Biophysics 230: Nuclear Magnetic Resonance
Math/FT Review and Haacke Chapter 9

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Math/FT Review

Complex Numbers

A complex number z has a real part x and an imaginary part y is

$$z = x + iy$$

where i is the imaginary unit (electrical engineers use j)

$$i = \sqrt{-1} \; .$$

Complex numbers can represent two real values simultaneously.

The angular frequency is defined to be

$$\omega = 2\pi\nu$$

where ω is in radians/sec and ν is in Hz.

Complex Numbers

Euler's formula is

$$e^{i2\pi\nu t} = \cos(2\pi\nu t) + i\sin(2\pi\nu t)$$

and also

$$e^{-i2\pi\nu t} = \cos(2\pi\nu t) - i\sin(2\pi\nu t)$$

which by addition and subtraction can be used to find

$$\cos(2\pi\nu t) = \frac{e^{i2\pi\nu t} + e^{-i2\pi\nu t}}{2}$$

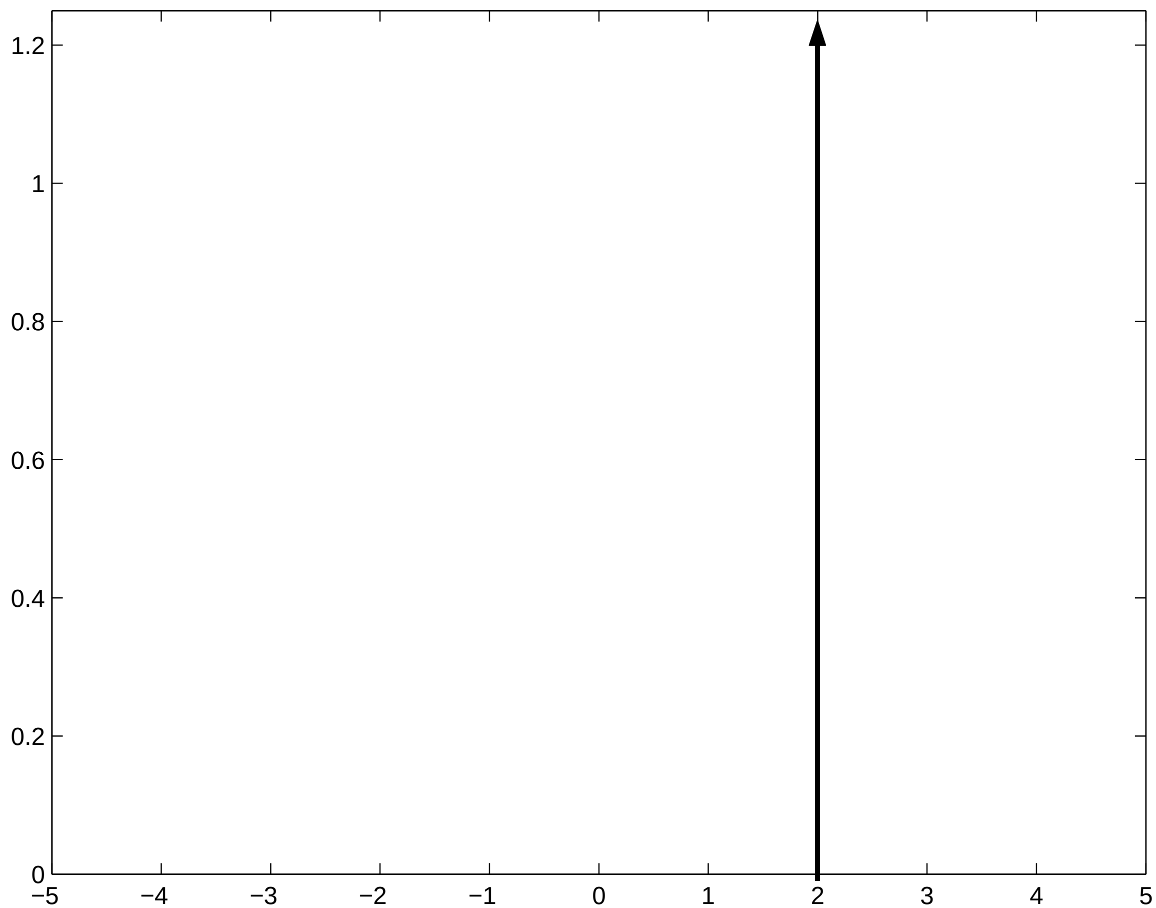
and

$$\sin(2\pi\nu t) = \frac{e^{i2\pi\nu t} - e^{-i2\pi\nu t}}{2i}.$$

Delta Functions

The Dirac delta function is defined to be

$$\delta(\nu - \nu_0) = \begin{cases} \infty & \text{if } \nu = \nu_0 \\ 0 & \text{if } \nu \neq \nu_0 \end{cases}$$



$$\delta(\nu - 2).$$

Delta Function

It is simultaneously infinitely narrow and infinitely high.

$$\int_{-\infty}^{+\infty} \delta(\nu - \nu_0) d\nu = 1.$$

The Dirac delta function may also be represented in terms of the integral

$$\delta(\nu - \nu_0) = \int_{-\infty}^{+\infty} e^{-i2\pi(\nu-\nu_0)t} \, dt$$

or it can be defined in terms of its effect on other functions:

$$\int_{-\infty}^{+\infty} F(\nu)\delta(\nu - \nu_0) \, d\nu = F(\nu_0).$$

This is similar to the “selecting” property of the Fourier Transform operation that was mentioned earlier.

One Dimensional FT-Continuous

The FT of a continuous function $f(t)$ is

$$F(\nu) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi\nu t} \, dt$$

also denoted as $\mathcal{F}\{f(t)\}$ and its inverse to be

$$f(t) = \int_{-\infty}^{+\infty} F(\nu) e^{+i2\pi\nu t} \, d\nu$$

also denoted as $\mathcal{F}^{-1}\{F(\nu)\}$.

Don't forget that

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \; .$$

One Dimensional FT-Continuous

$$
\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} [f(t)][\cos(2\pi\nu t) - i\sin(2\pi\nu t)]\; dt \\
&= \int_{-\infty}^{\infty} [f(t)]\cos(2\pi\nu t)\; dt - i\int_{-\infty}^{\infty} [f(t)]\sin(2\pi\nu t)\; dt \\
&= F_C(\nu) - iF_S(\nu)
\end{aligned}
$$

$$
\begin{aligned}
F_C(\nu) &= \int_{-\infty}^{\infty} \left[\sum_j A_j\cos(2\pi\nu_j t) + \sum_j B_j\sin(2\pi\nu_j t)\right]\cos(2\pi\nu t)\; dt \\
F_S(\nu) &= \int_{-\infty}^{\infty} \left[\sum_j A_j\cos(2\pi\nu_j t) + \sum_j B_j\sin(2\pi\nu_j t)\right]\sin(2\pi\nu t)\; dt
\end{aligned}
$$

The $\cos()\sin()$ and $\sin()\cos()$ cross terms are zero.

One Dimensional FT-Continuous

$$
\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} [f(t)][\cos(2\pi\nu t) - i\sin(2\pi\nu t)]\, dt \\
&= \int_{-\infty}^{\infty} [f(t)]\cos(2\pi\nu t)\, dt - i\int_{-\infty}^{\infty} [f(t)]\sin(2\pi\nu t)\, dt \\
&= F_C(\nu) - iF_S(\nu)
\end{aligned}
$$

$$
\begin{aligned}
F_C(\nu) &= \int_{-\infty}^{\infty} \left[\sum_j A_j \cos(2\pi\nu_j t)\right] \cos(2\pi\nu t)\, dt \\
F_S(\nu) &= \int_{-\infty}^{\infty} \left[\sum_j B_j \sin(2\pi\nu_j t)\right] \sin(2\pi\nu t)\, dt
\end{aligned}
$$

Can move the integral past the sum.

