Biophysics 230: Nuclear Magnetic Resonance
Haacke Chapter 10

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10.1: Imaging in More Dimensions

The 1D signal $s(k)$ given by

$$s(k) = \int \rho(z) e^{-i2\pi kz} \, dk \tag{9.15}$$

is generalized to a signal $s(\vec{r})$ in 3D $\vec{r}$ from a single RF excitation

$$s(\vec{k}) = \int \rho(\vec{r}) e^{-i2\pi \vec{k}\cdot\vec{r}} \, d^3r \tag{10.1}$$

or in terms of Cartesian coordinates

$$\begin{aligned} s(k_x, k_y, k_z) &= \int\int\int \rho(x, y, z) e^{-i2\pi(k_x x + k_y y + k_z z)} \, dxdydz \\ &= \mathcal{F}[\rho(x, y, z)] \end{aligned} \tag{10.2}$$

this is the 3D *imaging equation*.

10.1: Imaging in More Dimensions

The three time t and gradient $G(t)$ dependent k-space coordinates are

$$k_x(t) = \gamma \int^t G_x(t')\, dt',$$
$$k_y(t) = \gamma \int^t G_y(t')\, dt',$$
$$k_z(t) = \gamma \int^t G_z(t')\, dt'$$

which are the areas between the gradient curves and the primed time axis.

10.1: Imaging in More Dimensions

Need to apply the gradients G_x, G_y, G_z in a particular fashion

so that the (frequency and phase) k-space can be sufficiently covered

and sampled to reconstruct (estimate) the spin density $\hat{\rho}(\vec{r})$

with the use of the inverse (discrete) Fourier transform

$$\hat{\rho}(\vec{r}) = \int s_m(\vec{k}) e^{i2\pi \vec{k}\cdot\vec{r}}\, d^3k \tag{10.3}$$

10.1: Imaging in More Dimensions

Consider the covering of (k_x, k_y) space in 3D imaging for a particular k_z.

But let's first take a look at 1D imaging, $k_y = 0$ line.

The 2D signal with $k_y = 0$ (and $k_z = 0$) will be according to

$$s(k_x, 0) = \int \rho(\vec{r}) e^{-i2\pi(k_x x + \overbrace{k_y}^{=0} y + \overbrace{k_z}^{=0} z)} \, d^3r \tag{10.4}$$

