sum_{q=-\infty}^{\infty} \rho_{tri}(x - q/\Delta k).
\end{aligned}$$

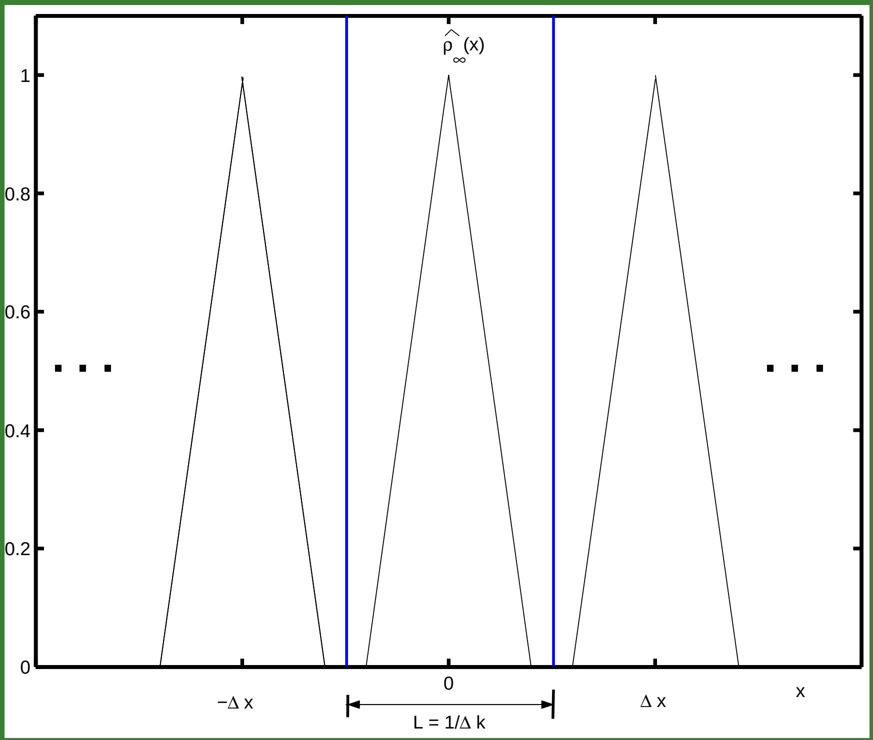
That is, an infinite number of copies of $\rho_{tri}(x)$ every $L = 1/\Delta k$. L=FOV

Don't forget that $\Delta k = \bar{\gamma} G_R \Delta t$. Decrease/increase G_R & Δt.

Sampling $s(k)$ in k-space generates copies of $\rho(x)$ in image space.

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

The IFT of the sampled spatial frequencies constists of evenly spaced copies of the IFT of the unsampled (continuous) spatial frequencies.