One Dimensional FT-Continuous

$$
\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} [f(t)][\cos(2\pi\nu t) - i\sin(2\pi\nu t)]\, dt \\
&= \int_{-\infty}^{\infty} [f(t)]\cos(2\pi\nu t)\, dt - i\int_{-\infty}^{\infty} [f(t)]\sin(2\pi\nu t)\, dt \\
&= F_C(\nu) - iF_S(\nu)
\end{aligned}
$$

$$
\begin{aligned}
F_C(\nu) &= \sum_j A_j \int_{-\infty}^{\infty} \cos(2\pi\nu_j t)\cos(2\pi\nu t)\, dt \\
F_S(\nu) &= \sum_j B_j \int_{-\infty}^{\infty} \sin(2\pi\nu_j t)\sin(2\pi\nu t)\, dt
\end{aligned}
$$

The $\cos()\cos()$ and $\sin()\sin()$ integrals are nonzero only when $\nu = \nu_j$.

Nonzero values at constituent frequencies where A_j and B_j nonzero.

One Dimensional FT-Continuous

$$
\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} [f(t)][\cos(2\pi\nu t) - i\sin(2\pi\nu t)]\ dt \\
&= \int_{-\infty}^{\infty} [f(t)]\cos(2\pi\nu t)\ dt - i\int_{-\infty}^{\infty} [f(t)]\sin(2\pi\nu t)\ dt \\
&= F_C(\nu) - iF_S(\nu)
\end{aligned}
$$

$$
\begin{aligned}
F_C(\nu) &= \sum_j \frac{1}{2} A_j[\delta(\nu+\nu_j) + \delta(\nu-\nu_j)] \\
F_S(\nu) &= \sum_j \frac{1}{2} B_j[\delta(\nu+\nu_j) - \delta(\nu-\nu_j)]
\end{aligned}
$$

The $\cos()\cos()$ and $\sin()\sin()$ integrals are δ functions at $\nu = \nu_j$.

The A_j and B_j amplitudes represent the strength of the cosines and sines.

One Dimensional FT-Continuous

Fourier Transform properties.

Property	Function	Transform
Linearity	$af(x) + bg(x)$	$aF(k) + bG(k)$
Similarity	$f(ax)$	$\frac{1}{\lvert a \rvert} F(\frac{k}{a})$
Shifting	$f(x - a)$	$e^{-i2\pi ka} F(k)$
Derivative	$\frac{d^\ell f(x)}{dx^\ell}$	$(i2\pi k)^\ell F(k)$

One Dimensional FT-Continuous

Convolution of functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(\alpha)\ g(x-\alpha)\ d\alpha\ .$$

Further

$$\mathcal{F}\ \{f(x) * g(x)\} = F(k) \cdot G(k)\ .$$

and

$$\mathcal{F}\ \{f(x) \cdot g(x)\} = F(k) * G(k)\ ,$$

One Dimensional FT-Continuous

Convolution properties.

$$f(x) * g(x) = g(x) * f(x) \quad \text{commutative}$$

$$f(x) * [g(x) * h(x)] = [f(x) * g(x)] * h(x) \quad \text{associative}$$

$$f(x) * [g_1(x) + g_2(x)] = f(x) * g_1(x) + f(x) * g_2(x) \quad \text{distributive}$$

$$\frac{d\ f(x)*g(x)}{dx} = \frac{d\ f(x)}{dx} * g(x) = f(x) * \frac{d\ g(x)}{dx} \quad \text{derivative}$$

$$h(x - x_0) = f(x - x_0) * g(x) = f(x) * g(x - x_0) \quad \text{shift}$$

if $h(x) = f(x) * g(x)$

One Dimensional FT-Continuous

Now you know everything about Fourier transforms.

Questions?

Chapter 9: One-Dimensional Fourier Imaging, k-space and Gradient Echos

9.1: Signal and Effective Spin Density

9.1.1 Complex Demodulated Signal

Recall Equation (7.28).

$$s(t) \propto \omega_0 \int e^{-t/T_2(\vec{r})} M_\perp(\vec{r},0) B_\perp(\vec{r}) e^{(i(\Omega-\omega_0)t+\phi_0(\vec{r})-\theta_B(\vec{r}))}\ d^3r \tag{7.28}$$

Assumptions:

It is assumed that the RF coils are uniform so that:
1) The initial magnetization ϕ_0,
2) The initial receive field direction θ_B, and
3) The receive field amplitude $B_\perp$

are independent of position, $\vec{r}$.

4) The total sampling time $T_s \ll T_2^*$ and thus $e^{-t/T_2} \approx 1$.

The e^{-t/T_2}, $e^{i\phi_0}$, and $e^{-\theta_B}$ are incorporated into $\Lambda = e^{-t/T_2} e^{i\phi_0} e^{-\theta_B}$ and $B_\perp$ taken out of the integral. Also define $\phi(\vec{r},t) = -\omega_0 t$.

Chapter 9.1: 1D Fourier Imaging

Having done the aforementioned, Equation (7.28) becomes

$$s(t) = \omega_0 \Lambda B_\perp \int M_\perp(\vec{r},0) e^{i(\Omega t + \phi(\vec{r},t))} \, d^3 r \tag{9.1}$$

The signal is generalized to include a position and time dependent $\omega(\vec{r},t)$ so that the accumulated phase $\phi(\vec{r},t) = -\omega_0 t$ is generalized to be

$$\phi(\vec{r},t) = -\int_0^t \omega(\vec{r},t') \, dt'. \tag{9.2}$$

In the presence of a uniform static field, $\omega(\vec{r},t') = \omega_0$ and

$$\phi(\vec{r},t) = -\omega_0 t \tag{9.3}$$

where $\omega_0 = \gamma B_0$ from Equation (1.1), see also Equation (5.22).

Chapter 9.1: 1D Fourier Imaging

9.1.2 Magnetization and Effective Spin Density

In Chapter 6, using quantum mechanical arguments it was shown (Equation 6.11, also Equation 1.3) that the initial proton magnetization for $\hbar\omega_0 \ll \kappa T$ is

$$M_0 \simeq \frac{1}{4}\rho_0\frac{\gamma^2\hbar}{\kappa T}B_0 \tag{6.11}$$

before the gradient field is turned on.

ρ_0 is the ‘spin density’, spins per unit volume
γ is the gymagnetic ratio, $\gamma = 2.68 \times 10^8 rad/s/T$
h is Plancks constant, $\hbar = h/(2\pi)$, $h = 1.05 \times 10^{-34} J \cdot s$
κ is Boltzmann’s constant, $\kappa = 1.38 \times 10^{-23} J/K$
T is the temperature in Kelvin
B_0 is the external (main) magnetic field.

Chapter 9.1: 1D Fourier Imaging

This is generalized to be position dependent through the spin density

$$M_{\perp}(\vec{r},0) = M_0(\vec{r}) = \frac{1}{4}\rho_0(\vec{r})\frac{\gamma^2\hbar}{\kappa T}B_0 \tag{9.3}$$

and combined with Equation (9.1) to obtain

$$s(t) = \int \rho(\vec{r})e^{i(\Omega t+\phi(\vec{r},t))}\ d^3r \tag{9.4}$$

where the 3D spin density $\rho(\vec{r})$ is

$$\rho(\vec{r}) \equiv \omega_0\Lambda B_{\perp}M_0(\vec{r}) = \frac{1}{4}\omega_0\Lambda B_{\perp}\rho_0(\vec{r})\frac{\gamma^2\hbar}{\kappa T}B_0 \tag{9.5}$$

Let's focus interest on one dimension, say z.

Chapter 9.1: 1D Fourier Imaging

The signal in Equation (9.4) becomes

$$s(t) = \int \rho(z) e^{i(\Omega t + \phi(z,t))}\, dz \tag{9.6}$$

where

$$\rho(z) = \int \rho(\vec{r})\, dxdy. \tag{9.7}$$

Note:

Equation (9.6) holds for multiple RF pulses
when $T_R \gg T_1$ and $T_E \ll T_2$!

Because $e^{-T_R/T_1} \approx 0$ and $e^{-T_E/T_2} \approx 1$.

Otherwise use $\rho(z, T_1, T_2)$ or in general $\rho(\vec{r}, T_1, T_2)$.

Chapter 9.2: Frequency Encoding and the FT

The objective is to determine $\rho(z)$.

9.2.1 Frequency Encoding of the Spin Position

The Larmor frequency of a spin will be linearly proportional to its position along the z direction with the addition of a linearly varying field.