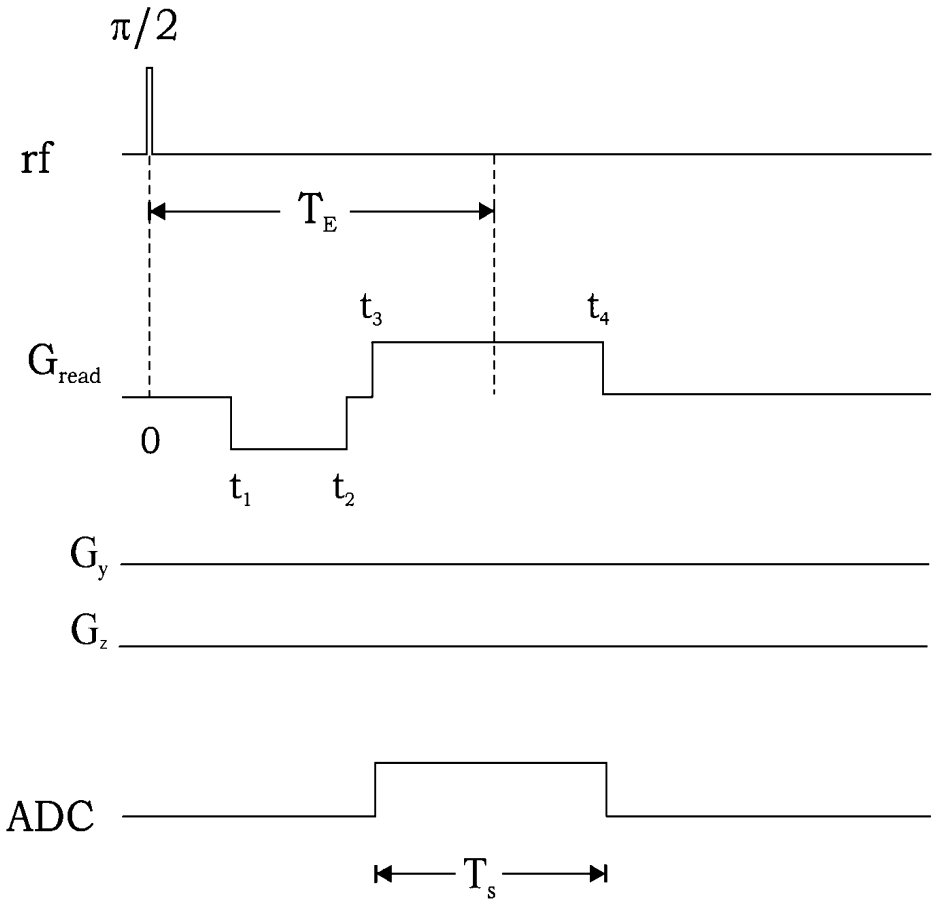
10.1: Imaging in More Dimensions

The inverse FT of the signal data along this line (as in Chapter 9) is*

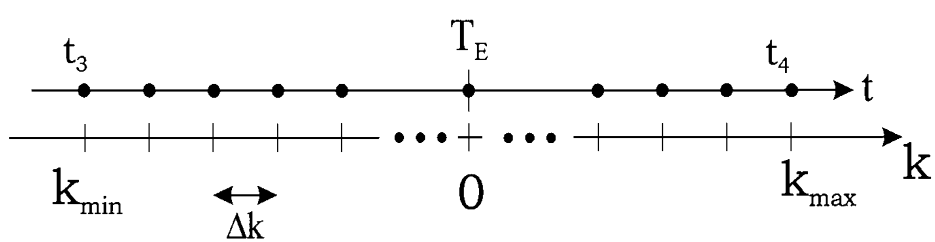
$$
\begin{aligned}
\hat{\rho}_{1D}(x) &= \int_{k_x} s(k_x,0) e^{i2\pi(k_x x+0y+0z)}\, dk_x && \text{insert}(10.5) \\
&= \int_{k_x} \left[\int_{r'} \rho(\vec{r'}) e^{-i2\pi k_x x'}\, d^3r' \right] e^{i2\pi k_x x}\, dk_x && \text{exponential inside} \\
&= \int_{k_x} \left[\int_{r'} \rho(\vec{r'}) e^{-i2\pi k_x (x-x')}\, d^3r' \right]\, dk_x && \text{cartesian} \\
&= \int_{k_x} \left[\int_{z'} \int_{y'} \int_{x'} \rho(\vec{r'}) e^{-i2\pi k_x (x-x')}\, dx'dy'dz' \right]\, dk_x && \text{switch order} \\
&= \int_{z'} \int_{y'} \int_{x'} \rho(\vec{r'}) \left[\int_{k_x} e^{-i2\pi k_x (x-x')}\, dk_x \right]\, dx'dy'dz' && \text{is delta} \\
&= \int_{z'} \int_{y'} \int_{x'} \rho(x',y',z') \delta(x-x')\, dx'dy'dz' && \text{only for } x=x' \\
&= \int_{z} \int_{y} \rho(x,y,z)\, dydz && (10.6)
\end{aligned}
$$

* Note: " ' " is not rotating frame

10.1: Imaging in More Dimensions, 1D Coverage



(a)



(b)

The $(\frac{\pi}{2})_x$ RF pulse at $t = 0$ tips the spins in the y direction.

The negative gradient between t_1 and t_2 moves us to k_{min} (left).

The positive gradient between t_3 and t_4 moves us to the right.

We measure the signal $s(t)$ between t_3 and t_4.

Note that $t_2 - t_3 = (t_4 - t_3)/2$.

10.1: Imaging in More Dimensions

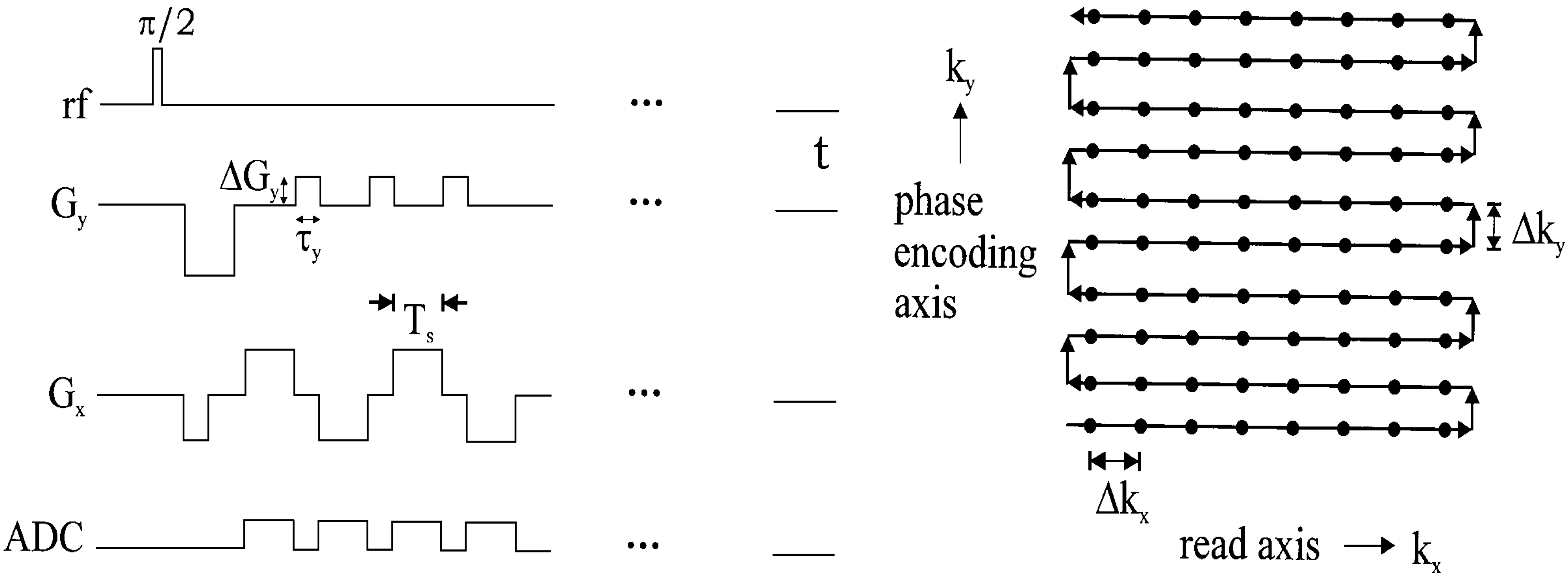
The generalization of the signal equation for all k_y (and a particular k_z) is

$$s(k_x, k_y) = \int_z \int_y \int_x \rho(x, y, z) e^{-i2\pi(k_x x + k_y y)} \, dx dy dz \qquad (10.7)$$

$\hat{\rho}(x, y)$ can be found in a similar fashion.