$$\hat{\rho}_\infty(x) = \sum_{q=-\infty}^{\infty} \rho_{tri}(x - q/\Delta k).$$

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

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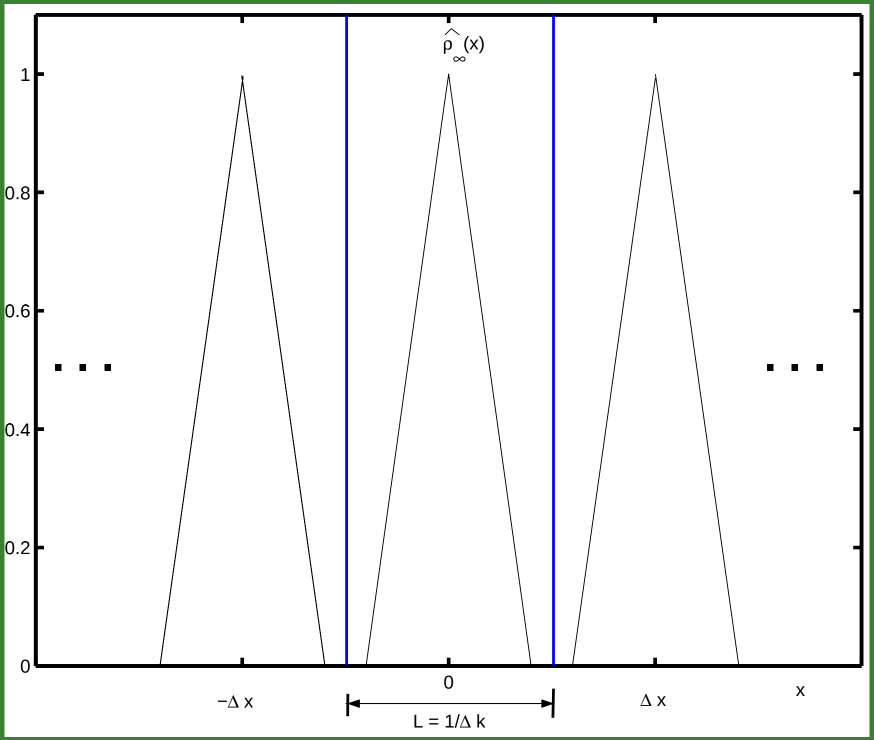
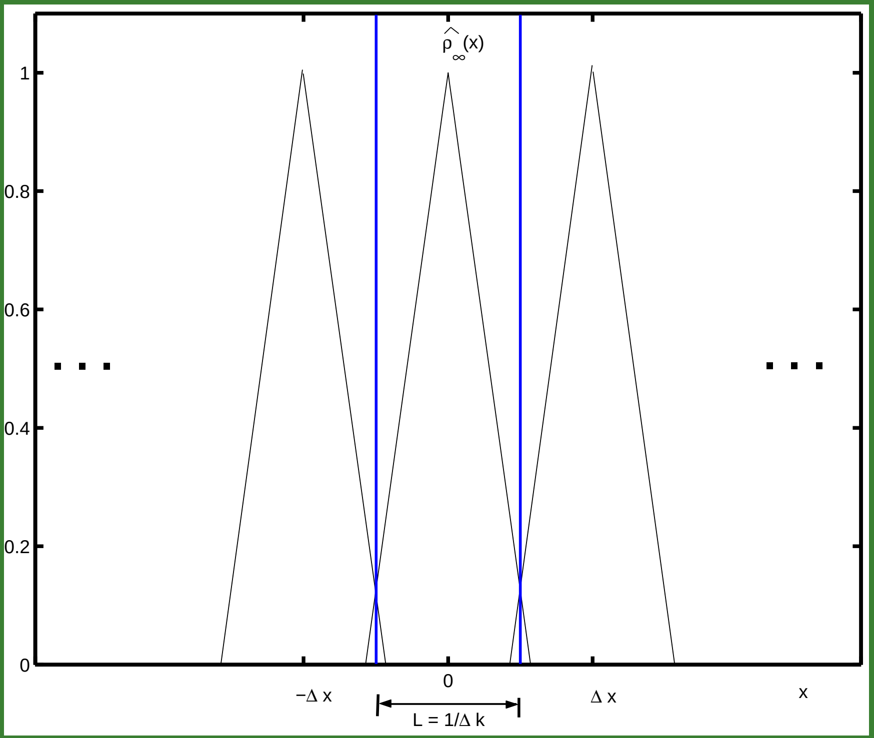