If a linearly varying field is added to the static field, then

$$B_z(z,t) = B_0 + zG(t) \tag{9.8}$$

is the z component. And note that the derivative of the magnetic field is

$$G_z = \frac{\partial B_z}{\partial z} \ . \tag{9.9}$$

Chapter 9.2: Frequency Encoding and the FT

In the presence of the linearly varying magnetic field along the z axis, the variation in the angular frequency of the spins is

$$\begin{aligned} \omega(z,t) &= \gamma B_0 + \gamma z G(t) \\ &= \omega_0 + \omega_G(z,t) \ . \end{aligned} \tag{9.10}$$

For a linearly varying magnetic field, according to Equation (9.8),
the deviation $\omega_G(z,t)$ from the Larmor frequency ω_0 (Equation 2.27) is

$$\omega_G(z,t) = \gamma z G(t) \tag{9.11}$$

This is refereed to as "frequency encoding."
The accumulated phase is

$$\phi_G(z,t) = -\int_0^t \omega_G(z,t')\ dt' \tag{9.12}$$

$$= -\gamma z \int_0^t G(t')\ dt' \tag{9.13}$$

Chapter 9.2.2: The 1D Imaging equation and the FT

Using the precessing frequency $\omega(z,t)$ as in Equation (9.10) and
Assumption:
The demodulating frequency is $\Omega = \omega_0$, the signal becomes

$$s(t) = \int \rho(z) e^{i\phi_G(z,t)} \, dz \tag{9.14}$$

Note: Look at Equation (9.6)

$$\begin{aligned} s(t) &= \int \rho(z) e^{i(\Omega t + \phi(z,t))} \, dz \qquad (9.6) \\ &= \int \rho(z) e^{i(\omega_0 t - \omega_0 t - \phi_G(z,t))} \, dz \\ &= \int \rho(z) e^{i\gamma z \int_0^t G(t') \, dt'} \, dz \end{aligned}$$

Chapter 9.2.2: The 1D Imaging equation and the FT

For the linear gradient field this leads to

$$
\begin{aligned}
s(t) &= \int \rho(z) e^{-i\gamma z \int_0^t G(t')dt'} \, dz \\
s(k) &= \int \rho(z) e^{-i2\pi k z} \, dz
\end{aligned}
\tag{9.15}
$$

where the spatial frequency k (analogous to ν in FT Review) is

$$
k(t) = \gamma\!\!\!- \int_0^t G(t') \, dt' \; . \tag{9.16}
$$

The signal $s(k)$ is the Fourier transform of the spin density!

Chapter 9.2.2: The 1D Imaging equation and the FT

This means that the spin density $\rho(z)$ can be found as the inverse Fourier transform of the signal $s(k)$

$$\rho(z) = \int s(k)e^{+i2\pi kz}\, dk \tag{9.17}$$

Measure $s(k)$ then compute $\rho(z)$.

Chapter 9.2.2: The 1D Imaging equation and the FT

9.2.3 The Coverage of k-Space

When the gradient field is constant over time, $G_z(t) = G$, Equation (9.16)

$$k(t) = \gamma \int_0^t G(t')\, dt' \qquad (9.16)$$

becomes

$$k = \gamma\, Gt. \qquad (9.18)$$

We are going to sample at a regular points in space which means we only need to sample at constant time intervals.

Chapter 9.2.2: The 1D Imaging equation and the FT

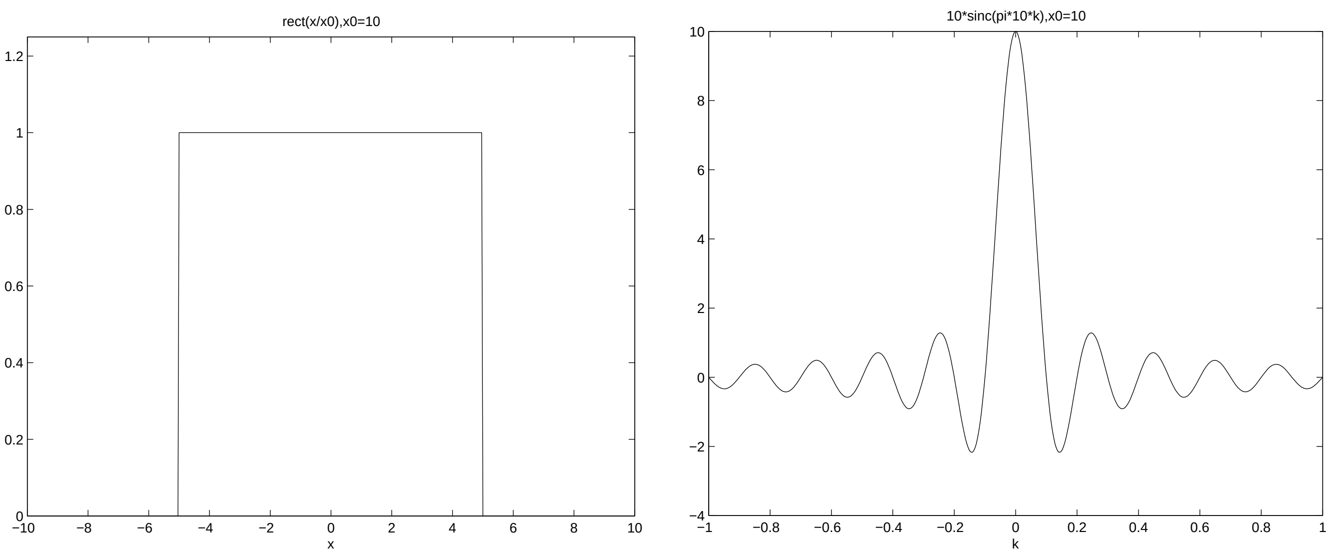
9.2.4 Rect and Sinc Functions

The boxcar or rect function of width z_0 is

$$\mathrm{rect}\left(\frac{z}{z_0}\right) \equiv \begin{cases} 0 & z < -\frac{z_0}{2} \\ 1 & -\frac{z_0}{2} < z < \frac{z_0}{2} \\ 0 & z > \frac{z_0}{2} \end{cases} \tag{9.19}$$

and the its Fourier transform is

$$F(k) = z_0 \mathrm{sinc}(\pi z_0 k) \ . \tag{9.20}$$



These two functions form a Fourier transform pair.

Chapter 9.2.2: The 1D Imaging equation and the FT

$$
\begin{aligned}
F(k) &= \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{z}{z_0}\right) e^{-i2\pi kz} dz \\
&= \int_{-\frac{z_0}{2}}^{+\frac{z_0}{2}} e^{-i2\pi kz} dz \\
&= -\frac{1}{i2\pi k}[e^{-i2\pi k\frac{z_0}{2}} - e^{-i2\pi k\frac{-z_0}{2}}] \\
&= -\frac{1}{i2\pi k}[\ \cos(\pi k z_0) + i\sin(\pi k z_0) - \cos(-\pi k z_0) - i\sin(-\pi k z_0)\ ] \\
&= \frac{\sin(\pi k z_0)}{\pi k}\ .
\end{aligned}
$$