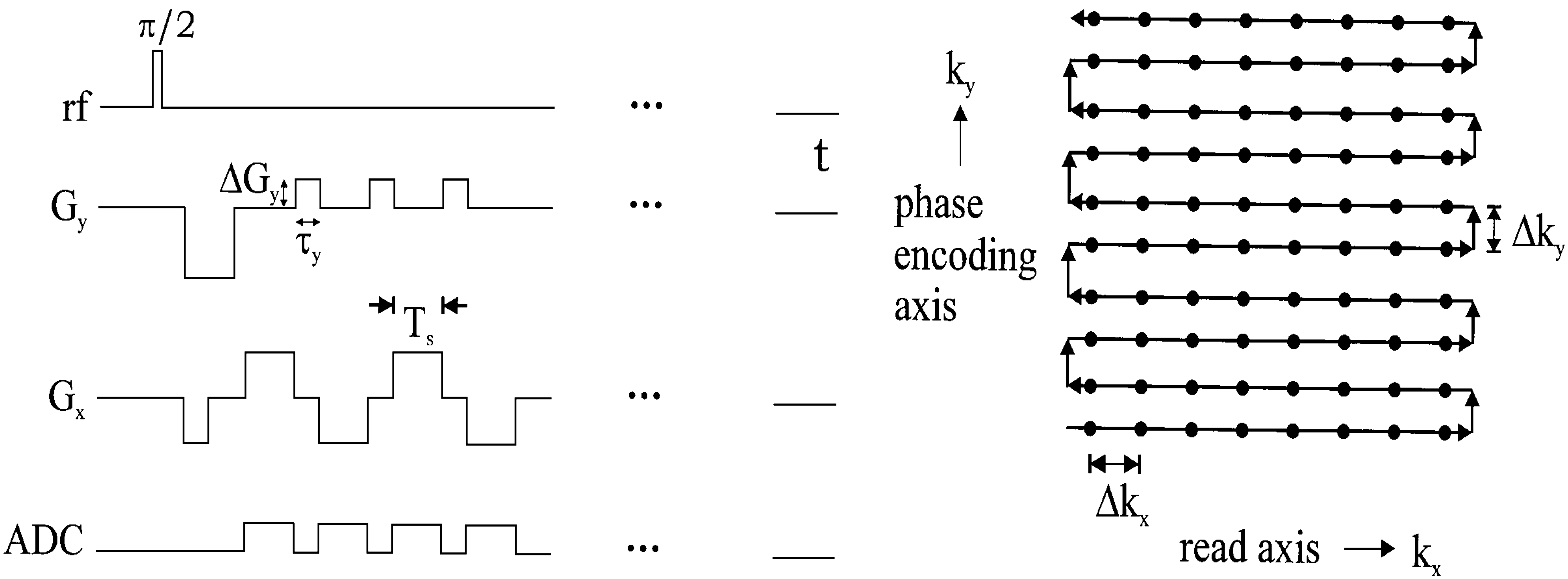
$$\begin{aligned}
\hat{\rho}(x, y) &= \int_{k_y} \int_{k_x} s(k_x, k_y) e^{i2\pi(k_x x + k_y y)} \, dk_x dk_y \\
&= \int_{k_y} \int_{k_x} \left[\int_{z'} \int_{y'} \int_{x'} \rho(x', y', z') e^{-i2\pi(k_x x + k_y y)} \, dx' dy' dz' \right] e^{i2\pi(k_x x + k_y y)} \, dk_x dk_y \\
&= \int_{k_y} \int_{k_x} \left[\int_{z'} \int_{y'} \int_{x'} \rho(x', y', z') e^{-i2\pi[k_x(x - x') + k_y(y - y')]} \, dx' dy' dz' \right] dk_x dk_y \\
&= \int_{z'} \int_{y'} \int_{x'} \rho(x', y', z') \left[\int_{k_y} \int_{k_x} e^{-i2\pi[k_x(x - x') + k_y(y - y')]} \, dk_x dk_y \right] dx' dy' dz' \\
&= \int_{z'} \int_{y'} \int_{x'} \rho(x', y', z') \delta(x - x', y - y') \, dx' dy' dz' \\
&= \int_z \rho(x, y, z) \, dz \qquad (10.8)
\end{aligned}$$

10.1: Imaging in More Dimensions, 2D Coverage



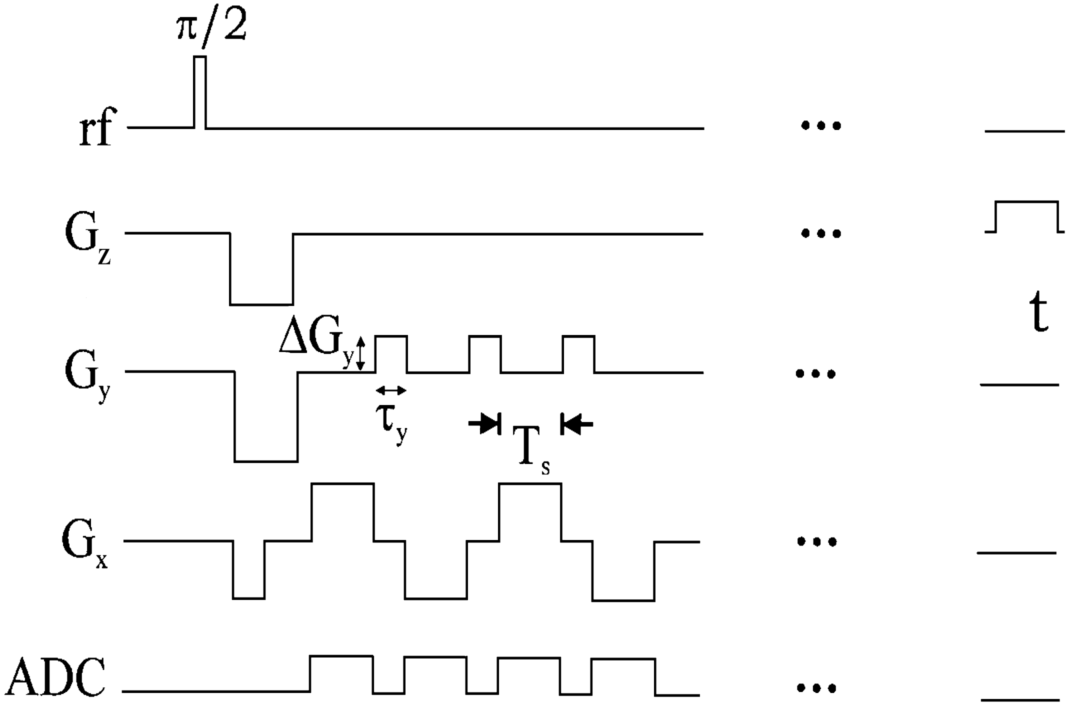
1. The $(\frac{\pi}{2})_x$ RF pulse at $t=0$ tips the spins in the y direction.
2. The negative gradient $-G_y$ moves us to $k_{y,min}$ (bottom).
3. The negative gradient $-G_x$ moves us to $k_{x,min}$ (left).
4. The $+G_x$ gradient moves us from $k_{x,min}$ to $k_{x,max}$. Data every Δt or $\Delta k_x = -\gamma \Delta G_x \Delta t$.