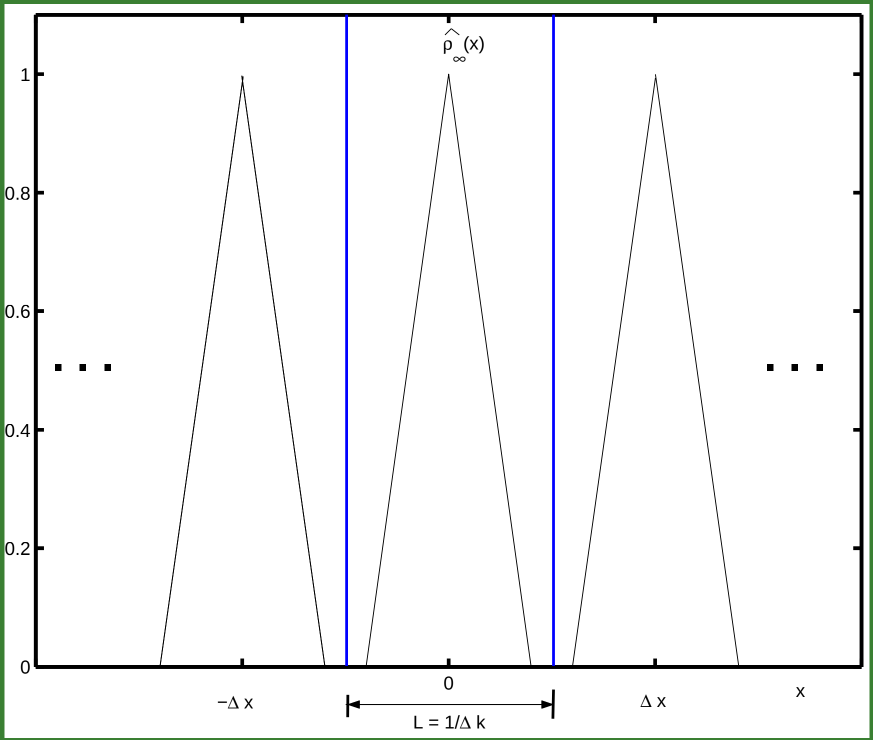


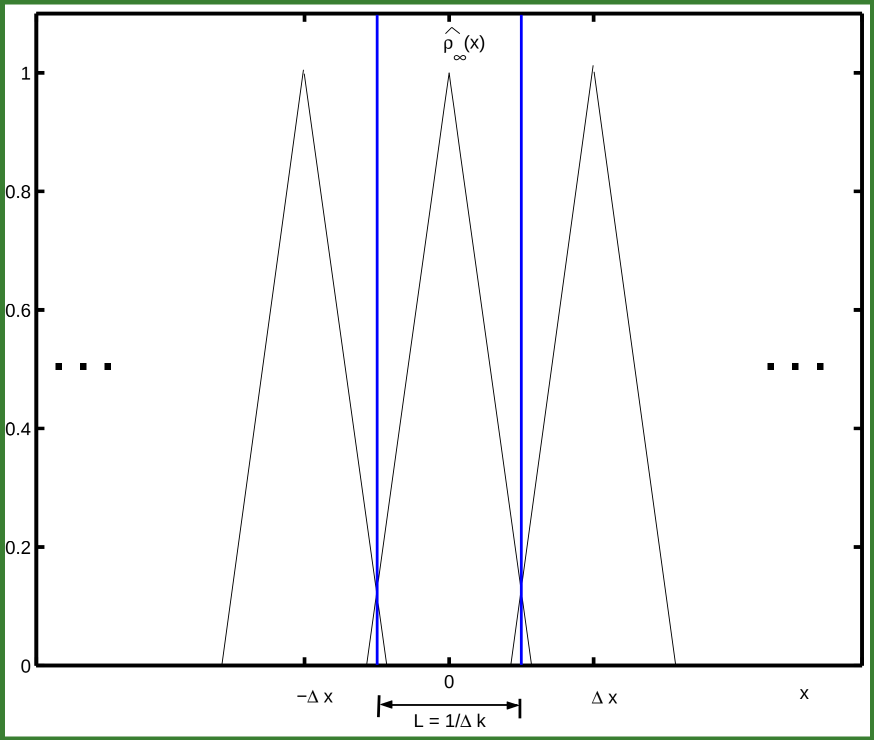
If $\rho(x)$'s are placed less than A apart they will overlap.