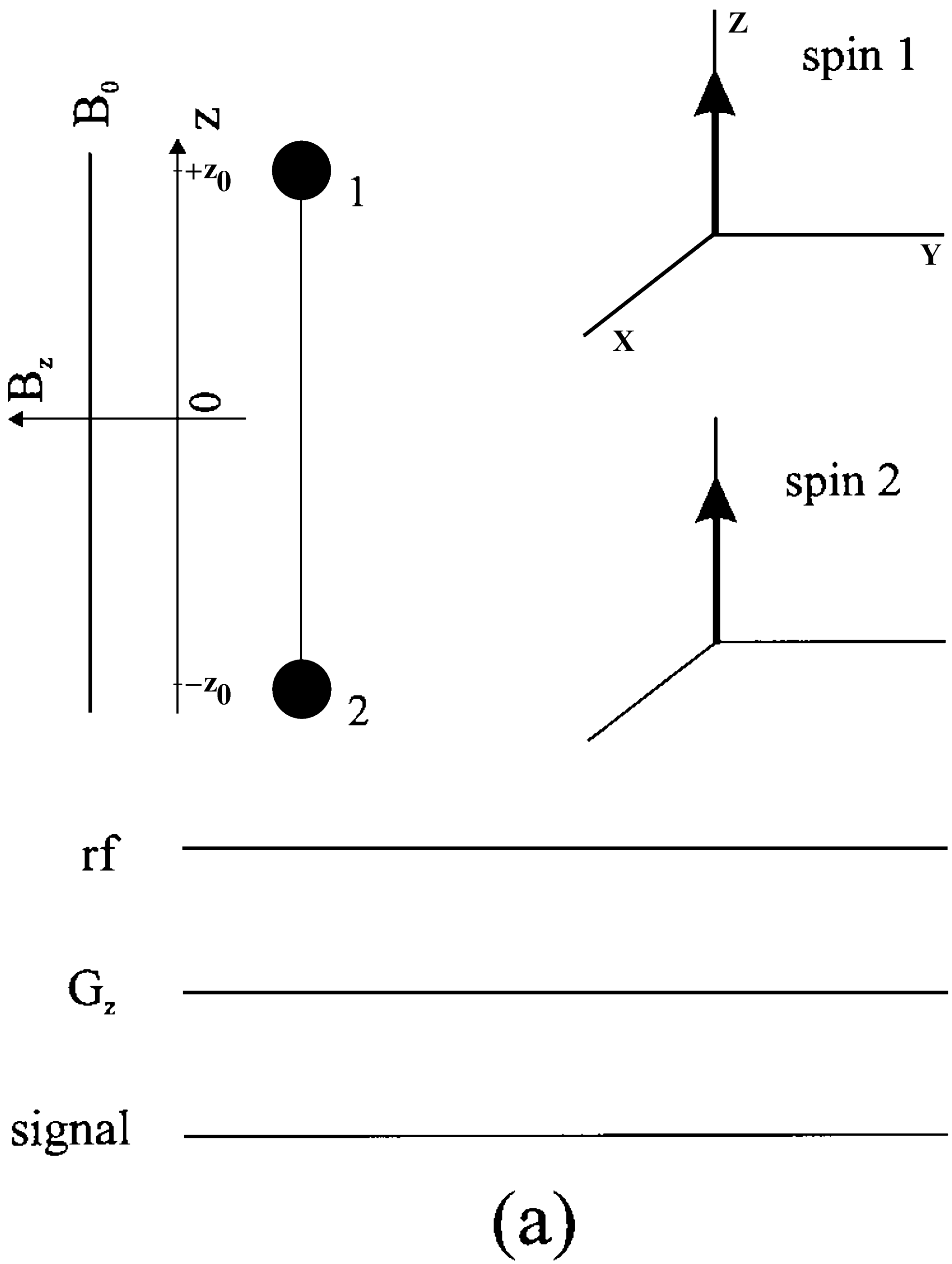
Using the definition:

$$\mathrm{sinc}(\pi z_0 k) = \frac{\sin(\pi z_0 k)}{(\pi z_0 k)}, \qquad k \in \mathbb{R}.$$

Therefore the Fourier transform of the rect function is:

$$F(k) = \mathcal{F}\left\{\mathrm{rect}\left(\frac{z}{z_0}\right)\right\} = z_0 \mathrm{sinc}(\pi z_0 k)\ .$$

Chapter 9.3: Simple Two-Spin Example



Consider two spins at $z = \pm z_0$.

Refer to Figure 9.1a.

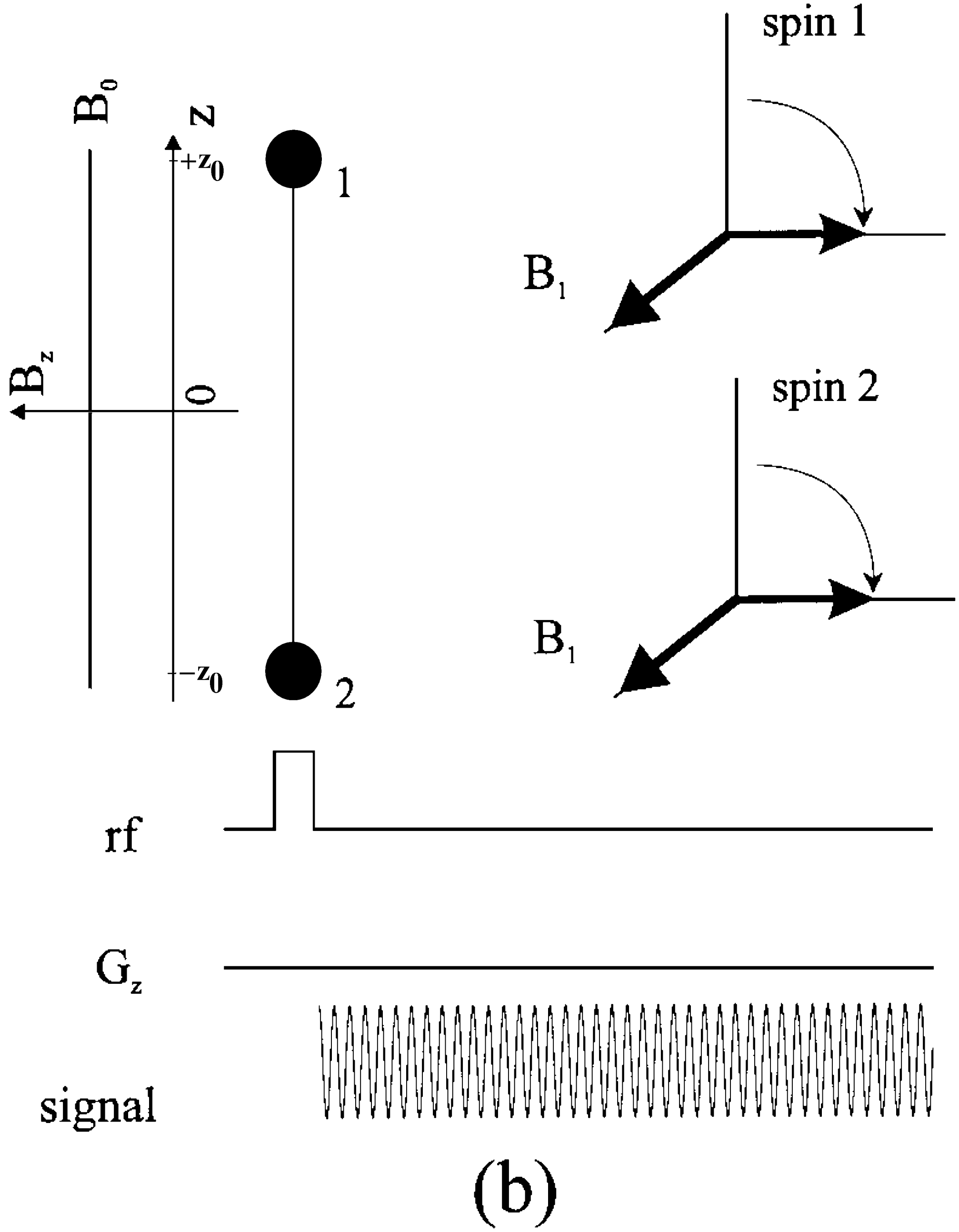
Nothing is going on.

The two spins are in equilibrium.

The equilibrium magnetization is

$$M_0(z) = \frac{1}{4}\omega_0 \Lambda B_\perp \rho_0(z) \frac{\gamma^2 \hbar}{\kappa T} B_0$$

