Biophysics 230: Nuclear Magnetic Resonance
Haacke Chapter 11

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11: The Continuous and Discrete Fourier Transforms

The Continuous Fourier Transform

We've already discussed the continuous 1D Fourier transform

$$F(\nu) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi\nu x}\, dx$$

and its inverse

$$f(x) = \int_{-\infty}^{+\infty} F(\nu) e^{+i2\pi\nu x}\, d\nu$$

The 1D signal which arises from the application of magnetic (gradient) fields in the x direction is

$$s(k) = \int_{-\infty}^{+\infty} \rho(x) e^{-i2\pi k x}\, dx \qquad (11.1)$$

11: The Continuous and Discrete Fourier Transforms

If we were able to continuously measure $s(k)$ for all k,
then we could compute the inverse Fourier transform

$$\rho(x) = \int_{-\infty}^{+\infty} s(k) e^{+i2\pi kx} \, dk \tag{11.7}$$

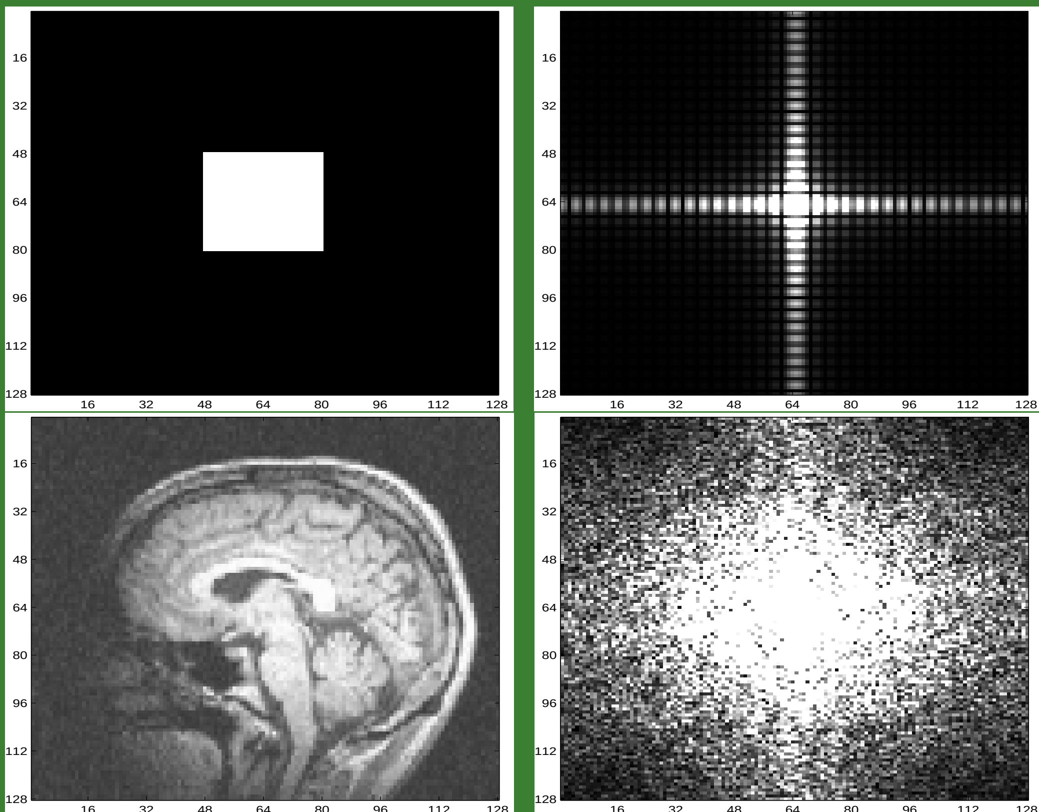
and obtain our spin density.