10.1: Imaging in More Dimensions, 2D Coverage



5. When we reach $k_{x,max}$, turn on $+\Delta G_y$ for time τ_y. Up 1 line $\Delta k_y = \bar{\gamma}\,\Delta G_y \tau_y$.
6. The $-G_x$ gradient moves us from $k_{x,max}$ to $k_{x,min}$. Data at Δt or $\Delta k_x = \bar{\gamma}\,\Delta G_x \Delta t$.
7. When we reach $k_{x,min}$, turn on $+\Delta G_y$ on for time τ_y. Up 1 line $\Delta k_y = \bar{\gamma}\,\Delta G_y \tau_y$
8. Repeat steps 4.-7. until the (k_x, k_y) space is covered.

10.1: Imaging in More Dimensions, 3D Coverage

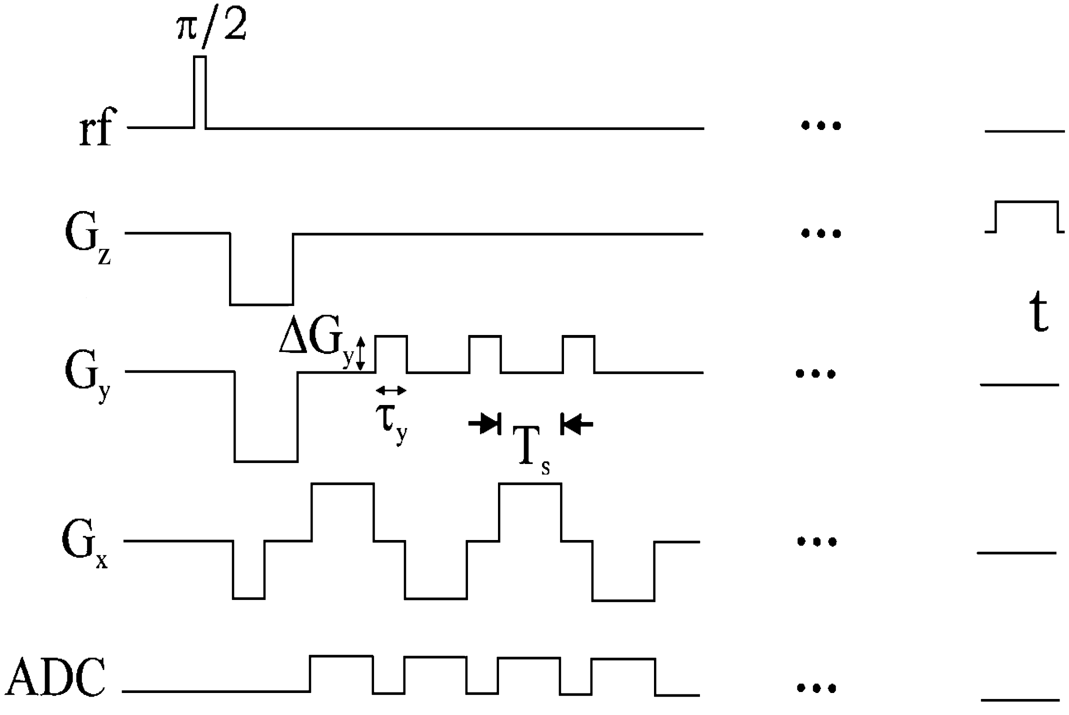


3D imaging similar to 2D.

Don't like Figure 10.4.
Take Figure 10.2
Add negative G_x, G_y, and G_z.
Add positive G_z blip.

1. The $(\frac{\pi}{2})_x$ RF pulse at $t=0$ tips the spins in the y direction.
2. The negative gradient $-G_z$ moves us to $k_{z,min}$ (bottom z).
3. The negative gradient $-G_y$ moves us to $k_{y,min}$ (bottom).
4. The negative gradient $-G_x$ moves us to $k_{x,min}$ (left).
5. The $+G_x$ gradient moves us from $k_{x,min}$ to $k_{x,max}$. Data every Δt or $\Delta k_x = -\gamma \Delta G_x \Delta t$.

10.1: Imaging in More Dimensions, 3D Coverage



3D imaging similar to 2D .

Don't like Figure 10.4.
Take Figure 10.2
Add negative G_x, G_y, and G_z.
Add positive G_z blip.