Chapter 9.3: Simple Two-Spin Example



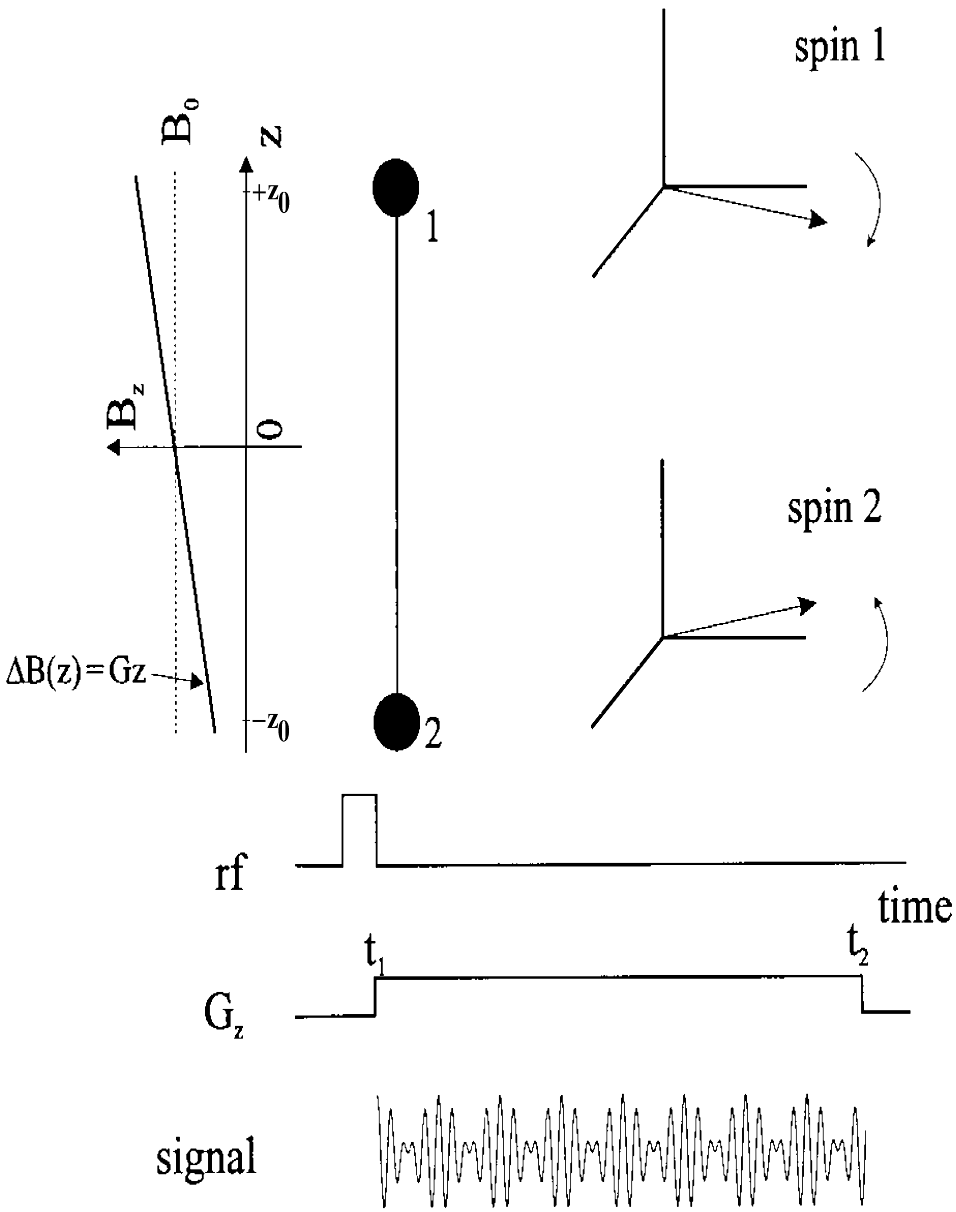
Apply a 90° RF pulse x direction.

Spins tipped in y direction

Into the transverse plane,

A single frequency results as in Figure 9.1b.
(T_2 decay neglected)

Chapter 9.3: Simple Two-Spin Example



(c)

Apply the RF field again followed by a gradient G_z in the z direction in the interval t_1 to t_2.

The spin at $+z_0$ will rotate clockwise and the spin at $-z_0$ will precess counterclockwise at the same rate. (Fan out.)

While G is applied (i.e. $t_1 < t < t_2$), the spins will have rotated through angles $\phi(z_0,t) = -\gamma G z_0(t-t_1)$ and $\phi(-z_0,t) = \gamma G z_0(t-t_1)$.

Chapter 9.3: Simple Two-Spin Example

Recall Equations (9.15) and (9.16). Note that Equation (9.16) becomes

$$k(t) = \gamma\!\!\!\!-\, G \cdot (t - t_1)$$

because of a constant gradient and integration from t_1 to t, while Equation (9.15) with the integral is replaced by a sum to becomes

$$s(t) = \sum_{z=\pm z_0} \rho(z) e^{-i2\pi kz}$$

which is with $t_1 = 0$

$$\begin{aligned} s(t) &= \rho(-z_0)e^{i\gamma Gtz_0} + \rho(+z_0)e^{-i\gamma Gtz_0} \\ &= s_0(e^{i\gamma Gtz_0} + e^{-i\gamma Gtz_0}) \\ &= 2s_0 \cos\gamma Gtz_0 \quad t_1 < t < t_2 \end{aligned} \tag{9.21}$$

where $\rho(-z_0) = \rho(+z_0) = s_0$ and Euler's cosine formula have been used. This is the signal in Figure 9.1c.

Chapter 9.3: Simple Two-Spin Example

This can also be expressed as

$$s(k) = 2s_0 \cos 2\pi k z_0 \quad 0 < k < k_2 \equiv \gamma\!\!\!\!-\ G t_2 \tag{9.22}$$

with $k = \gamma\!\!\!\!-\ G t$. From $s(k)$ and G, use Equation (9.17)

$$\begin{aligned}
\rho(z) &= \int_{-\infty}^{+\infty} s(k) e^{+i2\pi k z}\, dk \\
&= \int_{-\infty}^{+\infty} 2s_0 \cos(2\pi k z_0) e^{+i2\pi k z}\, dk \\
&= s_0 \int_{-\infty}^{+\infty} \left(e^{i2\pi k z_0} + e^{-i2\pi k z_0} \right) e^{+i2\pi k z}\, dk \\
&= s_0 \int_{-\infty}^{+\infty} \left(e^{i2\pi k (z - z_0)} + e^{i2\pi k (z + z_0)} \right) dk \\
&= s_0 \int_{-\infty}^{+\infty} e^{i2\pi k (z - z_0)}\, dk + \int_{-\infty}^{+\infty} e^{i2\pi k (z + z_0)}\, dk \\
&= s_0 [\delta(z - z_0) + \delta(z + z_0)]
\end{aligned} \tag{9.23}$$

Chapter 9.3.1: Dirac Delta Function

A Dirac delta function is such that

$$\delta(z-a) = 0 \text{ if } z \neq a \tag{9.24}$$

and

$$\int_{z_1}^{z_2} \delta(z-a)\, dz = \begin{cases} 1 & a \in (z_1, z_2) \\ 0 & a \notin (z_1, z_2) \end{cases} \quad . \tag{9.25}$$

The delta function picks out a particular value of the function

$$\int_{-\infty}^{+\infty} \delta(z-a) f(z)\, dz = f(a)\ . \tag{9.26}$$

Chapter 9.3.1: Dirac Delta Function

Consider the inverse Fourier transform of the rect function $rect\left(\frac{z}{2K}\right)$