So we have to have $L > A$ or $\Delta k < \frac{1}{A}$. (12.11)

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

The IFT of the sampled spatial frequencies constists of evenly spaced copies of the IFT of the unsampled (continuous) spatial frequencies.





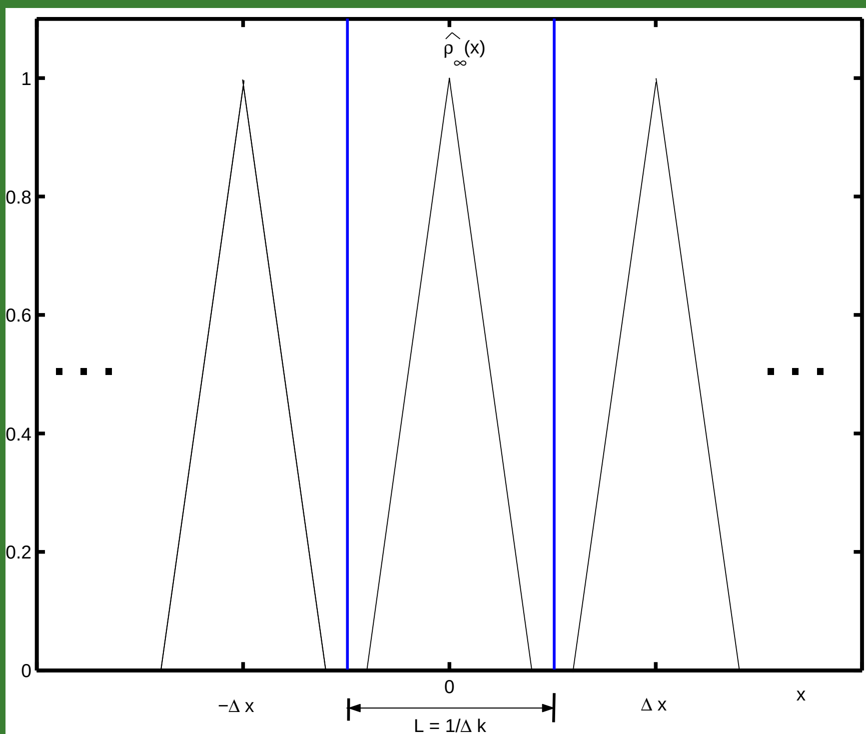
With $\Delta k_R = \bar{\gamma} G_R \Delta t,$ (12.13)

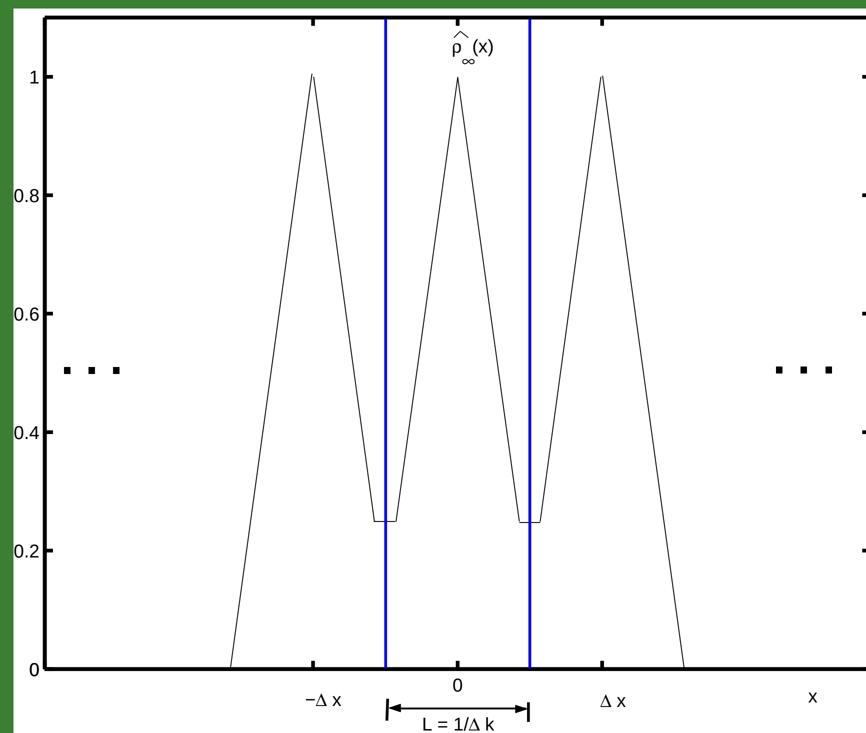
$f_R \equiv BW_R \equiv \frac{1}{\Delta t_R} = \bar{\gamma} G_R L_R > \bar{\gamma} G_R A_R.$ (12.15)