6. When we reach $k_{x,max}$, turn on $+\Delta G_y$ for time τ_y. Up 1 line $\Delta k_y = -\gamma\, \Delta G_y \tau_y$.
7. The $-G_x$ gradient moves us from $k_{x,max}$ to $k_{x,min}$. Data at Δt or $\Delta k_x = -\gamma\, \Delta G_x \Delta t$.
8. When we reach $k_{x,min}$, turn on $+\Delta G_y$ on for time τ_y. Up 1 line $\Delta k_y = -\gamma\, \Delta G_y \tau_y$.
9. Repeat steps 5.-8. until (k_x, k_y) space covered.
10. Turn $+\Delta G_z$ on for τ_z. Up one slice in $\Delta k_z = -\gamma \Delta G_z \tau_z$. Repeat 1.-9 until k_z covered.

10.1: Imaging in More Dimensions, Multiple RF

Time Constraints and Collecting Data over Multiple Cycles

Typically only one slice of k space is collected following each RF excitation.

This is because of signal loss due to decay.

This means multiple excitations at every T_R.

When doing this, all k_x are collected from left to right.

Look at Figures 10.4 and 10.5.
Let N_y and N_z denote the number of phase encoding steps. Then it takes

$$T_{acq} = N_y N_z T_R \tag{10.11}$$

for 3D and

$$T_{acq} = N_y T_R \tag{10.12}$$

for 2D. These are increased by N_{acq} if we repeat N_{acq} to reduce noise.

10.1: Imaging in More Dimensions, Multiple RF

Variations in k-Space Coverage

The way in which k-space is covered can be varied as long as each point is covered (according to the book).

Methods were developed so that every every k-space point is not needed.

Instead of sampling at time Δt while a constant $G_x = G_0$ is applied,

May instead change the gradient by the amount ΔG_x each time we sample

to generate the same Δk_x.

These are identical.

10.2: Slice Selection with Boxcar Excitations

Slice Selection

The 'slice select axis' is perpendicular to the desired slice.
$z-axis$ gradient selects transverse
$y-axis$ gradient selects coronal
$x-axis$ gradient selects sagittal

Let's take $z-axis$ or transverse slices.
The frequency along this direction (with a constant linear gradient) is

$$f(z)=f_0+\gamma\!\!\!{}^{-}\, G_z z \tag{10.15}$$

where $f_0=\gamma\!\!\!{}^{-}\, B_0$ is the Larmor frequency (see figure 10.7).

To select (excite) the slice $(z_0-\Delta z/2)$ to $(z_0+\Delta z/2)$, the RF pulse

must be uniform over $(\gamma\!\!\!{}^{-}\, G_z z_0-\gamma\!\!\!{}^{-}\, G_z\Delta z/2)$ to $(\gamma\!\!\!{}^{-}\, G_z z_0+\gamma\!\!\!{}^{-}\, G_z\Delta z/2)$,

which is shown in Figure 10.8.

10.2: Slice Selection with Boxcar Excitations

The bandwidth Δf is

$$\begin{aligned} BW_{rf} &\equiv \Delta f & (10.16)\\ &= (\gamma\!\!\!^{-} G_z z_0 + \gamma\!\!\!^{-} G_z \Delta z/2) - (\gamma\!\!\!^{-} G_z z_0 - \gamma\!\!\!^{-} G_z \Delta z/2) & (10.17)\\ &= \gamma\!\!\!^{-} G_z \Delta z & (10.18)\end{aligned}$$

Denote

$$\Delta z \equiv \mathrm{TH} \qquad (10.19)$$

and as a result

$$TH = \frac{BW_{rf}}{\gamma\!\!\!^{-} G_z} \qquad (10.20)$$

In order to get a uniform flip, the frequency profile must be a boxcar function rect$\left(\frac{f}{\Delta f}\right)$ of bandwidth Δf.

This means the temporal envelope of the rf pulse $B_1(t)$ is a sinc function

$$B_1(t) \propto \mathrm{sinc}(\pi \Delta f t) \qquad (10.21)$$

This is in Chapter 16. Just accept it for now.

10.2: Slice Selection with Boxcar Excitations

The sinc function is extremely important in fMRI.

You can set it equal to zero and find that its first zero crossing is $t_1 = \frac{1}{\Delta f}$.

In general, the zeros of sinc(x) =sin$(x)/x$ are $x = n\pi$ for all nonzero integers n.

Realistic sinc pulses are only on for time τ_{rf} and thus truncated with

$$\begin{aligned} n_{zc} &= \left[\frac{\tau_{rf}}{t_1}\right] \\ &= \left[\Delta f \tau_{rf}\right] \end{aligned} \tag{10.22}$$

where the brackets denote the largest integer less than its value.

The longer the pulse is on, the more zero crossings and the closer the real approximate pulse is to its theoretical value.

10.2: Slice Selection with Boxcar Excitations

Examples:

$G_z = 20mT/m$.