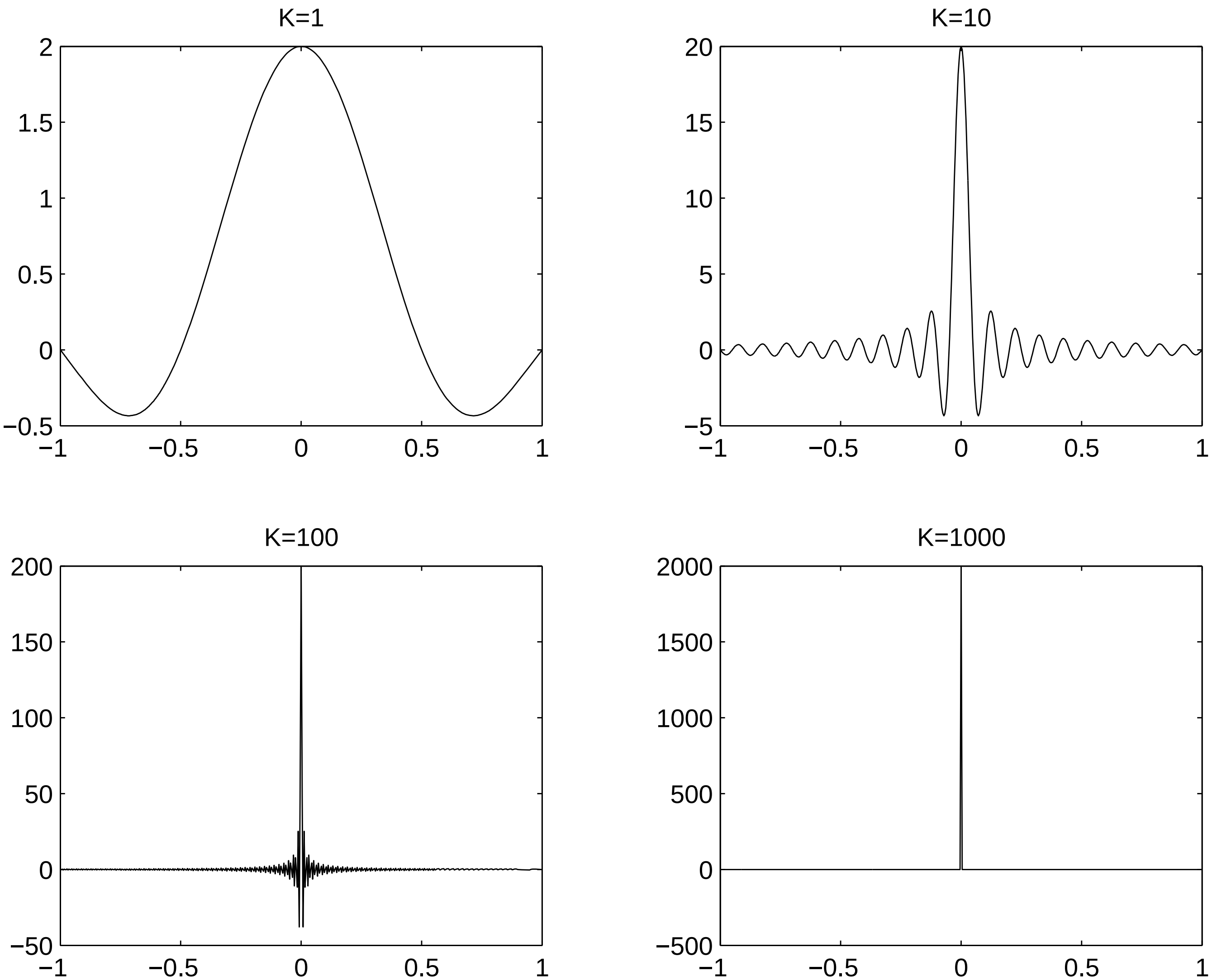
$$\begin{aligned} I(z,K) &\equiv \int_{-K}^{+K} rect\left(\frac{z}{2K}\right) e^{i2\pi kz}\, dk \\ &= \frac{1}{i2\pi z} e^{i2\pi kz}\Big|_{-K}^{K} \\ &= \frac{1}{\pi z 2i}\left(e^{i2\pi Kz} - e^{-i2\pi kz}\right) \\ &= \frac{sin(2\pi Kz)}{\pi z} \\ &= 2K sinc(2\pi Kz) \end{aligned} \quad (9.27)$$

Now let $K \longrightarrow +\infty$

$$\lim_{K \rightarrow +\infty} I(z,K) = \delta(z) \quad (9.28)$$

Which is Equation (9.24).

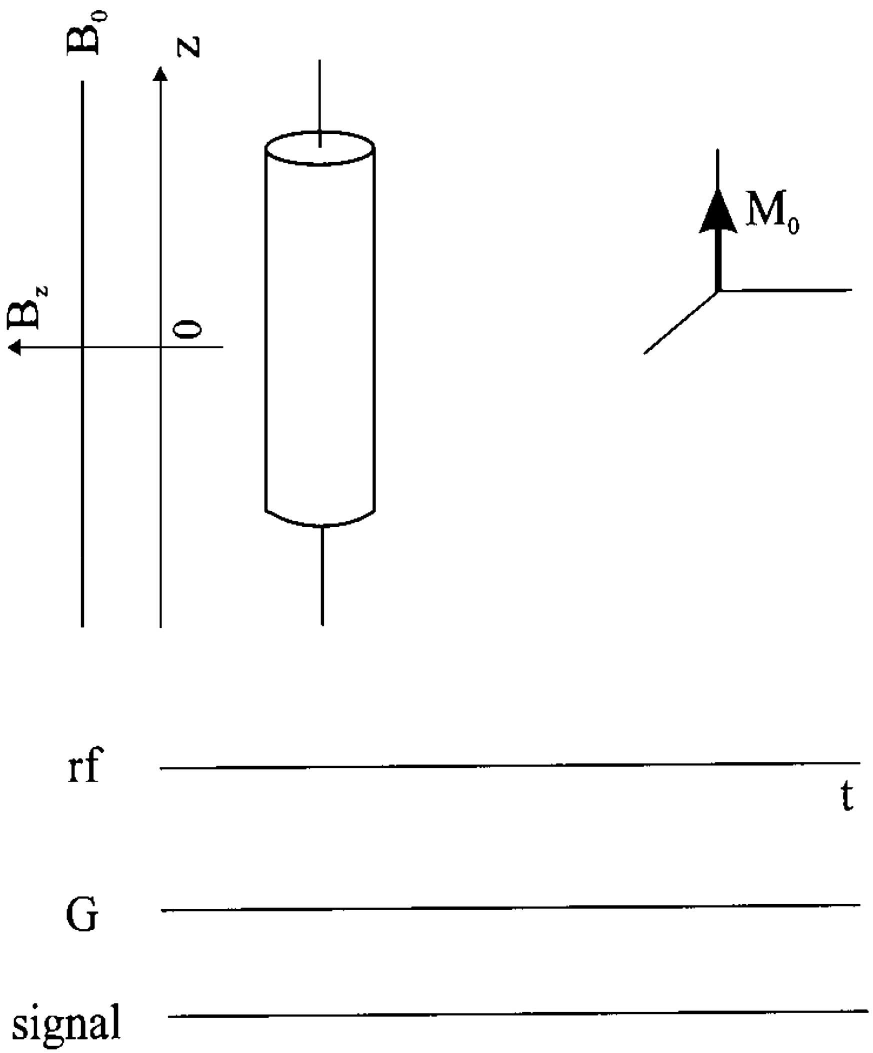
Chapter 9.3.1: Dirac Delta Function



$$I(z,K) = 2K\,sinc(2\pi K z)$$

So the Fourier transform of a constant function (rect of infinite width) is a Dirac δ-function. And vise versa.

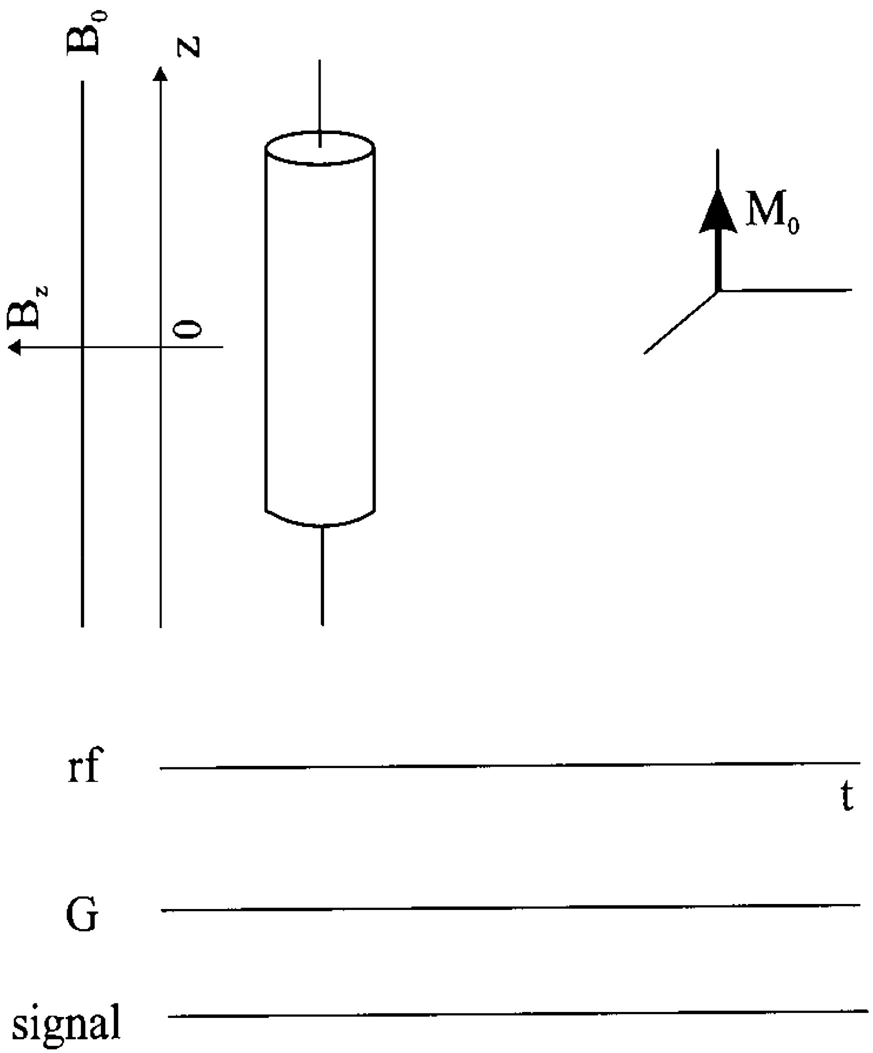
Chapter 9.4: Gradient Echo and k-Space Diagrams



(a)

Replace the dumbbell two-spin example with a cylinder with an arbitrary z-distribution of spins $\rho(z)$ as in Figure 9.2 (on left).