With $\Delta k_{PE} = \bar{\gamma} \Delta G_{PE} \tau_{PE},$ (12.17)

12.1: Infinite Sampling, Aliasing and the Nyquist Criterion

The IFT of the sampled spatial frequencies constists of evenly spaced copies of the IFT of the unsampled (continuous) spatial frequencies.





With $\Delta k_R = \gamma G_R \Delta t$, (12.13)

$$f_R \equiv BW_R \equiv \frac{1}{\Delta t_R} = \gamma G_R L_R > \gamma G_R A_R. \quad (12.15)$$

With $\Delta k_{PE} = \gamma \Delta G_{PE} \tau_{PE}$, (12.17)

12.2: Finite Sampling, Image Reconstruction & the DFT

Finite Sampling

We talked sbout the effect of $s(k) \cdot u(k)$.

But we can't infinitely sample $s(k)$.

So we need to 'window' to sample k-space over a finite interval

Let W be the width of k-space coverage

$$\text{rect}\left(\frac{k+\Delta k/2}{W}\right) = \begin{cases} 0 & k < -\frac{W+\Delta k}{2} \\ 1 & -\frac{W+\Delta k}{2} \leq k \leq \frac{W-\Delta k}{2} \\ 0 & k > \frac{W-\Delta k}{2} \end{cases}$$

12.2: Finite Sampling, Image Reconstruction & the DFT

Then when we multiply our $u(k)$ by $\mathrm{rect}\left(\frac{k+\Delta k/2}{W}\right)$ in k-space

$$u(k) \cdot \mathrm{rect}\left(\frac{k + \Delta k/2}{W}\right) = \Delta k \sum_{p=-n}^{n-1} \delta(k - p\Delta k) \qquad (12.18)$$

where $W = 2n\Delta k = N\Delta k$ and $N = 2n$ is the number of points sampled.

And so finally

$$\begin{aligned} s_m(k) &= s(k) \cdot u(k) \cdot \mathrm{rect}\left(\frac{k + \Delta k/2}{W}\right) \\ &= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k)\delta(k - p\Delta k) \qquad (12.20) \end{aligned}$$

12.2: Finite Sampling, Image Reconstruction & the DFT

Reconstructed Spin Density

The spin density from sampling and windowing is

$$\hat{\rho}(x) = \int_{-\infty}^{\infty} s_m(k) e^{i2\pi kx} dk \tag{12.21}$$

$$= \int_{-\infty}^{\infty} s(k) \cdot u(k) \cdot \mathrm{rect}\left(\frac{k + \Delta k/2}{W}\right) e^{i2\pi kx} dk$$

$$= \Delta k \int_{-\infty}^{\infty} \sum_{p=-n}^{n-1} s(p\Delta k) \delta(k - p\Delta k) e^{i2\pi kx} dk$$

$$= \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta kx} \tag{12.22}$$

12.2: Finite Sampling, Image Reconstruction & the DFT

The inverse Fourier transform of $\mathrm{rect}(k/W)$ is

$$\mathcal{F}^{-1}\left\{\mathrm{rect}\left(\frac{k}{W}\right)\right\} = W\mathrm{sinc}(\pi W x)$$

however, we have shifted it by $\Delta k/2$, so

$$\mathrm{rect}\left(\frac{k+\Delta k/2}{W}\right)$$

this means that we need to use the shift theorem

$$\mathcal{F}^{-1}\left\{\mathrm{rect}\left(\frac{k+\Delta k/2}{W}\right)\right\} = e^{-2i\pi x\Delta k/2}W\mathrm{sinc}(\pi W x)$$