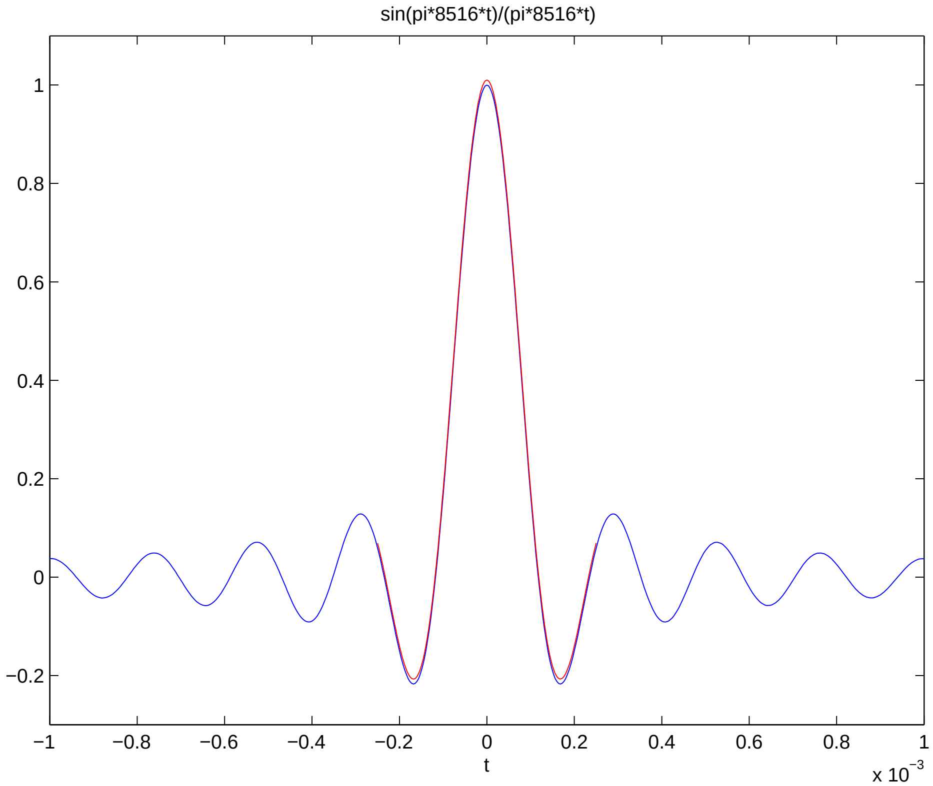
$\tau_{rf} = .002s$ vs $\tau_{rf} = .0005s$.

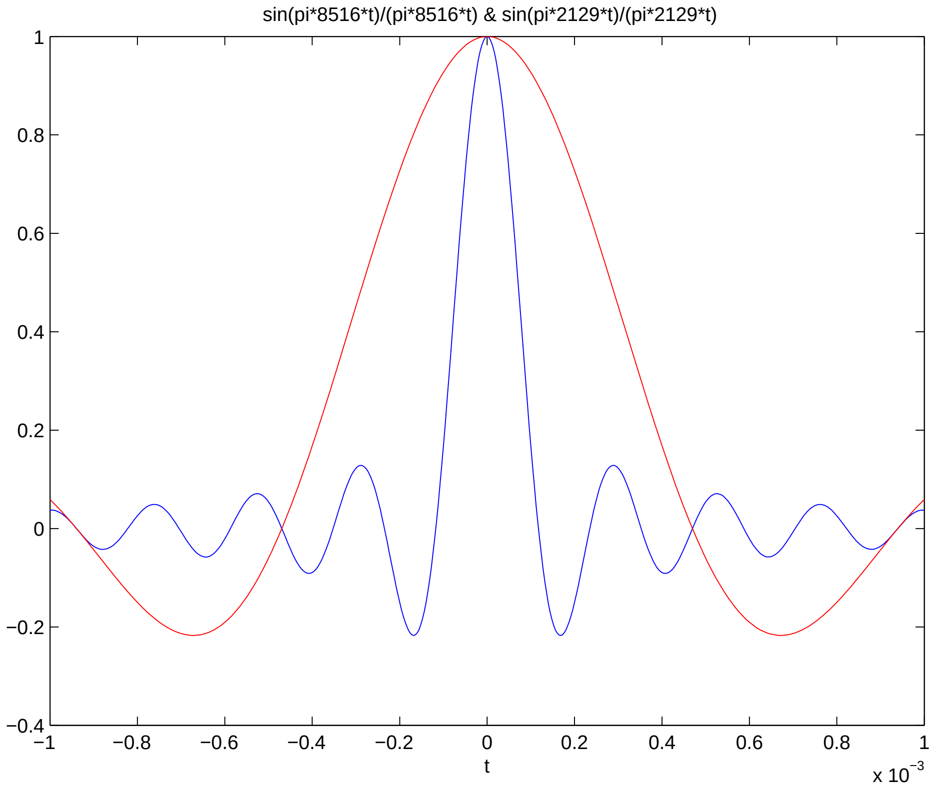
(Blue) (red)

$\tau_{rf} = .002$s.

$G_z = 20mT/m$ vs $G_z = 5mT/m$.

(Blue) (red)





The longer more precise the sinc, the more ideal square slice profile.

10.2: Slice Selection with Boxcar Excitations

Gradient Rephasing

As in 1D GRE imaging case, get an echo by adding a rephasing gradient.

Look at Figure 10.9

If slice is instantaneously excited at $t = 0$ as in Figure 10.9 with constant gradient strength $G_z = G_{SS}$, then the phase of the transverse magnetization from Equation 9.13 is

$$\phi(z,t) = -\gamma G_{SS} z t \tag{10.23}$$

The signal $s(t)$ in the slice of width Δz at t follows Equation (9.14)

$$s(t) = \int \rho(z) e^{i\phi_G(z,t)}\, dz \tag{9.14}$$

It effectively has a one-dimensional spin density $\rho(z)$ as defined in Equation (9.7) given by

$$\rho(z) \equiv \int\int \rho(\vec{r})\, dx dy \tag{9.7}$$

10.2: Slice Selection with Boxcar Excitations

Gradient Rephasing

The spin density can also be assumed to be constant over time (but not ϕ which is a function of z) resulting in

$$s(t) \simeq \rho(z_0) \int_{z_0-\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}} e^{i\phi(z,t)}\, dz \tag{10.24}$$