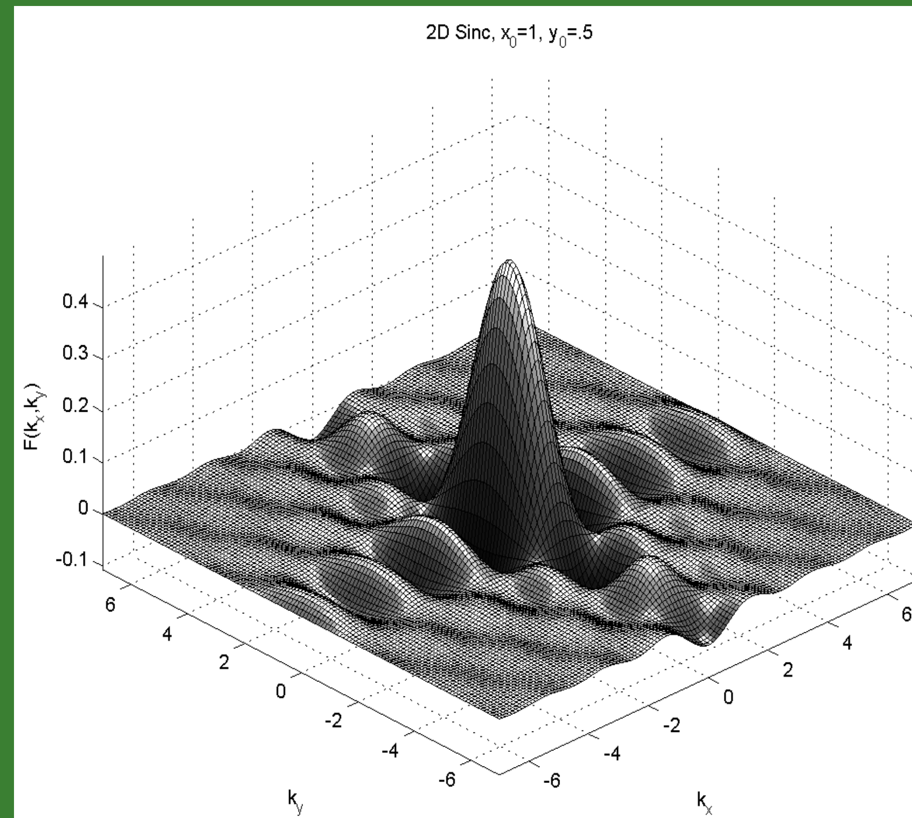
Unfortunately or fortunately we can only measure/record/observe the signal at discrete time intervals. Every Δt.

If the measured signal is $s_m(k)$, the inverse Fourier transform which is the estimated spin density $\hat{\rho}(x)$ is

$$\hat{\rho}(x) = \int_{-\infty}^{+\infty} s_m(k) e^{+i2\pi kx} \, dk \tag{11.8}$$

11.2: Continuous Transform Properties & Phase Imaging





I can do this too!

Actually the MR image was a .jpg (joint photographic experts group).

11.2: Continuous Transform Properties & Phase Imaging

Complexity of the Reconstructed Image

Our true image $\rho(x,y)$ is always real,
our reconstructed image $\hat{\rho}(x,y)$ is not in general real.

One obvious reason is a constant phase shift.

$$\tilde{s}(k) = e^{i\phi_0} s(k) \tag{11.9}$$

which can arise from the real and imaginary channels being switched or from incorrect demodulation and leads to

$$\hat{\rho}(x) = e^{i\phi_0} \rho(x) \tag{11.10}$$

11.2: Continuous Transform Properties & Phase Imaging

The Shift Theorem

Signal shifted by k_0 in k-space so truly not $s(k)$ but truly $s(k-k_0)$.

Can happen because of improper demodulation or
center of echo at $k=k_0$ instead of $k=0$

The effect on the 1D spin density is

$$\begin{aligned}\hat{\rho}(x) &= \int_{-\infty}^{+\infty} s_m(k-k_0)e^{i2\pi kx}\,dk \\ &= e^{i2\pi k_0 x}\int_{-\infty}^{+\infty} s_m(k')e^{i2\pi kx}\,dk' \\ &= e^{i2\pi k_0 x}\hat{\rho}_{expected}(x) \qquad (11.11)\end{aligned}$$

in the third line change of variable $k'=k-k_0$ and $dk'=dk$.
The magnitude is not affected, $|\hat{\rho}(x)|=\hat{\rho}(x)$.

11.2: Continuous Transform Properties & Phase Imaging

There can also be a spatial shift due to an improper read gradient as in Figure 11.2.

Phase Imaging and Phase Aliasing

Read. Not discussing. I think it is important.

Duality

The shift theorem also applies to the delta function.

$$\begin{aligned} h(x) &= \delta(x - x_0) \\ H(k) &= e^{-i2\pi k x_0} \end{aligned} \tag{11.14}$$

11.2: Continuous Transform Properties & Phase Imaging

Convolution Theorem (The most important theorems in MRI!)

Get a modified image due to multiplication by function (‘filter’).
The Fourier transform of a product

$$\mathcal{F}\left\{g(x) \cdot h(x)\right\} = G(k) * H(k) \tag{11.16}$$

where

$$G(k) * H(k) \equiv \int G(k')H(k-k')\ dk' \ . \tag{11.17}$$

the reverse is also true,

$$\mathcal{F}\left\{g(x) * h(x)\right\} = G(k) \cdot H(k)$$

The details of the convolution are learned on Biophysics 03240.