12.2: Finite Sampling, Image Reconstruction & the DFT

When $s(k)$ is sampled with $u(k)$ with spacing and 'height' Δk over W.

$$u(k) \cdot \mathrm{rect}\left(\frac{k+\Delta k/2}{W}\right) = \Delta k \sum_{p=-n}^{n-1} \delta(k - p\Delta k)$$

The IFT of this is

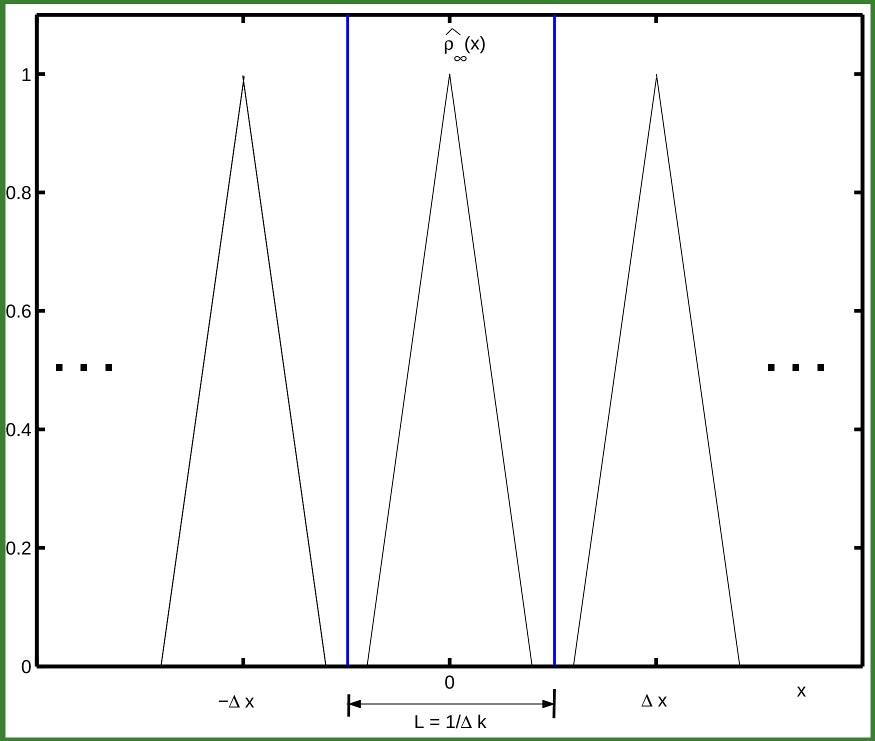
$$\begin{aligned}\hat{\rho}(x) &= \rho(x) * U(x) * W\mathrm{sinc}(\pi W x)e^{-i\pi x \Delta k} \\ &= \sum_{q=-n}^{n-1} \int \rho(x') * W\mathrm{sinc}(\pi W x')e^{-i\pi x' \Delta k}\delta(x - x' - q/\Delta k)dx' \\ &= \sum_{q=-n}^{n-1} \rho(x - qL) * W\mathrm{sinc}(\pi W(x - qL))e^{-i\pi(x-qL)\Delta k}.\end{aligned}$$