Chapter 9.4: Gradient Echo and k-Space Diagrams



(a)

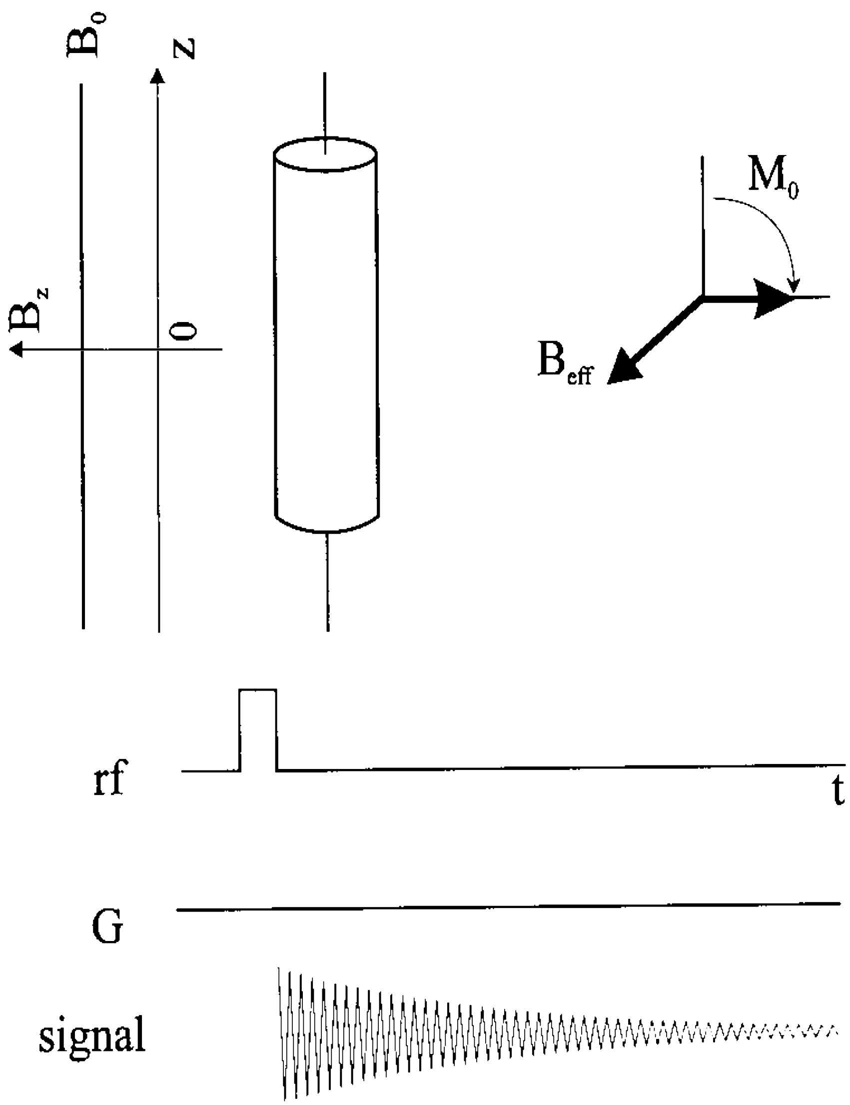
Nothing is going on.

Everything is in equilibrium.

The equilibrium magnetization is

$$M_0(z) = \frac{1}{4}\omega_0 \Lambda B_\perp \rho_0(z) \frac{\gamma^2 \hbar}{\kappa T} B_0$$

Chapter 9.4: Gradient Echo and k-Space Diagrams



(b)

Apply a 90° RF pulse x direction.

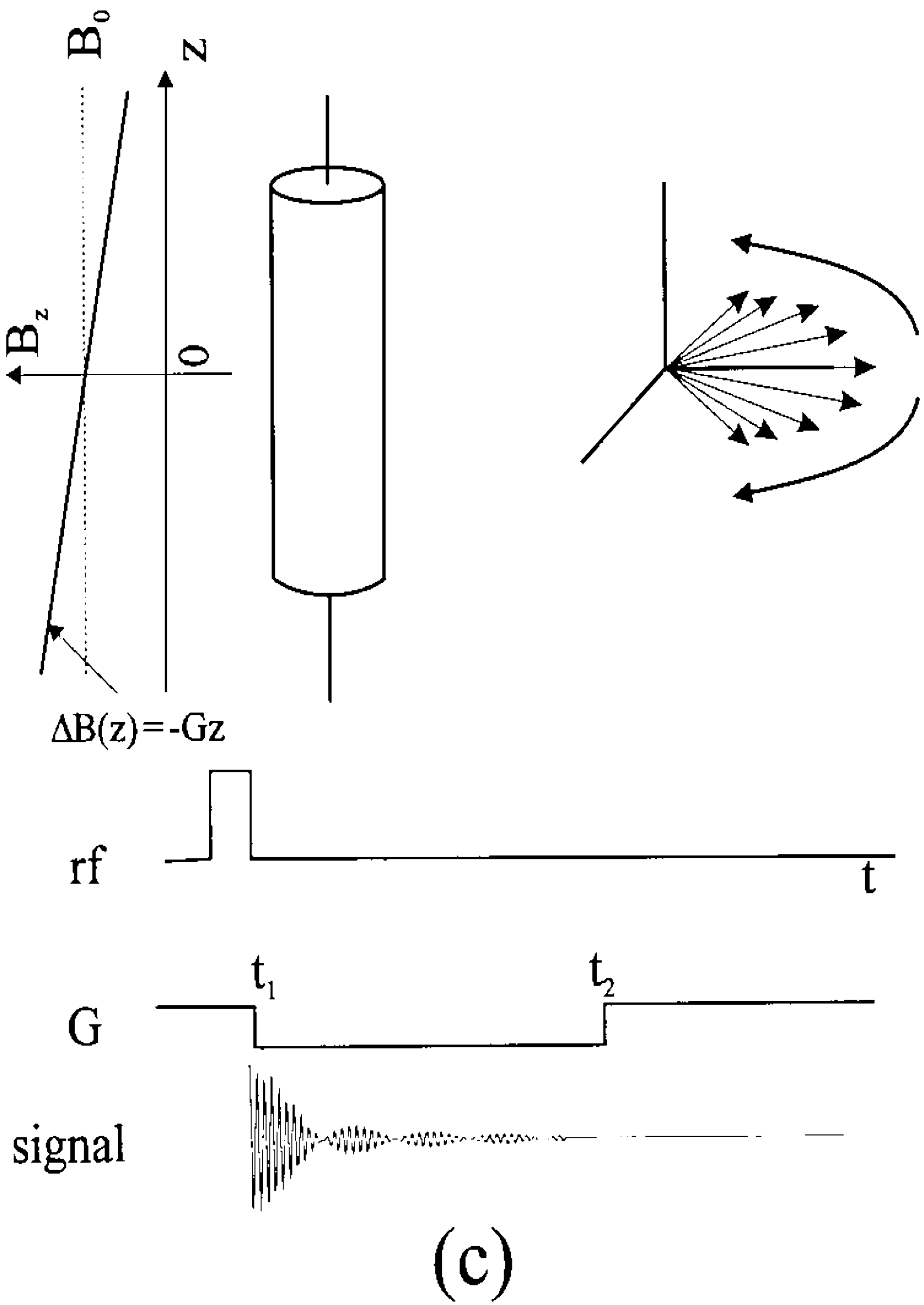
Spins tipped in y direction

Into the transverse (x, y) plane,

Produce $M_\perp(z)$ as in Figure 9.2b.

Note the decaying signal.
(T_2 decay not neglected)

Chapter 9.4: Gradient Echo and k-Space Diagrams



Apply a 90° RF pulse x direction.