But do know the theorem! There are very important practical implications.
See Figure 11.5.

11.2: Continuous Transform Properties & Phase Imaging

Convolution Associativity

$a(x) * (b(x) * c(x)) = (a(x) * b(x)) * c(x)$

Other Convolution Properties

$$\begin{aligned} g(x) * h(x) &= h(x) * g(x) && \text{commutative} \\ g(x) * (h_1(x) + h_2(x)) &= g(x) * h_1(x) + g(x) * h_2(x) && \text{distributive} \end{aligned}$$

11.2: Continuous Transform Properties & Phase Imaging

Derivative Theorem (We already discussed this)

$$\begin{aligned}
\mathcal{F}\{f'(x)\} &= \mathcal{F}\left\{\frac{df(x)}{dx}\right\} \\
&= \int_{-\infty}^{+\infty} \left[\frac{df(x)}{dx}\right] e^{-i2\pi kx}\, dx \\
&= \left[f(x)e^{-i2\pi kx}\Big|_{-\infty}^{+\infty} - (-i2\pi k)\int_{-\infty}^{+\infty} f(x)e^{-i2\pi kx}\, dx \right] \\
&= i2\pi k F(k) \qquad (11.22)
\end{aligned}$$

in the second to last line we integrated by parts

$$\int g(x)h'(x)\, dx = g(x)h(x) - \int g'(x)h(x)dx$$

where $g(x) = e^{-i2\pi kx}$ and $h'(x) = \frac{df(x)}{dx}$.
FT of derivative (1D image) $f'(x)$ by multiplying $F(k)$ by $i2\pi k$.

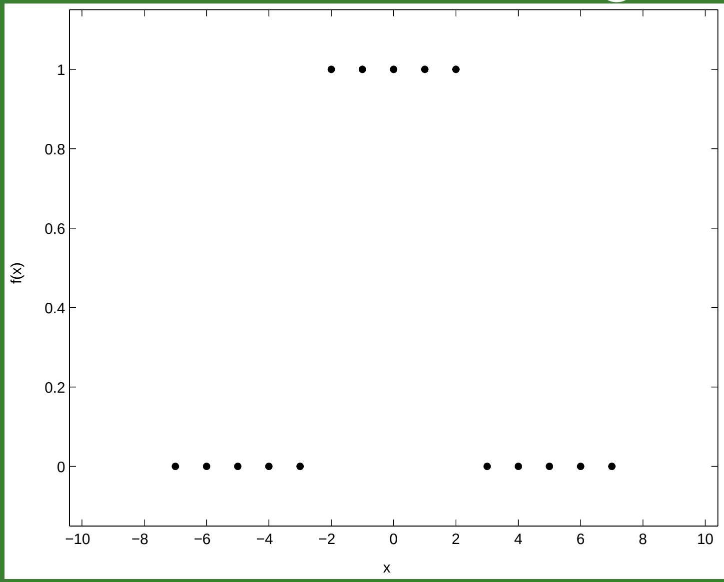
11.2: Continuous Transform Properties & Phase Imaging

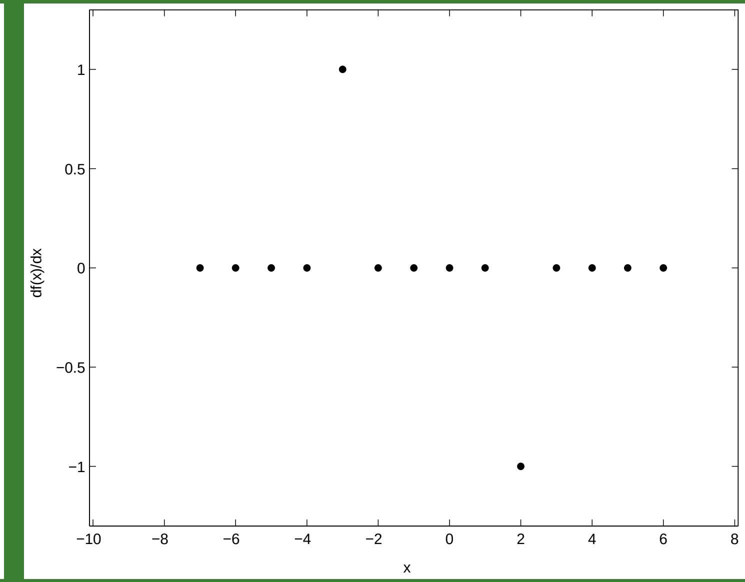
Example:

When taking the derivative of an image, it is actually discrete differences.
Consider the following discrete function $f(x)$.
Define the derivative at a point x_i to be

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Then the function along with its derivative are





This is also true in two dimensions.