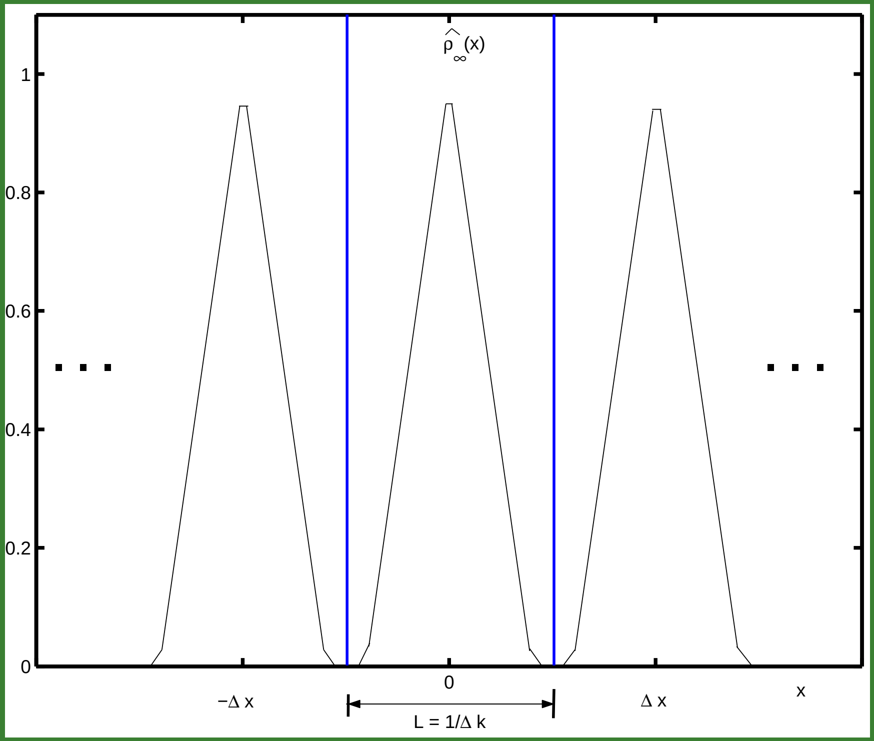
A finite number of copies of $\rho(x) * W\mathrm{sinc}(\pi W x)e^{-i\pi x \Delta k}$ every $L = 1/\Delta k$.

Don't forget that $\Delta k = ⩒ G_R \Delta t$. Decrease/increase G_R & Δt.

Finite sampling $s(k)$ generates copies of $\rho(x) * W\mathrm{sinc}(\pi W x)e^{-i\pi x \Delta k}$.

12.2: Finite Sampling, Image Reconstruction & the DFT





$\rho(x) * U(x)$

$\rho(x) * U(x) * W\text{sinc}(\pi W x)$

Also a phase shift $e^{-i\pi x \Delta k}$.

12.2: Finite Sampling, Image Reconstruction & the DFT

Discrete and Truncated Sampling of $\hat{\rho}(x)$: Resolution

Imagine that we sample the reconstructed image.
The sampling function would be

$$\tilde{U}(x) = \Delta x \sum_{q=-\infty}^{\infty} \delta(x - q\Delta x) \tag{12.24}$$

and the measured spin density would be

$$\begin{aligned}\hat{\rho}_m(x) &= \hat{\rho}(x) \cdot \tilde{U}(x) \cdot \text{rect}\left(\frac{x + \Delta x/2}{L}\right) \\ &= \Delta x \sum_{q=-n'}^{n'-1} \hat{\rho}(q\Delta x)\delta(x - q\Delta x)\end{aligned} \tag{12.25}$$

where $L = 2n'\Delta x$. (12.26)

12.2: Finite Sampling, Image Reconstruction & the DFT

The Fourier transform of the measured spin density is

$$\begin{aligned}\hat{s}(k) &= \int \hat{\rho}_m(x) e^{-i2\pi kx} dx \\ &= \Delta x \sum_{q=-n'}^{n'-1} \hat{\rho}(q\Delta x) e^{-i2\pi kq\Delta x} \qquad (12.27)\end{aligned}$$

Can see that for:
-Large n' and small Δx, in (12.27) $\hat{s}(k)$ is continuous FT of $\hat{\rho}(x)$ and that $\hat{\rho}(x)$ is continuous IFT of $\hat{s}(k)$

But for:
-Large n and small Δk, in (12.21) $\hat{\rho}(x)$ is continuous IFT of $\hat{s}(k)$.

$$\hat{\rho}(x) = \Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta kx} dk \qquad (12.21)$$

When is $\hat{s}(k) = s(k)$?