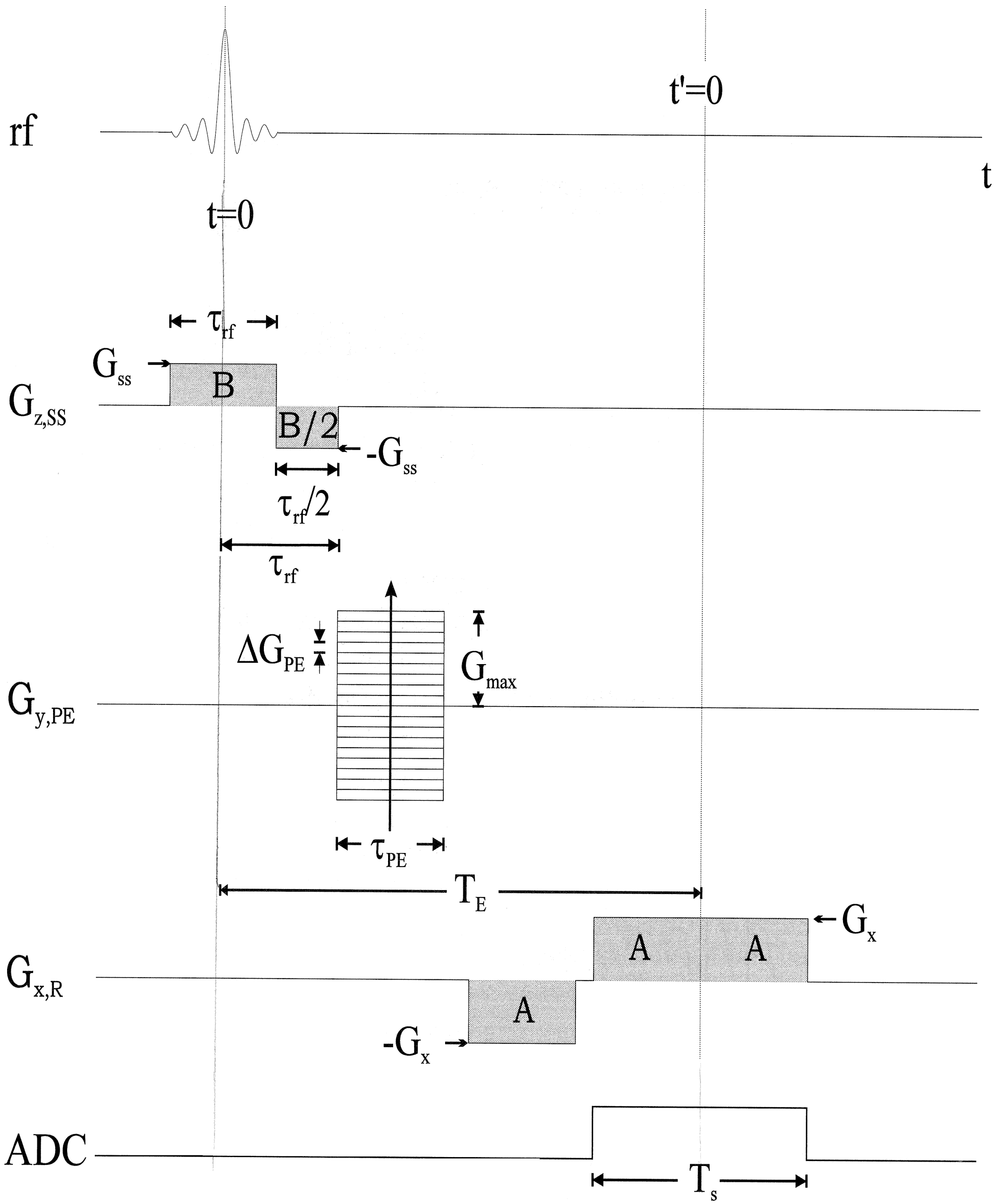
As time increases, the spins at different positions z accumulate differing amounts of phase. The signal decreases as a result of the integral

$$\int_{z_0-\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}} e^{i\phi(z,t)}\, dz \overrightarrow{\text{dephasing}}\ 0 \tag{10.25}$$

Arbitrary Slice Orientation

Not Covering.