11.2: Continuous Transform Properties & Phase Imaging

Now look at Figure 11.6 in the book.

A 1D derivative of the magnitude image is taken in the vertical direction.

The magnitude means derivative is strictly positive.

We could also take a 2D derivative, the gradient.
The derivative or gradient tells us where the edges are in the image.
Can also do the Laplacian.

What about integrals?

Fourier Transform Symmetries
Read.

11.2: Continuous Transform Properties & Phase Imaging

Summary of 1D Fourier Transform Properties

Property	Function	Transform
Linearity	$af(x) + bg(x)$	$aF(k) + bG(k)$
Similarity	$f(ax)$	$\frac{1}{\|a\|}F(\frac{k}{a})$
Shifting	$f(x-a)$	$e^{-i2\pi ka}F(k)$
Derivative	$\frac{d^\ell f(x)}{dx^\ell}$	$(i2\pi k)^\ell F(k)$

11.3: Fourier Transform Pairs

Heaviside Function

The Heaviside function is

$$\Theta(k) = \begin{cases} 1 & k > 0 \\ \frac{1}{2} & k = 0 \\ 0 & k < 0 \end{cases} . \tag{11.24}$$

Here is a picture of the Heaviside function. Often it doesn't have the $\frac{1}{2}$.

The inverse Fourier transform of the Heaviside function can be found as

$$h_{\Theta}(x) = \mathcal{F}^{-1}\{\Theta(k)\}$$

11.3: Fourier Transform Pairs

$$
\begin{aligned}
h_\Theta(x) &= \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \Theta(k) e^{i2\pi kx} e^{-2\pi\epsilon|k|} \, dk \\
&= \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{0^-} \overbrace{\Theta(k)}^{=0} e^{i2\pi kx} e^{-2\pi\epsilon(-k)} \, dk \right. \\
&\quad \left. + \int_0^{+\infty} \Theta(k) e^{i2\pi kx} e^{-2\pi\epsilon k} \, dk \right] \\
&= \lim_{\epsilon \to 0^+} \int_0^{+\infty} \Theta(k) e^{-2\pi(\epsilon - ix)k} \, dk \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi(\epsilon - ix)} e^{-2\pi(\epsilon - ix)k} \Big|_0^{+\infty}
\end{aligned}
$$

*$e^{-2\pi|k|x}$ to eliminate convergence ambiguities

11.3: Fourier Transform Pairs

$$\begin{aligned}
h_{\Theta}(x) &= \begin{matrix}\lim \\ \epsilon \rightarrow 0^{+}\end{matrix} \frac{1}{2\pi(\epsilon - ix)} \\
&= \begin{matrix}\lim \\ \epsilon \rightarrow 0^{+}\end{matrix} \frac{1}{2\pi}\frac{1}{(\epsilon - ix)}\frac{(\epsilon + ix)}{(\epsilon + ix)} \\
&= \begin{matrix}\lim \\ \epsilon \rightarrow 0^{+}\end{matrix} \frac{1}{2\pi}\frac{(\epsilon + ix)}{(\epsilon^2 + x^2)} \\
&= \begin{matrix}\lim \\ \epsilon \rightarrow 0^{+}\end{matrix} \frac{1}{2\pi}\left[\frac{\epsilon}{(\epsilon^2 + x^2)} + \frac{ix}{(\epsilon^2 + x^2)}\right] \\
&= \frac{1}{2\pi}\delta(x) + \frac{i}{2\pi}P(\frac{1}{x}) \qquad (11.25)
\end{aligned}$$

11.3: Fourier Transform Pairs

Lorentzian Form

The two sided (double exponential)

$$f(k) = e^{-a|k|}$$

inverse Fourier transform of this is

$$\begin{aligned} F(x) &= \int_{-\infty}^{+\infty} e^{-a|k|} e^{i2\pi kx} \, dk \\ &= \int_{-\infty}^{+\infty} e^{-a|k|} e^{i2\pi kx} \, dk \\ &= \int_{-\infty}^{0} e^{(i2\pi x + a)k} \, dk + \int_{0}^{+\infty} e^{(i2\pi x - a)k} \, dk \\ &= \frac{1}{(i2\pi x + a)} e^{(i2\pi x + a)k} \Big|_{-\infty}^{0} + \frac{1}{(i2\pi x - a)} e^{(i2\pi x - a)k} \Big|_{0}^{+\infty} \end{aligned} \tag{11.26}$$