12.2: Finite Sampling, Image Reconstruction & the DFT

Discrete Fourier Transform

For $s(p\Delta k)$ and $\hat{\rho}(q\Delta x)$ to be a DFT pair need $n = n'$. (12.28)

$$\hat{s}(r\Delta k) = \Delta x \sum_{q=-n'}^{n'-1} \left[\hat{\rho}(q\Delta x)\right] e^{-i2\pi r\Delta kq\Delta x} \quad (12.27)$$

$$= \Delta x \sum_{q=-n'}^{n'-1} \left[\Delta k \sum_{p=-n}^{n-1} s(p\Delta k) e^{i2\pi p\Delta kq\Delta x}\right] e^{-i2\pi r\Delta kq\Delta x}$$

$$= \Delta x\Delta k \sum_{p=-n}^{n-1} \sum_{q=-n'}^{n'-1} s(p\Delta k) e^{i2\pi(p-r)\Delta kq\Delta x} \quad (12.29)$$

12.2: Finite Sampling, Image Reconstruction & the DFT

If $n = n'$, then

$$\Delta k \Delta x = \frac{1}{L} \cdot \frac{L}{2n} = \frac{1}{2n} \tag{12.30}$$

and (12.29)

$$\hat{s}(r\Delta k) = \Delta x \Delta k \sum_{p=-n}^{n-1} \sum_{q=-n'}^{n'-1} s(p\Delta k) e^{i2\pi(p-r)\Delta k q \Delta x} \tag{12.29}$$

becomes

$$\hat{s}(r/L) = \frac{1}{2n} \sum_{p=-n}^{n-1} s(p/L) \sum_{q=-n}^{n-1} e^{i\frac{2\pi q(p-r)}{2n}} \tag{12.31}$$

then using

$$\sum_{q=-n}^{n-1} e^{i\frac{2\pi q(p-r)}{2n}} = 2n\delta_{pr} \tag{12.32}$$

we get

$$\hat{s}(r/L) = s(r/L). \tag{12.33}$$

12.2: Finite Sampling, Image Reconstruction & the DFT

So then we can see that

$$s\left(\frac{p}{L}\right) = \Delta x \sum_{q=-n}^{n-1} \hat{\rho}\left(\frac{qL}{2n}\right) e^{-i\frac{2\pi pq}{2n}}$$

$$\hat{\rho}\left(\frac{qL}{2n}\right) = \Delta k \sum_{p=-n}^{n-1} s\left(\frac{p}{L}\right) e^{i\frac{2\pi pq}{2n}} \tag{12.34}$$

then we can define

$$\hat{\rho}_{MRI}\left(\frac{qL}{2n}\right) = \hat{\rho}\left(\frac{qL}{2n}\right) \Delta x \tag{12.35}$$

$$s\left(\frac{p}{L}\right) = \sum_{q=-n}^{n-1} \hat{\rho}_{MRI}\left(\frac{qL}{2n}\right) e^{-i\frac{2\pi pq}{2n}}$$

$$\hat{\rho}_{MRI}\left(\frac{qL}{2n}\right) = \frac{1}{2n} \sum_{p=-n}^{n-1} s\left(\frac{p}{L}\right) e^{i\frac{2\pi pq}{2n}} \tag{12.36}$$

these last two define a DFT pair.

Read on own.

12.3: RF Coils, Noise and Filtering

12.4 Nonuniform Sampling

Homework

Do 12.1, 12.2, 12.3, 12.4