10.3: 2D Imaging and k-space, Gradient Echo Example



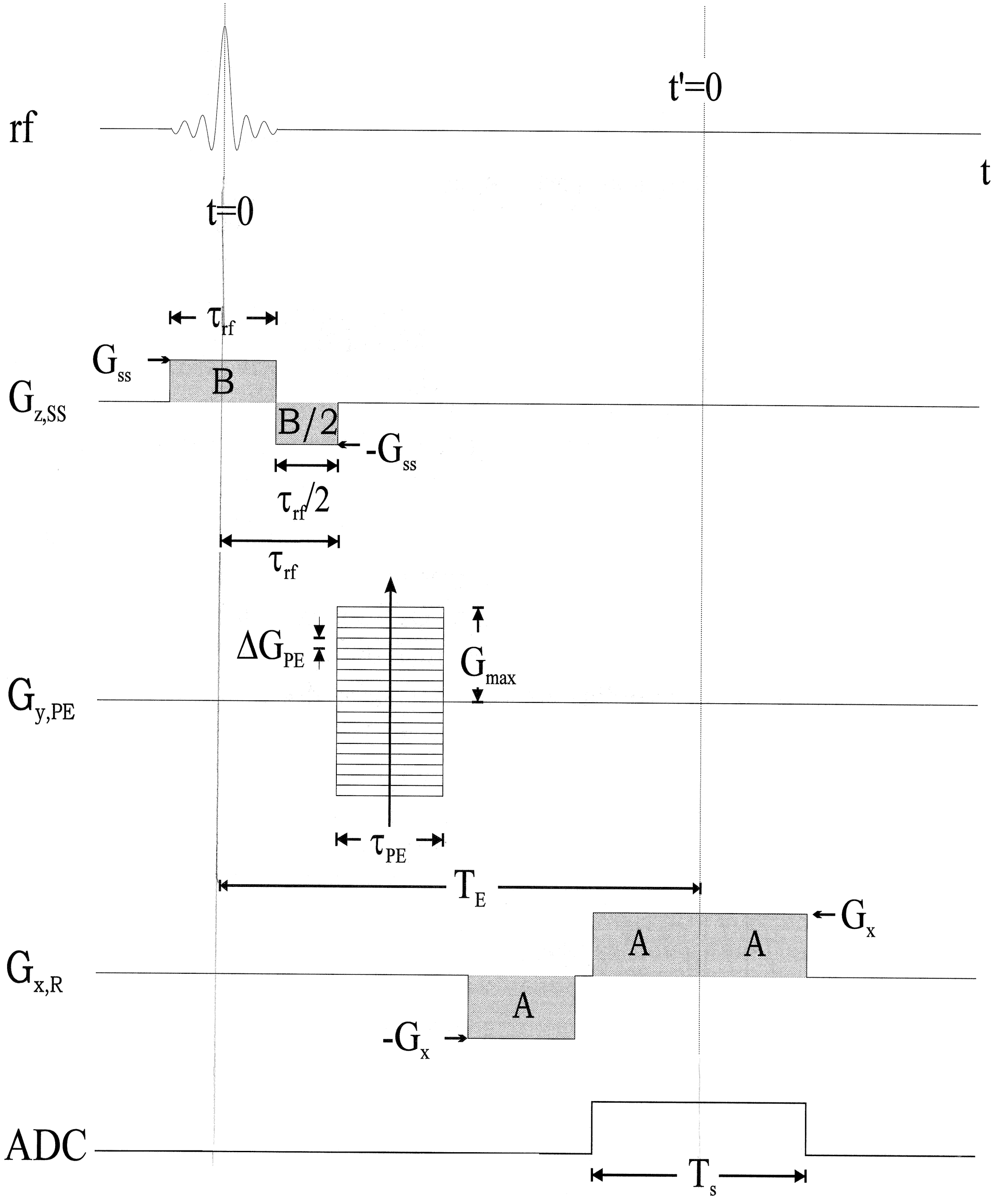
Slice Selecting

After the rephase lobe of the slice select gradient, the signal is described by

$$s(\tau_{rf}) = \int\int\left[\int_{z_0-\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}} \rho(x,y,z)\, dz\right]\, dxd \tag{10.29}$$

We now want to explore the spin density within the z slices.

10.3: 2D Imaging and k-space, Gradient Echo Example



Phase Encoding

Phase encoding y gradient is applied and while on, the signal described by

$$s(\tau_{rf}+\tau_y) = \int \left[\int [\quad] \, e^{-i2\pi\bar{\gamma} G_y \tau_y y} \, dy \right] dx$$

$$[\quad] = \left[\int_{z_0-\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}} \rho(x,y,z) \, dz \right] \tag{10.30}$$

which is a y dependent adaptation of Equation (10.23)

and in terms of k-space with

$$k_y(G_y) = \bar{\gamma}\, G_y \tau_y. \tag{10.31}$$

The y gradient G_y is varied in steps ΔG_{PE} as in 10.1.4.

10.3: 2D Imaging and k-space, Gradient Echo Example

Reading the Data

As in Chapter 9, an x gradient is applied in order to obtain an echo at T_E.

In terms of $t' = t - T_E$ with boxcar gradient of Figure 10.13, the signal is

$$s(t', G_y) = \int \left[\int \left[\int_{z_0 - \frac{\Delta z}{2}}^{z_0 + \frac{\Delta z}{2}} \rho(x,y,z) dz \right] e^{-i2\pi \gamma\!\!\!\!- G_y \tau_y y} dy \right] e^{-i2\pi \gamma\!\!\!\!- G_x t' x} dx$$

$$-T_s/2 < t' < T_s/2 \tag{10.32}$$

and in terms of k space with

$$k_x(t') = \gamma\!\!\!\!- \, G_x t' \tag{10.33}$$

and Equation (10.31) leads to

$$s(k_x, k_y) = \int \int \left[\int_{z_0 - \frac{\Delta z}{2}}^{z_0 + \frac{\Delta z}{2}} \rho(x,y,z) \, dz \right] e^{-i2\pi\gamma(k_x x + k_y y)} \, dx dy. \tag{10.34}$$

10.3: 2D Imaging and k-space, Gradient Echo Example

Assumption:

The average spin density over the z slice is often approximated by

the spin density at the center of the slice z_0

$$s(k_x, k_y) = \int\int \rho(x, y, z_0) e^{-i2\pi\gamma(k_x x + k_y y)}\, dx dy \tag{10.35}$$

By repeating the gradients in Figure 10.13 with the y gradient stepped up

each time gives additional k_y lines in (k_x, k_y) space for the selected z slice.

Repeat for each z slice.

10.3: 2D Imaging and k-space, Gradient Echo Example

Superposition of Phase Effects

Look at Figure 10.14

The phase encoding and the dephasing gradient of the read lobe

along with the rephasing lobe of the slice select gradient

can be simultaneously applied as in Figure 10.14.

The results are the same.

This does not apply during the application of the rf pulse or data acquisition.

10.3: 2D Imaging and k-space, Gradient Echo Example

k-Space coverage

The gradient echo sequences given in Figures 10.13 and 10.14 cover k space as in Figure 10.15.

Refer to Figure 10.14

Read rest of Chapter 10.3 on own.

Spin Echo Example

Read Chapter 10.4 on own.

Homework

Do 10.1, 10.2, 10.3 (Use H^1 and $\gamma = 42.58\mathrm{Mhz/T}$ and watch your units.)
Read the rest of the Chapter, 10.4 on.