11.3: Fourier Transform Pairs

Lorentzian Form

$$\begin{aligned} F(x) &= \frac{1}{(i2\pi x + a)} - \frac{1}{i2\pi x - a} \\ &= \frac{1}{(i2\pi x + a)}\frac{(i2\pi x + a)}{(i2\pi x + a)} - \frac{1}{(i2\pi x - a)}\frac{(-i2\pi x + a)}{(-i2\pi x + a)} \\ &= \frac{i2\pi x + a}{(2\pi x)^2 + a^2} + \frac{-i2\pi x + a}{(2\pi x)^2 + a^2} \\ &= \frac{2a}{(2\pi x)^2 + a^2} \end{aligned}$$

which is called “Lorentzian” form.

11.3: Fourier Transform Pairs

The Sampling Function

The sampling or comb function $u(k)$ is defined to be the sum of a (doubly) infinite number of delta functions each Δk apart.

$$u(k) = \Delta k \sum_{p=-\infty}^{+\infty} \delta(k - p\Delta k) \tag{11.28}$$

When we measure/record/observe/sample the signal $s(k)$

at discrete k-space (time) points Δk (Δt) apart,

this is equivalent to multiplying the continuous signal by the comb function.

11.4 The Discrete Fourier Transforms

The inverse Fourier transform of the comb function is

$$\begin{aligned} U(x) &= \mathcal{F}\{u(k)\} \\ &= \Delta k \sum_{p=-\infty}^{+\infty} \int \delta(k - p\Delta k) e^{i2\pi kx}\, dk \\ &= \Delta k \sum_{p=-\infty}^{+\infty} e^{i2\pi p\Delta kx} \end{aligned} \quad (11.29)$$

where

$$\sum_{n=-\infty}^{+\infty} e^{i2\pi na} = \sum_{m=-\infty}^{+\infty} \delta(a - m) \quad (11.30)$$

and thus

$$U(x) = \sum_{q=-\infty}^{+\infty} \delta(x - \frac{q}{\Delta k}) \quad (11.31)$$

11.4 The Discrete Fourier Transforms

The DFT is an approximation to the continuous FT.

Often in theory assume $s_m(k)$ is continuous so we could perform an IFT.

In reality we sample the signal $s_m(k)$ at discrete k-space (time) points.

With a length scale L, the discrete Fourier transform

$$G\left(\frac{p}{L}\right) \equiv \mathcal{D}(g) = \sum_{q=-n}^{n-1} g\left(\frac{qL}{2n}\right) e^{-\frac{i2\pi pq}{2n}} \qquad (11.32)$$

$\Delta k = 1/L$ and $L = 2n\Delta x$

$$G\left(p\Delta k\right) = \sum_{q=-n}^{n-1} g\left(q\Delta x\right) e^{-\frac{i2\pi pq\Delta x\Delta k}{2n}} \qquad (11.33)$$

$$g\left(\frac{ql}{2n}\right) \equiv \mathcal{D}^{-1}(G) = \frac{1}{2n}\sum_{p=-n}^{n-1} G\left(\frac{p}{L}\right) e^{\frac{i2\pi pq}{2n}} g\left(q\Delta x\right) \qquad (11.34)$$

$$= \frac{1}{2n}\sum_{p=-n}^{n-1} G\left(p\Delta k\right) e^{\frac{i2\pi pq}{2n}} \qquad (11.35)$$

11.4 The Discrete Fourier Transforms

Other DFT Pair Parameterizations

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi ux/N}$$

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{+i2\pi ux/N}$$

$$F(u) = \sum_{x=1}^{N} f(x) e^{-i2\pi(u-1)(x-1)/N}$$

$$f(x) = \frac{1}{N} \sum_{u=1}^{N} F(u) e^{+i2\pi(u-1)(x-1)/N}$$

$*$ Matlab's "fft" and "ifft" use the second one.

11.5 Discrete Transform Properties

The Discrete Convolution Theorem

The convolution theorem holds for the discrete convolution just as for the continuous convolution. Using a shorthand notation,

$$g_1(q)g_2(q) \overset{\mathcal{D}}{-} G_1(p) * G_2(p) \tag{11.39}$$

and

$$G_1(p) * G_2(p) = \frac{1}{2n} \sum_{r=-n}^{n-1} G_1(r)G_2(p-r) \tag{11.40}$$

also

$$g_1(q) * g_2(q) \overset{\mathcal{D}}{-} G_1(p)G_2(p) \tag{11.41}$$

11.5 Discrete Transform Properties

Summary of Discrete Fourier Transform Properties

Table 11.4 summarizes the DFT properties.