Spins tipped in y direction

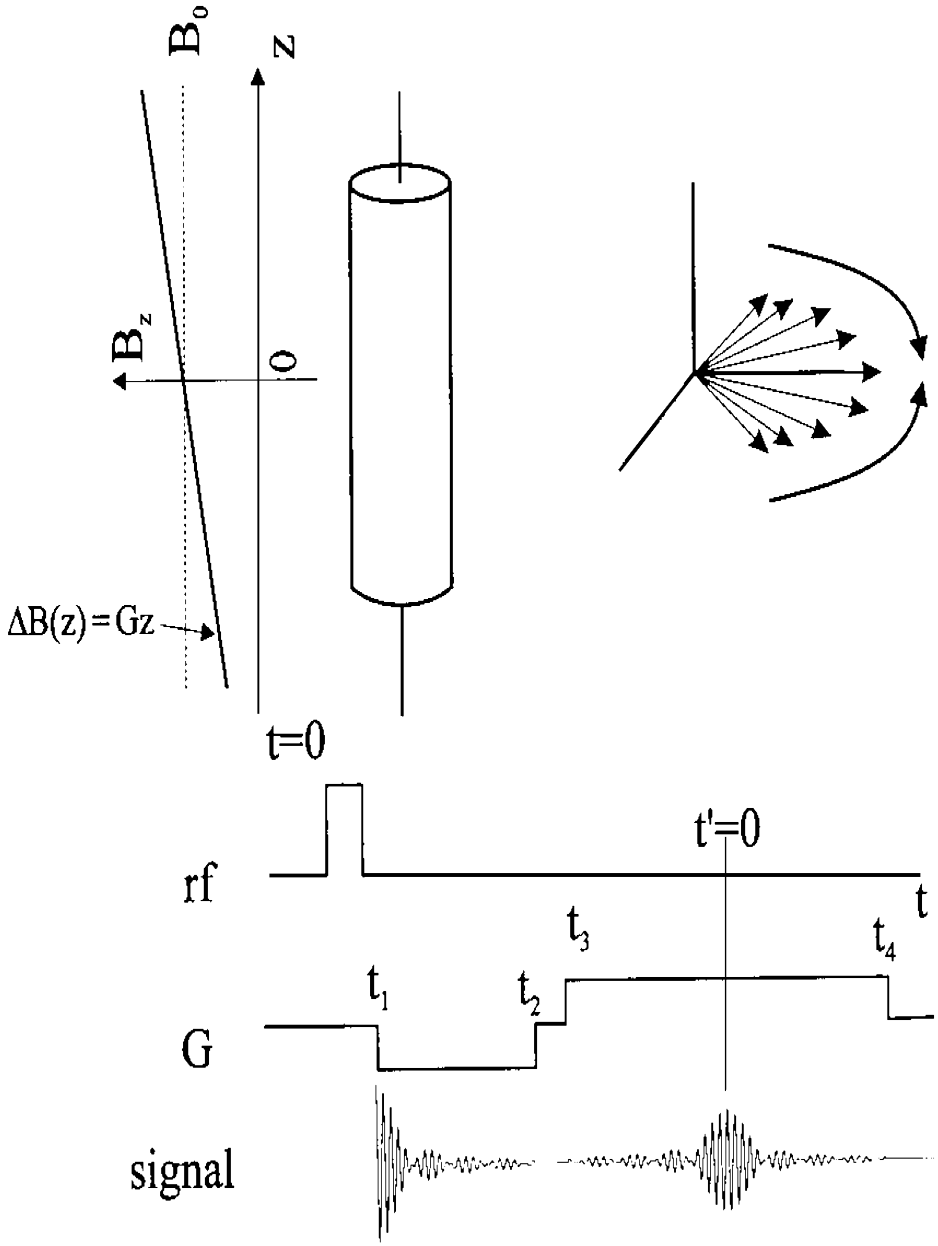
Into the transverse (x, y) plane,

Produce $M_{\perp}(z)$ as in Figure 9.2b.

Apply G_z between t_1 and t_2 as in Figure 9.2c.

Note the more rapidly decaying signal and the dephasing.

Chapter 9.3: Simple Two-Spin Example



(d)

Apply a 90° RF pulse x direction.

Spins tipped in y direction

Into the transverse (x, y) plane,

Produce $M_{\perp}(z)$ as in Figure 9.2b.

Apply G_z between t_1 & t_2 as in Figure 9.2c.

Note the more rapidly decaying signal and the dephasing.

Reverse gradient between t_3 & t_4. An echo is formed at $t' = 0$

Chapter 9.4.1: The Gradient Echo

Look at Figure 9.2.

While the first gradient lobe is applied, the phase is of the form

$$\phi_G(z,t) = +\gamma G z(t-t_1) \qquad t_1 < t < t_2 \tag{9.31}$$

(for $+z$ spins, negative for $-z$ spins).

While the second gradient lobe is applied, the phase accumulation is of the form

$$\phi_G(z,t) = +\gamma G z(t_2-t_1) - \gamma G z(t-t_3) \qquad t_3 < t < t_4 \tag{9.32}$$

By selecting $(t_4 - t_3)/2 = t_2 - t_1$, the time at which the spins rephase, the echo, is at

$$t = t_3 + (t_2 - t_1) \equiv T_E \tag{9.33}$$

for all z.

Chapter 9.4.1: The Gradient Echo

The echo occurs at the point when the area under the second lobe just cancels out the area under the first lobe. (The gradients do not have to be constant or of same height or length.)

$$\int G(t)\, dt = 0 \tag{9.34}$$

Let's reparameterize time so that it is zero at T_E.

$$t' \equiv t - t_3 - (t_2 - t_1) = t - T_E \tag{9.35}$$

Having reparameterized time, the phase during the second gradient lobe can be written as

$$\phi_G(z,t) = +\gamma G z t' \qquad -(t_4 - t_3)/2 < t' < (t_4 - t_3)/2 \tag{9.36}$$

Chapter 9.4.1: The Gradient Echo

The signal in Equation (9.14) in terms of t' that we will "record" during the second gradient lobe is

$$
\begin{aligned}
s(t') &= \int \rho(z) e^{-i\gamma G z t'}\, dz \\
&= \int \rho(z) e^{-i2\pi(\bar{\gamma} G t')z}\, dz \\
&= \int \rho(z) e^{-i2\pi k(t')z}\, dz. \qquad -(t_4 - t_3)/2 < t' < (t_4 - t_3)/2
\end{aligned}
\tag{9.37}
$$

where we noted that $k = \bar{\gamma}\, G t'$ and $k(t')$ denotes that k is a function of t'.

Chapter 9.4.1: The Gradient Echo

The signal can be written solely in terms of the variable k as

$$s(k) = \int \rho(z) e^{-i2\pi kz}\, dz \qquad -k_{max} < k < k_{max} \tag{9.38}$$

where $k_{max} = \gamma\!\!\!- G(t_4 - t_3)/2$.

This means that when we observe the signal $s(t)$ from $t = t_3$ to $t = t_4$ which is the same as observing the signal $s(t')$ from time $t' = -(t_4 - t_3)/2$ to $t' = (t_4 - t_3)/2$ we are observing the signal $s(k)$ at the different k-space values from $k = -k_{max}$ to $k = k_{max}$.

We cover that range of k-space.

Voila, Equation (9.38) is a Fourier transform.

Take the inverse Fourier transform to get $\rho(z)$.

Summary:

Because $B(z) = B_0 + zG$ (assuming $G(z) = G$) is changing along z,

the Larmor frequency $\omega_z = \omega_0 + \omega_G(z)$ is also changing along z,

so the signal is changing along the z direction with $\omega_G(z) = \gamma z G$.

The observed signal at time t is $s(t) = \int \rho(z) e^{-i\gamma z G t}\, dz$.

The observed signal at spatial frequency k is $s(k) = \int \rho(z) e^{-i2\pi k z}\, dz$,
where $k = \gamma\!\!\!\!-\, G t'$, and $t' = t - t_E$.

An inverse FT $\rho(z) = \int s(k) e^{+i2\pi k z}\, dk$ gives us $\rho(z)$,
the (proton) spin density which changes z which is our “intensity” image.

Homework

Do 9.1, Look at 9.3, Do 9.4, Read the rest of the chapter, 9.4.2-.